SOME OF MY FAVORITE OPEN PROBLEMS

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To the memory of Antonio Ambrosetti, a dearly missed friend, and master of Nonlinear Analysis

ABSTRACT. This is a selection of the main open problems I raised throughout my career and that have resisted so far. This is not an exhaustive list, but striking questions fairly easy to state; some were raised 40 years ago, others quite recently.

1. An elliptic equation involving the critical exponent in 3D.

Let Ω be the unit ball in $\mathbb{R}^3$. Consider the equation

$$\begin{cases} -\Delta u = u^5 + \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where the unknown $u : \Omega \to \mathbb{R}$ is a smooth function and $\lambda \in \mathbb{R}$ is a parameter.

A natural question is whether (1.1) admits a non-trivial solution, $u \not\equiv 0$. Note that the exponent 5 corresponds to the critical Sobolev exponent $(N+2)/(N-2)$ when $N = 3$, which produces notorious difficulties. The answer, which depends on λ, is known for a large class of λ's ; however, for one interval of λ's the answer has remained undecided over the past forty years. Let $\lambda_1 = \pi^2$ be the first eigenvalue of $-\Delta$ on Ω with zero Dirichlet conditions.

Open Problem 1.1 (implicit in ([BrNi1]). Assume that

$$0 < \lambda \leq \lambda_1/4. \tag{1.2}$$

Does there exist a solution $u \not\equiv 0$ of (1.1)?

Several comments are in order:

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a) The answer is not known even if (1.2) is replaced by a sub-interval, e.g. $0 < \lambda < \delta$ with δ small.

b) When $\lambda \le 0$ the only solution of (1.1) is $u \equiv 0$; this is a celebrated result of Pohozaev (1965).

c) When

$$\lambda_1/4 < \lambda < \lambda_1 \tag{1.3}$$

there exists a radial solution $u > 0$ in Ω of (1.1). This is a central result from Brezis-Nirenberg [BrNi1, Theorem 1.2] (see also [Br1] and [Dr]).

d) When

$$0 < \lambda \le \lambda_1/4, \tag{1.4}$$

any radial solution u of (1.1) must be $u \equiv 0$ (see [BrNi1, proof of Lemma 1.4]). In particular (via Gidas-Ni-Nirenberg) there exists no solution $u > 0$ in Ω of (1.1). Therefore if (1.4) holds and a solution $u \not\equiv 0$ of (1.1) exists, it must be *non-radial and sign-changing.*

e) When

$$\lambda \ge \lambda_1 \tag{1.5}$$

there exist sign-changing solutions of (1.1) - but no solution $u > 0$ of (1.1), (see [Co]). In the bifurcation diagram, branches of solutions emanate from the eigenvalues associated with non-radial sign-changing eigenfunctions. It would be interesting to decide whether such branches "reach" the interval $(0, \lambda_1/4)$; they might instead admit e.g. vertical asymptotes at values of $\lambda \ge \lambda_1/4$.

2. Questions of uniqueness and radial symmetry arising from the Ginzburg-Landau system.

Let Ω be the unit disc in $\mathbb{R}^2$. Consider the system

$$\begin{cases} -\Delta u = \frac{1}{\varepsilon^2} u(1 - |u|^2) & \text{in } \Omega, \\ u(x) = x & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\varepsilon > 0$ is a given parameter and the unknown u maps Ω into $\mathbb{R}^2$.

It is easy to check that (2.1) admits a solution u of the form

$$u(x) = \frac{x}{|x|} f_\varepsilon(|x|) \tag{2.2}$$

where $f_\varepsilon : [0,1] \to \mathbb{R}$ satisfies the ODE

$$\begin{cases} -f'' - \frac{1}{r} f' + \frac{1}{r^2} f = \frac{1}{\varepsilon^2} f(1-f^2) \text{ in } (0,1), \\ f(0) = 0 \text{ and } f(1) = 1. \end{cases} \tag{2.3}$$

In fact (2.3) admits a unique solution (see [BBH, Appendix II]). We will denote by $U_\varepsilon(x)$ the solution of (2.1) given by (2.2)-(2.3), and we call it the radially symmetric (or just the radial) solution of (2.1). A long-standing open problem is whether $U_\varepsilon(x)$ is also the unique solution of (2.1):

Open Problem 2.1 ([BBH, Problem 10 in Chapter XI]). Is the radial solution U_ε the only solution of (2.1)?

A positive answer would, in particular, imply that solutions of some specific nonlinear *systems* of PDEs inherit the radial symmetry of the data - a property reminiscent of the celebrated Gidas-Ni-Nirenberg result relative to *positive* solutions of some *scalar* PDEs.

Note that (2.1) has a variational structure: the solutions of (2.1) are the critical points of the Ginzburg-Landau energy.

$$E_\varepsilon(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_\Omega (|u|^2 - 1)^2,$$

subject to the boundary condition $u \in H^1_g(\Omega; \mathbb{R}^2)$, where $g(x) = x$ on $\partial\Omega$.

The answer to Open Problem 2.1 is known to be positive in two "opposite" cases:

a) When ε is sufficiently *large*; more precisely $\varepsilon \geq 1/\sqrt{\lambda_1}$ where λ_1 is the first eigenvalue of $-\Delta$ under zero Dirichlet condition. Indeed it is easy to check that E_ε is strictly convex when $\varepsilon \geq 1/\sqrt{\lambda_1}$ and its unique minimizer is also its unique critical point.

b) When ε is sufficiently *small*: $\varepsilon < \varepsilon_0$ for some appropriate ε_0. This result is due to Pacard-Rivière [PR]. Their proof is highly non-trivial and fills a significant part of the monograph [PR]; it would be interesting to find a simpler proof.

The intermediate range $\varepsilon_0 \leq \varepsilon < 1/\sqrt{\lambda_1}$ is totally open. An easier question still unresolved is:

Open Problem 2.2. Is U_ε a minimizer of E_ε on $H^1_g(\Omega; \mathbb{R}^2)$ for any $\varepsilon > 0$?

Note that a positive answer to Open Problem 2.1 implies a positive answer to Open Problem 2.2, since any minimizer of E_ε on $H^1_g(\Omega; \mathbb{R}^2)$ is a solution of (2.1), and by uniqueness it would coincide with U_ε.

The next result provides substantial evidence that the answer to Open Problem 2.2 is positive.

Theorem 2.1 (Mironescu [Mi1], see also Lieb-Loss [LL]). *For every $\varepsilon > 0, U_\varepsilon$ is a local minimizer of E_ε; moreover $D^2 E_\varepsilon(U_\varepsilon)$ is positive definite.*

Following Brezis [Br3, Open Problem 6] one may ask similar questions when Ω is the unit ball in $\mathbb{R}^N, N \geq 3$ and $u : \Omega \to \mathbb{R}^2$; the counterpart of (2.3) is

$$\begin{cases} -f'' - \frac{(N-1)}{r} f' + \frac{N-1}{r^2} f = \frac{1}{\varepsilon^2} f(1-f^2) \text{ in } (0,1), \\ f(0) = 0 \text{ and } f(1) = 1. \end{cases} \tag{2.4}$$

Ignat-Nguyen [IN] established the analogue of Theorem 2.1 in any dimension $N \geq 3$, while Ignat-Nguyen-Slastikov-Zarnescu [INSZ] proved that in dimension $N \geq 7$, U_ε is a *global* minimizer of E_ε on $H^1_g(\Omega); \mathbb{R}^N)$ for any $\varepsilon > 0$; in fact U_ε is the unique global minimizer of E_ε.

One can also raise identical questions for the $p-$GL energy

$$E_{\varepsilon,p}(u) := \frac{1}{p} \int_\Omega |\nabla u|^p + \frac{1}{4\varepsilon^2} \int_\Omega (|u|^2 - 1)^2,$$

where $p > 1$, Ω is the unit disc in $\mathbb{R}^2$ and $u \in W^{1,p}_g(\Omega; \mathbb{R}^2)$, with $g(x) = x$ on $\partial\Omega$.

A somewhat related direction concerns the "asymptotic radial symmetry" as $\varepsilon \to 0$; more specifically:

Open Problem 2.3 ([BM, Open Problem 30]). Assume $p > 2$ and let u_ε be any minimizer (resp. critical point) of $E_{\varepsilon,p}$ on $W^{1,p}_g(\Omega, \mathbb{R}^2)$. Does $u_\varepsilon(x) \to x/|x|$ in $\Omega \setminus \{0\}$ as $\varepsilon \to 0$?

Note that the answer to the same problem when $p = 2$ is positive (see [BBH] for minimizers and [PR] for critical points). When $p < 2$ the answer is positive for minimizers (this is an immediate consequence of Theorem 13.6 in [BM]) and is open for general critical points.

Entire solutions of the Ginzburg-Landau equation are also of interest. Consider the system

$$-\Delta u = u(1 - |u|^2) \text{ on } \mathbb{R}^2, \tag{2.5}$$

and the condition at infinity

$$\lim_{|x| \to \infty} |u(x)| = 1, \tag{2.6}$$

where u is a smooth function from $\mathbb{R}^2$ into $\mathbb{R}^2 \simeq \mathbb{C}$. Here the parameter ε is irrelevant since it can be scaled out .

Property (2.6) allows to define the degree of u at infinity

$$\deg(u,\infty) := \deg\left(\frac{u(Rx)}{|u(Rx)|}; x \in \mathbb{S}^1\right) \text{ for } R \text{ sufficiently large.} \tag{2.7}$$

Given any $q \in \mathbb{Z}, q \neq 0$, there exists a distinguished solution u of (2.5)-(2.6) given in polar coordinates by

$$u(r,\theta) = e^{iq\theta} g_q(r), \tag{2.8}$$

where $g_q : [0,\infty) \to [0,1)$ satisfies the ODE

$$\begin{cases} -g'' - \frac{1}{r}g' + \frac{q^2}{r^2}g = g(1-g^2) \text{ on } (0,\infty), \\ g(0) = 0 \text{ and } \lim_{r\to\infty} g(r) = 1. \end{cases} \tag{2.9}$$

In fact, (2.9) admits a unique solution denoted g_q (see [BBH, Appendix III]). When $q \in \mathbb{Z}, q \neq 0$ we set $V_q(r,\theta) := e^{iq\theta} g_q(r)$. We also set $V_0 \equiv 1$. Note that

$$\deg(V_q,\infty) = q \quad \forall q \in \mathbb{Z}.$$

A long-standing open problem is whether the functions $V_q, q \in \mathbb{Z}$, are the *only* solutions of (2.5)-(2.6). More precisely

Open Problem 2.4 ([BBH, Problem 14], [BMR], [Br3]). Let u be any solution of (2.5) - (2.6). Does u coincide with V_q modulo rotation and translation, where $q = \deg(u,\infty)$? I.e., is $u(x) = \alpha V_q(x - x_0)$ for some $\alpha \in \mathbb{C}, |\alpha| = 1$, and $x_0 \in \mathbb{R}^2$?

Two partial results are known so far

Theorem 2.2 ([BMR]). *Assume that u is a solution of (2.5)-(2.6) such that*

$$\deg(u,\infty) = 0, \tag{2.10}$$

and which satisfies in addition

$$\int_{\mathbb{R}^2} (|u|^2 - 1)^2 < \infty. \tag{2.11}$$

Then $u = V_0$ modulo rotation and translation, i.e., u is a constant of modulus 1.

Theorem 2.3 ([Mi2]). *Assume that u is a solution of (2.5)-(2.6) such that*

$$\deg(u,\infty) = \pm 1, \tag{2.12}$$

and which satisfies in addition (2.11).

Then $u = V_{\pm 1}$ modulo rotation and translation.

Addressing Open Problem 2.4 when $|\deg(u,\infty)| \geq 2$, Ovchinnikov and Sigal [OS] have devised a strategy to construct non-radial solutions of (2.5)-(2.6), thereby providing a negative answer to Open Problem 2.4. However their proposed construction has been criticized (see Esposito [Es] and Kurzke [Ku]), and the problem remains open.

In another direction we point out that it is not known whether the conclusions of Theorem 2.2 and Theorem 2.3 remain true if one removes assumption (2.11). More generally

Open Problem 2.5 ([BMR, Problem 2], [Br3, Open Problem 2]). Assume that u satisfies (2.5) and (2.6). Does (2.11) hold?

Finally, we mention that property (2.11) appears quite naturally in connection with solutions of (2.5). In particular the functions $V_q, q \in \mathbb{Z}$, satisfy (2.11); more precisely (see e.g. [Sh1]), for any $q \in \mathbb{Z}$,

$$|V_q(x)| = 1 - \frac{q^2}{2|x|^2} + o\left(\frac{1}{|x|^2}\right) \text{ as } |x| \to \infty.$$

Also, one can show (see [BMR] and [Sh1]) that any solution of (2.5) satisfying (2.11) enjoys the following properties:

a)
$$\lim_{|x|\to\infty} |u(x)| = 1, \text{ so that } q = \deg(u,\infty) \text{ is well-defined,}$$

b)
$$|u(x)| = 1 - \frac{q^2}{2|x|^2} + o\left(\frac{1}{|x|^2}\right) \text{ as } |x| \to \infty,$$

c)
$$\int_{\mathbb{R}^2} (|u|^2 - 1)^2 = 2\pi q^2,$$

d)
$$\lim_{|x|\to\infty} |u(x) - \alpha V_q(x)| = 0 \text{ for some } \alpha \in \mathbb{C}, |\alpha| = 1.$$

3. Harmonic maps from the disc to $\mathbb{S}^2$.

Let Ω be the unit disc in $\mathbb{R}^2$. A harmonic map to $\mathbb{S}^2$ is a smooth map $u : \Omega \to \mathbb{R}^3$ satisfying

$$-\Delta u_i = u_i |\nabla u|^2, \quad i = 1, 2, 3, \text{ in } \Omega, \tag{3.1}$$

$$|u(x)| = 1 \quad \text{in } \Omega. \tag{3.2}$$

Given a smooth map $g : \partial\Omega \to \mathbb{S}^2$ we add the boundary condition

$$u = g \quad \text{on } \partial\Omega. \tag{3.3}$$

One could also define a concept of weak harmonic map, i.e., a map $u \in H^1_g(\Omega; \mathbb{S}^2)$ satisfying (3.1) in the sense of distributions. A celebrated result of F. Hélein (1990) asserts that weak harmonic maps are smooth.

The solutions of (3.1)-(3.3) correspond to critical points of the energy

$$E(u) = \int_\Omega |\nabla u|^2 \text{ on } \mathcal{E} = H^1_g(\Omega; \mathbb{S}^2). \tag{3.4}$$

It is easy to produce a solution of (3.1)-(3.3) by minimizing E on $\mathcal{E}$. Denote by $\underline{u}$ such a minimizer (it need not be unique). A natural question is whether there are other solutions. In fact one might expect that (3.1)-(3.3) admits infinitely many solutions. The reason is that $\mathcal{E}$ has infinitely many connected components classified by a topological degree. More precisely, given $u \in \mathcal{E}$, let $v : \mathbb{S}^2 \to \mathbb{S}^2$ be defined by

$$v(x, y, z) := \begin{cases} u(x, y) & \text{if } (x, y, z) \in \mathbb{S}^2 \text{ and } z \geq 0, \\ \underline{u}(x, y) & \text{if } (x, y, z) \in \mathbb{S}^2 \text{ and } z \leq 0. \end{cases}$$

Note that $v \in H^1(\mathbb{S}^2; \mathbb{S}^2)$ since $u = \underline{u}$ on $\partial\Omega$. Hence $\deg v$ is well-defined (see [BrNi2]) and in fact

$$\deg v = \frac{1}{4\pi} \int_\Omega u \cdot u_x \wedge u_y - \frac{1}{4\pi} \int_\Omega \underline{u} \cdot \underline{u}_x \wedge \underline{u}_y,$$

by Kronecker's formula. One may thus decompose $\mathcal{E}$ into its connected components

$$\mathcal{E} = \bigcup_{k \in \mathbb{Z}} \mathcal{E}_k, \text{ where } \mathcal{E}_k = \{u \in \mathcal{E}; \deg v = k\}. \tag{3.5}$$

It is tempting to minimize E in each class $\mathcal{E}_k$. However $\operatorname*{Inf}_{\mathcal{E}_k} E$ need not be achieved (except when $k = 0$ since $\underline{u} \in \mathcal{E}_0$); the reason being that the degree is *not* continuous under weak convergence in H^1. Thus, if (u_n) is a minimizing sequence in $\mathcal{E}_k$ and $u_n \rightharpoonup u_\infty$ weakly in H^1, the limit u_∞ might "jump" to another class $\mathcal{E}_\ell, \ell \neq k$, and will not be a minimizer of E in $\mathcal{E}_k$. This scenario can really occur. For example, if $g \equiv C$ is a constant, a result of Lemaire [Le] asserts that $u \equiv C$ is the only solution of (3.1)-(3.3). As a consequence $\operatorname*{Inf}_{\mathcal{E}_k} E$ is achieved only when $k = 0$.

Here is a general result in the positive direction.

Theorem 3.1 ([BrCo], [Jo]). *Assume $g \not\equiv C$, then $\operatorname*{Inf}_{\mathcal{E}_k} E$ is achieved at least in one of the classes $\mathcal{E}_{+1}$ or $\mathcal{E}_{-1}$. As a consequence problem (3.1)-(3.3) admits at least two solutions (including $\underline{u}$).*

Little is known concerning the existence of additional solutions, even when g has a simple form (see however works by Jie Qing, A. Soyeur, Morgan Pierre, L. Oswald and G. Paulik). Consider the boundary condition

$$g(x,y) = \left(Rx, Ry, \sqrt{1-R^2}\right) \text{ for } (x,y) \in \partial\Omega, \tag{3.6}$$

with $0 < R < 1$. In this case one can write down two explicit solutions of (3.1)-(3.3):

$$\underline{u}(x,y) = \frac{2\lambda}{\lambda^2 + r^2}(x,y,\lambda) + (0,0,-1) \quad \text{ for } (x,y) \in \Omega, \tag{3.7}$$

and

$$\bar{u}(x,y) = \frac{2\mu}{\mu^2 + r^2}(x,y,-\mu) + (0,0,1) \quad \text{ for } (x,y) \in \Omega, \tag{3.8}$$

where $r^2 = x^2 + y^2, \lambda = \frac{1}{R} + \sqrt{\frac{1}{R^2} - 1}$ and $\mu = \frac{1}{R} - \sqrt{\frac{1}{R^2} - 1}$.

It is not difficult to check that $\underline{u}$ is a minimizer of E in $\mathcal{E}$ and that $\bar{u}$ is a minimizer of E in $\mathcal{E}_{-1}$.

More precisely $\underline{u}$ is the unique minimizer of E in $\mathcal{E}_0$ and $\bar{u}$ is the unique minimizer of E in $\mathcal{E}_{-1}$. Moreover $\operatorname*{Inf}_{\mathcal{E}_k} E$ is not achieved when $k \neq 0$ and $k \neq -1$. This does not exclude the possible existence of other solutions of (3.1)-(3.3):

Open Problem 3.1 ([BrCo]). Assume g is given by (3.6). Are there other solutions of (3.1)-(3.3) besides $\underline{u}$ and $\bar{u}$?

Either way, the answer to Open Problem 3.1 would be illuminating. A negative answer might possibly shed some light on the important question whether solutions of specific nonlinear systems inherit the symmetry of the data –assuming the first step in the proof establishes that any solution is radially symmetric. A positive answer (more than 2 solutions) might involve the development of new techniques for finding non-minimizing critical points in variational problems with lack of compactness.

4. Continuous harmonic maps from B^3 to $\mathbb{S}^2$.

Let $\Omega = B^3$ be the unit ball in $\mathbb{R}^3$. A (weak) harmonic map to $\mathbb{S}^2$ is a map $u \in H^1(\Omega; \mathbb{R}^3)$ satisfying

$$-\Delta u_i = u_i |\nabla u|^2 \quad i = 1, 2, 3, \text{ in } \Omega, \tag{4.1}$$

$$|u(x)| = 1 \quad \text{ in } \Omega. \tag{4.2}$$

Given a smooth map $g : \partial\Omega \simeq \mathbb{S}^2 \to \mathbb{S}^2$ we add the boundary condition

$$u = g \quad \text{ on } \partial\Omega. \tag{4.3}$$

Solutions of (4.1)-(4.3) correspond to critical points of the energy

$$E(u) = \int_\Omega |\nabla u|^2 \quad \text{ on } H^1_g(\Omega; \mathbb{S}^2). \tag{4.4}$$

Note that $H^1_g(\Omega; \mathbb{S}^2)$ is always non-empty since $u(x) = g(x/|x|) \in H^1_g(\Omega; \mathbb{S}^2)$.

It is therefore easy to produce solutions of (4.1)-(4.3) e.g. by considering minimizers of the problem

$$\text{Min}\left\{E(u); u \in H^1_g(\Omega; \mathbb{S}^2)\right\}. \tag{4.5}$$

In contrast with the 2D case (see Section 3) weak harmonic maps need not be smooth - and not even continuous. The optimal regularity result for minimizers is known from the works of Schoen-Uhlenbeck [SU] and Brezis-Coron-Lieb [BCL]: any minimizer u of (4.5) is smooth in $\bar{\Omega}$ except at a finite number of points (a_i) in Ω and near each a_i, u behaves like $\pm(x - a_i)/|x - a_i|$ modulo a rotation.

When $\deg g \neq 0$, $\left\{u \in C(\bar{\Omega}; \mathbb{S}^2); u = g \text{ on } \partial\Omega\right\} = \emptyset$, and thus singularities are unavoidable. Since we will be concerned with the existence of continuous harmonic maps satisfying (4.3) we assume throughout this section that

$$\deg g = 0. \tag{4.6}$$

Here is a long-standing open problem originally posed by R. Schoen in the mid-1980's:

Open Problem 4.1 ([HaLi], [Ha], [Br5, Open Problem 3]). Assume that (4.6) holds. Does there exist a continuous harmonic map satisfying (4.1)-(4.3)?

Even in the absence of a topological obstruction (i.e., when (4.6) holds), minimizers in (4.5) can still have singularities, and therefore will not provide a solution to Open Problem 4.1. This is a consequence of a remarkable gap phenomenon discovered by Hardt-Lin [HaLi] (see also [Br2]): There exist smooth maps $g : \partial\Omega \to \mathbb{S}^2$ satisfying (4.6) and such that

$$\text{Min } \left\{E(u); u \in H^1_g(\Omega; \mathbb{S}^2)\right\} < \text{ Inf } \left\{E(u); u \in H^1_g(\Omega; \mathbb{S}^2) \cap C(\bar{\Omega})\right\}. \tag{4.7}$$

In order to solve Open Problem 4.1 it is tempting to tackle

Open Problem 4.2 ([HaLi], [Ha]). Is the

$$\text{Inf } \left\{E(u); u \in H^1_g(\Omega; \mathbb{S}^2) \cap C(\bar{\Omega})\right\} \tag{4.8}$$

achieved?

Clearly, a positive answer to Open Problem 4.2 would provide a solution to Open Problem 4.1. But in principle it might happen that the answer to Open Problem 4.2 is negative while the answer to Problem 4.1 is positive. (Can this scenario occcur?)

A natural strategy to solve Open Problem 4.2 has been developed in Bethuel-Brezis-Coron [BBC] via the concept of *relaxed energy* defined as follows.

Fix $u \in H^1_g(\Omega; \mathbb{S}^2)$. In general there exist no sequence (u_n) in $H^1_g(\Omega; \mathbb{S}^2) \cap C(\bar{\Omega})$ such that $u_n \to u$ strongly in H^1. (This is e.g. a consequence of the gap phenomenon (4.7)). However there always exists a sequence (u_n) in $H^1_g(\Omega; \mathbb{S}^2) \cap C(\bar{\Omega})$ such that $u_n \rightharpoonup u$ weakly in H^1 (see [Be]). Set

$$R(u) := \text{ Inf } \left\{\liminf_{n\to\infty} E(u_n); u_n \in H^1_g(\Omega; \mathbb{S}^2) \cap C(\bar{\Omega}) \text{ and } u_n \rightharpoonup u \text{ weakly in } H^1\right\}, \tag{4.9}$$

where the first Inf is taken over all sequences (u_n) as above. (R stands for relaxed).

The functional R is well-defined on $H^1_g(\Omega; \mathbb{S}^2)$ and it is weakly lower semicontinuous. Therefore

$$\text{Min } \left\{R(u); u \in H^1_g(\Omega; \mathbb{S}^2)\right\} \text{ is achieved.} \tag{4.10}$$

We claim that if the Inf in (4.8) is achieved, say by some $\underline{u} \in H^1_g(\Omega; \mathbb{S}^2) \cap C(\bar{\Omega})$, then $\underline{u}$ is also a minimizer in (4.10).

Indeed, we clearly have

$$E(v) \le R(v) \quad \forall v \in H^1_g(\Omega; \mathbb{S}^2) \tag{4.11}$$

and

$$R(v) \le E(v) \quad \forall v \in H^1_g(\Omega; \mathbb{S}^2) \cap C(\bar{\Omega}) \tag{4.12}$$

(just take $u_n \equiv v$ in (4.9)).

Therefore

$$R(v) = E(v) \quad \forall v \in H^1_g(\Omega; \mathbb{S}^2) \cap C(\bar{\Omega}). \tag{4.13}$$

We also have

$$\operatorname*{Inf}_{v \in H^1_g} R(v) = \operatorname*{Inf}_{v \in H^1_g \cap C} E(v). \tag{4.14}$$

Indeed, it is clear that

$$\operatorname*{Inf}_{v \in H^1_g} R(v) \le \operatorname*{Inf}_{v \in H^1_g \cap C} R(v) = \operatorname*{Inf}_{v \in H^1_g \cap C} E(v) \text{ by (4.13).} \tag{4.15}$$

On the other hand, given any $w \in H^1_g(\Omega; \mathbb{S}^2)$, there exists (by definition of R) a sequence (w_n) in $H^1_g(\Omega; \mathbb{S}^2) \cap C(\bar{\Omega})$ such that $w_n \rightharpoonup w$ weakly in H^1 and

$$E(w_n) \to R(w). \tag{4.16}$$

Thus

$$\operatorname*{Inf}_{v \in H^1_g \cap C} E(v) \le E(w_n) \quad \forall n. \tag{4.17}$$

Passing to the limit in (4.17) using (4.16) gives

$$\operatorname*{Inf}_{v \in H^1_g \cap C} E(v) \le R(w) \quad \forall w \in H^1_g(\Omega; \mathbb{S}^2),$$

so that

$$\operatorname*{Inf}_{v \in H^1_g \cap C} E(v) \le \operatorname*{Inf}_{v \in H^1_g} R(v). \tag{4.18}$$

Combining (4.15) and (4.18) yields (4.14).

We may now return to the above claim concerning (4.8). Assume that $\operatorname*{Inf}_{v \in H^1_g \cap C} E(v)$ is achieved, say by some $\underline{u} \in H^1_g \cap C$, then by (4.13) and (4.14)

$$R(\underline{u}) = E(\underline{u}) = \operatorname*{Inf}_{v \in H^1_g \cap C} E(v) = \operatorname*{Inf}_{v \in H^1_g} R(v)$$

and hence $\underline{u}$ is a minimizer for the problem (4.10).

Therefore prospective solutions of Open Problem 4.2 are to be found among the minimizers of (4.10). This leads us to

Open Problem 4.3 ([BBC]). Are the minimizers of (4.10) continuous on $\bar{\Omega}$? (Is it true for at least one of the minimizers?)

If the answer to Open Problem 4.3 is positive and $\underline{u} \in H^1_g(\Omega; \mathbb{S}^2) \cap C(\bar{\Omega})$ is such a minimizer then $\underline{u}$ satisfies, by (4.13),

$$E(\underline{u}) = R(\underline{u}) = \operatorname*{Inf}_{v \in H^1_g} R(v) = \operatorname*{Inf}_{v \in H^1_g \cap C} E(v),$$

and therefore we have solved Open Problem 4.2 since $\underline{u}$ is a minimizer for Open Problem 4.2.

In tackling Open Problem 4.3 we have at our disposal explicit representation formulas for R. We first need some notations. Given $u \in H^1(\Omega; \mathbb{S}^2)$, consider the D-field (introduced in [BCL])

$$D(u) = \left(\det\left(u, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}\right), \det\left(\frac{\partial u}{\partial x_1}, u, \frac{\partial u}{\partial x_3}\right), \det\left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, u\right)\right),$$

so that $D(u) \in L^1(\Omega; \mathbb{R}^3)$, and one can define the distribution

$$Ju := \frac{1}{3} \operatorname{div} D(u) \in \mathcal{D}'(\Omega; \mathbb{R}). \tag{4.19}$$

J stands for Jacobian because Ju coincides with the usual Jacobian when $u \in C^2(\Omega; \mathbb{R}^3)$. Since u takes its values in $\mathbb{S}^2$, it follows that $Ju = 0$ in the region where u is smooth. As we are going to see below (in (4.23)), Ju carries important information about the location of the topological singularities of u.

Theorem 4.1 ([BBC]). *For every* $u \in H^1_g(\Omega; \mathbb{S}^2)$

$$R(u) = \int_\Omega |\nabla u|^2 + S(u), \tag{4.20}$$

where

(4.21)

$$S(u) := 2 \operatorname{Sup} \left\{ \int_\Omega D(u) \cdot \nabla\zeta - \int_{\partial\Omega} (\operatorname{Jac} g)\, \zeta;\ \zeta \in W^{1,\infty}(\Omega; \mathbb{R}), ||\nabla\zeta||_{L^\infty} \le 1 \right\},$$

and Jac g *denotes the Jacobian determinant of* $g : \partial\Omega \simeq \mathbb{S}^2 \to \mathbb{S}^2$.

As a consequence of (4.20) and (4.21) we see that R is continuous for the strong topology of H^1 (it is even locally Lipschitz). Thus it is helpful to know the value of R on a dense subset of $H^1_g(\Omega; \mathbb{S}^2)$ for the strong topology, in particular on the class

$$\mathcal{C} = \left\{ u \in H^1_g(\Omega; \mathbb{S}^2); u \text{ is continuous on } \bar{\Omega} \text{ except on a finite set } (a_i), 1 \le i \le k \text{ in } \Omega \right\}$$

which is dense in $H^1_g(\Omega;\mathbb{S}^2)$ (by a classical result of Bethuel).

When $u \in \mathcal{C}$ we have (see [BCL]) the important formula

$$Ju = \frac{4\pi}{3}\sum_{i=1}^{k}\deg(u,a_i)\delta_{a_i} \tag{4.22}$$

where $\deg(u,a_i)$ is the degree of u restricted to a small ball centered at a_i. Relabelling the points (a_i) as P_i and $N_i, 1 \le i \le \ell$, including multiplicities, we may write

$$Ju = \frac{4\pi}{3}\sum_{i=1}^{l}(\delta_{P_i} - \delta_{N_i}). \tag{4.23}$$

(Here we use the fact that $\sum_i \deg(u,a_i) = \deg g = 0$). Inserting (4.23) in (4.21) yields (after integration by parts)

$$S(u) = 8\pi \operatorname{Sup}\left\{\sum_{i=1}^{\ell}[\zeta(P_i) - \zeta(N_i)];\, \zeta \in W^{1,\infty} \text{ and } ||\nabla\zeta||_{L^\infty} \le 1\right\}, \tag{4.24}$$

which implies (see [BCL]) that

$$S(u) = 8\pi \operatorname*{Min}_{\sigma}\sum_{i=1}^{\ell}\left|P_i - N_{\sigma(i)}\right|, \tag{4.25}$$

where the $\operatorname*{Min}_{\sigma}$ is taken over all permutations σ of the integers $\{1,\dots,\ell\}$. This formula is closely connected to Optimal Transport as explained in [Br9], [BM].

Combining (4.20) and (4.25) we derive a remarkable explicit formula for R when $u \in \mathcal{C}$:

$$R(u) = \int_\Omega |\nabla u|^2 + 8\pi \operatorname*{Min}_{\sigma}\sum_{i=1}^{\ell}\left|P_i - N_{\sigma(i)}\right|. \tag{4.26}$$

In fact, there is a similar formula, just slightly more complicated for a general $u \in H^1_g(\Omega;\mathbb{S}^2)$. Namely one can show (see [BM]) that for every $u \in H^1_g(\Omega;\mathbb{S}^2)$ there exist sequences (P_i) and (N_i) such that $\sum_{i=1}^{\infty}|P_i - N_i| < \infty$, and

$$Ju = \frac{4\pi}{3}\sum_{i=1}^{\infty}(\delta_{P_i} - \delta_{N_i}).$$

Moreover

$$S(u) = 8\pi \text{ Inf } \left\{ \sum_{i=1}^{\infty} |\tilde{P}_i - \tilde{N}_i|; \sum (\delta_{\tilde{P}_i} - \delta_{\tilde{N}_i}) = \sum (\delta_{P_i} - \delta_{N_i}) \right\}.$$

The relaxed energy consists therefore of the usual energy $\int_\Omega |\nabla u|^2$ plus an additional term involving the "interaction of singularities" - a quantity which may possibly be of physical interest.

Some partial regularity results concerning the minimizers of the relaxed energy have been obtained by Giaquinta-Modica-Soucek, Hardt-Lin-Poon and others, but the answer to Open Problem 4.3 remains elusive. An easier question still unresolved is:

Open Problem 4.4. Let $\underline{u}$ be a minimizer of (4.10). Is it true that $S(\underline{u}) = 0$, i.e., $J\underline{u} = 0$? or equivalently that there exists a sequence (u_n) in $H^1_g(\Omega; \mathbb{S}^2) \cap C(\bar{\Omega})$ such that u_n converges to $\underline{u}$ *strongly* in H^1 ?

5. Degree, $VMO, W^{1/p,p}$, and Fourier.

Recall that

(5.1)

$$BMO(\mathbb{S}^1) := \left\{ f \in L^1(\mathbb{S}^1; \mathbb{C}); |f|_{BMO} := \sup_I \frac{1}{|I|^2} \int_I \int_I |f(x) - f(y)| dx dy < \infty \right\},$$

where $\sup\limits_I$ is taken over all arcs of circle in $\mathbb{S}^1$,

$$VMO(\mathbb{S}^1) := \left\{ f \in BMO(\mathbb{S}^1); \lim_{|I| \to 0} \frac{1}{|I|^2} \int_I \int_I |f(x) - f(y)| dx dy = 0 \right\}. \tag{5.2}$$

Obviously $L^\infty \subset BMO$ and $C \subset VMO$; moreover VMO is the closure of C in BMO.

For any $1 < p < \infty$

(5.3)

$$W^{1/p,p}(\mathbb{S}^1) := \left\{ f \in L^1(\mathbb{S}^1; \mathbb{C}); |f|^p_{W^{1/p,p}} := \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \frac{|f(x) - f(y)|^p}{|x - y|^2} dx dy < \infty \right\}.$$

As usual, $H^{1/2} = W^{1/2,2}$. It follows easily from Hölder that

$$W^{1/p,p}(\mathbb{S}^1) \subset VMO(\mathbb{S}^1) \quad \forall p \in (1, \infty). \tag{5.4}$$

Clearly the classes $W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$ increase with p on $(1, \infty)$; indeed if $p > q$

$$|f(x) - f(y)|^p \le 2^{p-q} |f(x) - f(y)|^q.$$

Brezis-Nirenberg [BrNi2] have established that degree theory persists in $VMO(\mathbb{S}^1; \mathbb{S}^1)$; in particular the new degree coincides with the classical degree on $C(\mathbb{S}^1; \mathbb{S}^1)$, and if $f_n, f \in VMO$ satisfy $f_n \to f$ in BMO, then $\deg f_n \to \deg f$.

The starting point in this section is the following estimate for the degree.

Theorem 5.1 ([BBM2, Corollary 0.5]). *For every $1 < p < \infty$ there exists a constant C_p (depending only on p) such that*

$$|\deg f| \leq C_p |f|^p_{W^{1/p,p}} \quad \forall f \in W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1). \tag{5.5}$$

Note that estimate (5.5) "deteriorates" as $p \searrow 1$ since $|f|_{W^{1/p,p}} \to \infty$ as $p \searrow 1$ unless f is a constant (see [Br4] and the beginning of Section 9 below). Therefore it is tempting to monitor the behavior of the constant C_p as $p \searrow 1$. A reasonable conjecture is:

Open Problem 5.1 ([Br7, Remark 7]). Does there exist a (universal) constant c such that, for every $1 < p \leq 2$,

$$|\deg f| \leq c(p-1) \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{|f(x)-f(y)|^p}{|x-y|^2} \quad \forall f \in W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)? \tag{5.6}$$

There is strong evidence in support of (5.6) as $p \searrow 1$. Indeed, we have (as a consequence of [BrNg1, Proposition 1]), when f is smooth,

$$\lim_{p\searrow 1}(p-1)\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{|f(x)-f(y)|^p}{|x-y|^2} \simeq \int_{\mathbb{S}^1} |\dot f|, \tag{5.7}$$

while

$$|\deg f| \leq \frac{1}{2\pi}\int_{\mathbb{S}^1} |\dot f| \text{ by the Cauchy formula.} \tag{5.8}$$

A far-reaching extension of (5.5) is the following striking estimate for the degree.

Theorem 5.2. *There exists a constant C such that for every $f \in C(\mathbb{S}^1;\mathbb{S}^1)$*

$$|\deg f| \leq C \underset{[|f(x)-f(y)|\geq\sqrt{3}]}{\int_{\mathbb{S}^1}\int_{\mathbb{S}^1}} \frac{dxdy}{|x-y|^2}. \tag{5.9}$$

A weaker version of (5.9) where $\sqrt{3}$ is replaced by a small constant $\delta_0 > 0$ was originally announced in [BBM2] and proved in [BBM3, Theorem 4]. Subsequently Bourgain-Brezis-Nguyen [BBNg] pushed the estimate up to $\delta_0 = \sqrt{2}$. Finally Nguyen [Ng1] established that (5.9) holds and that $\sqrt{3}$ is optimal (see also [BM, Section 12.5]); the current proof is quite involved and it is natural to raise:

Open Problem 5.2 ([BM, Open Problem 18]). Is there a simpler, more geometric proof of Theorem 5.2? What is the best constant C in (5.9)? Is it achieved?

Given $f \in C(\mathbb{S}^1;\mathbb{S}^1)$ and $\delta > 0$, set

$$I_\delta(f) := \underset{[|f(x)-f(y)|\geq\delta]}{\int_{\mathbb{S}^1}\int_{\mathbb{S}^1}} \frac{dxdy}{|x-y|^2}. \tag{5.10}$$

As a consequence of (5.9) we know that $\forall f$,

$$|\deg f| \leq C I_\delta(f) \quad \forall \delta \leq \sqrt{3}. \tag{5.11}$$

Note that this estimate deteriorates as $\delta \searrow 0$ since $I_\delta(f) \nearrow \infty$ as $\delta \searrow 0$ (unless f is a constant). Therefore it is natural to try to improve (5.11) by replacing C with a constant C_δ which tends to 0 as $\delta \searrow 0$. A reasonable conjecture is:

Open Problem 5.3 ([Br7, Open Problem 3]). Does there exist a (universal) constant c such that

$$|\deg f| \leq c\delta I_\delta(f) \quad \forall f \in C(\mathbb{S}^1;\mathbb{S}^1) \quad \forall \delta \leq \sqrt{3}? \tag{5.12}$$

There is strong evidence in support of (5.12) as $\delta \to 0$. Indeed, when f is smooth, we have, $\lim\limits_{\delta\to 0} \delta I_\delta(f) \simeq \int_{\mathbb{S}^1} |\dot{f}|$ (see [Br8, Theorem 3.1], and [BrNg2, Proposition 1]), while $|\deg f| \leq \frac{1}{2\pi}\int_{\mathbb{S}^1} |\dot{f}|$ by the Cauchy formula.

An $\mathbb{S}^N$-version of (5.12) was established by Nguyen [Ng2] for any $N \geq 2$, but, surprisingly, the case $N = 1$ remains elusive!

Nonlocal energies have become popular in recent years and it is of interest to study the "least amount of $W^{1/p,p}$-energy" necessary to produce a map $f : \mathbb{S}^1 \to \mathbb{S}^1$ of prescribed degree d (see [Br7, Remark 5]). More precisely, given $1 < p < \infty$ and $d \in \mathbb{Z}$ set

$$m_{p,d} := \operatorname{Inf} \left\{ |f|^p_{W^{1/p,p}}; f \in W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1), \deg f = d \right\}. \tag{5.13}$$

From Theorem 5.1 above and [BM, Theorem 12.9] we know that for every $1 < p < \infty$, there exist two positive constants $\underline{c}_p$ and $\bar{c}_p$ such that

$$\underline{c}_p |d| \leq m_{p,d} \leq \bar{c}_p |d| \quad \forall d \in \mathbb{Z}. \tag{5.14}$$

This suggests the following:

Open Problem 5.4 ([BM, Open Problems 22 and 23]). Let $1 < p < \infty$. Is true that

$$m_{p,d} = |d| m_{p,1} \quad \forall d \in \mathbb{Z}? \tag{5.15}$$

Is the inf in (5.13) achieved?

The answer to both questions is positive for $p = 2$ (see [BM, Theorems 12.9 and 12.10]). Assuming that (5.15) does not hold, is there an explicit formula for

$$\lim_{d\to+\infty} \frac{m_{p,d}}{d} = \inf_{d>0} \frac{m_{p,d}}{d}? \tag{5.16}$$

Recall that the Fourier coefficients of a function $f \in L^2(\mathbb{S}^1;\mathbb{C})$ are given, for every $n \in \mathbb{Z}$, by

$$a_n = a_n(f) := \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta = \frac{1}{2\pi}\int_{\mathbb{S}^1} f(z)\bar{z}^n d\ell, \tag{5.17}$$

so that,

$$f(e^{i\theta}) = \sum_{n\in\mathbb{Z}} a_n e^{in\theta} \quad \text{ in } L^2(0,2\pi). \tag{5.18}$$

An elementary computation (see [Br7, Lemma 5] or [BM, Lemma 12.5]) yields, for every $f \in H^{1/2}(\mathbb{S}^1)$,

$$|f|^2_{H^{1/2}(\mathbb{S}^1)} = \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{|f(x)-f(y)|^2}{|x-y|^2}dxdy = 4\pi^2 \sum_{n\in\mathbb{Z}} |n||a_n|^2. \tag{5.19}$$

The following striking formula connecting the degree of a map $f \in H^{1/2}(\mathbb{S}^1;\mathbb{S}^1)$ and its Fourier coefficients was brought to light in Brezis [Br7, Theorem 4] (see also [BM, Theorem 12.6]):

Theorem 5.3. *For every $f \in H^{1/2}(\mathbb{S}^1;\mathbb{S}^1)$ we have*

$$\deg f = \sum_{n\in\mathbb{Z}} n|a_n(f)|^2. \tag{5.20}$$

Equality (5.20) implies in particular that if $f, g \in H^{1/2}(\mathbb{S}^1;\mathbb{S}^1)$ satisfy $|a_n(f)| = |a_n(g)| \quad \forall n \in \mathbb{Z}$, then $\deg f = \deg g$. This formula has become the starting point of a challenging direction of research labeled "Can one hear the degree?" in [Br7]. More precisely:

$$\begin{cases} \text{Given two maps } f, g \in VMO(\mathbb{S}^1;\mathbb{S}^1) \text{ such that } |a_n(f)| = |a_n(g)| \\ \text{for all } n \in \mathbb{Z}, \text{can one conclude that } \deg f = \deg g? \end{cases} \tag{5.21}$$

The answer to question (5.21) turns out to be negative in general:

Theorem 5.4 ([BoKo]). *There exist two functions* $f, g \in C(\mathbb{S}^1; \mathbb{S}^1)$ *such that* $|a_n(f)| = |a_n(g)| \quad \forall n \in \mathbb{Z}$, *and* $\deg f \neq \deg g$.

The construction of Bourgain-Kozma in [BoKo] is quite elaborate and it would be desirable to find a simpler one.

On the other hand the answer is positive in some classes bigger than $H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$:

Theorem 5.5 ([Br7, Corollary 2], [BM, Corollary 12.3], [Ka]). *Assume that* $f, g \in W^{1/3,3}(\mathbb{S}^1; \mathbb{S}^1)$ *satisfy* $|a_n(f)| = |a_n(g)| \quad \forall n \in \mathbb{Z}$, *then* $\deg f = \deg g$. *In particular the conclusion holds if* $f, g \in C^{0,\alpha}(\mathbb{S}^1; \mathbb{S}^1)$ *with* $\alpha > 1/3$.

Note that the assertion in Theorem 5.5 is far from obvious since $W^{1/3,3}(\mathbb{S}^1; \mathbb{S}^1)$ is *strictly bigger* than $H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$ and thus the series $\sum_{n \in \mathbb{Z}} |n||a_n(f)|^2$ can be divergent for a general $f \in W^{1/3,3}(\mathbb{S}^1; \mathbb{S}^1)$. Note also that there is a wide "gap" between the positive result in Theorem 5.5 and the counterexample by Bourgain-Kozma in Theorem 5.4. It is not known whether $W^{1/3,3}$ is the sharp borderline:

Open Problem 5.5. What happens to Theorem 5.5 when $f, g \in W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$ with $p > 3$ (resp. $f, g \in C^{0,\alpha}(\mathbb{S}^1 : \mathbb{S}^1)$ with $\alpha \leq 1/3$)? The problem is open even under the stronger assumption that $f \in W^{1/p,p}$ with $2 < p \leq 3$ and $g \in W^{1/q,q}$ with $q > 3$ (resp. $f \in C^{0,\alpha}$ with $\alpha > 1/3$ and $g \in C^{0,\beta}$ with $\beta \leq 1/3$).

Theorem 5.5 is an immediate consequence of the following summation formula:

Theorem 5.6 ([Br7, Theorem 6], [BM, Theorem 12.7], [Ka]). *For every* $f \in W^{1/3,3}(\mathbb{S}^1; \mathbb{S}^1)$ *we have*

$$\deg f = \lim_{\varepsilon \searrow 0} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} n |a_n(f)|^2 \frac{\sin n\varepsilon}{n\varepsilon}. \tag{5.22}$$

In particular (5.22) holds if $f \in C^{0,\alpha}(\mathbb{S}^1; \mathbb{S}^1)$ *with* $\alpha > 1/3$.

It is not known whether different summation formulas might produce improvements of Theorem 5.5, e.g. assuming f belongs to a larger class $W^{1/p,p}$ with $p > 1/3$, or $C^{0,\alpha}$ with $\alpha \leq 1/3$. In fact, it is an open problem whether one can capture the degree of any map $f \in VMO(\mathbb{S}^1; \mathbb{S}^1)$ via a summation process involving only $|a_n(f)|, n \in \mathbb{Z}$. More precisely, by a summation process we mean a family $(\sigma_{n,\varepsilon}), n \in \mathbb{Z}, 0 < \varepsilon < 1$, satisfying:

$$\forall \varepsilon \in (0,1), \sup_{n \in \mathbb{Z}} |n||\sigma_{n,\varepsilon}| < \infty, \tag{5.23}$$

$$\forall n \in \mathbb{Z}, \quad \lim_{\varepsilon \searrow 0} \sigma_{n,\varepsilon} = 1. \tag{5.24}$$

Note that if (5.23)-(5.24) hold, then $\sum_{n\in\mathbb{Z}} n|a_n(f)|^2\sigma_{n,\varepsilon}$ is well-defined $\forall\varepsilon\in(0,1)$ and the question of interest is whether

$$\lim_{\varepsilon\searrow 0}\sum_{n\in\mathbb{Z}} n|a_n(f)|^2\sigma_{n,\varepsilon} = \deg f? \tag{5.25}$$

Open Problem 5.6. Given any $p > 3$ (resp. $\alpha \leq 1/3$) does there exist a summation process $(\sigma_{n,\varepsilon})$, depending only on p (resp. α) such that (5.25) holds for every $f \in W^{1/p,p}$ (resp. $f \in C^{0,\alpha}$)?

In the same vein one can raise:

Open Problem 5.7. Given any $f \in C(\mathbb{S}^1;\mathbb{S}^1)$ (resp. $f \in VMO(\mathbb{S}^1;\mathbb{S}^1)$) does there exist a summation process $(\sigma_{n,\varepsilon})$ (depending on f) such that (5.25) holds?

We call attention to the following assertion:

Corollary 5.1. *Given any summation process $(\sigma_{n,\varepsilon})$ there exists some $f \in C(\mathbb{S}^1;\mathbb{S}^1)$ (depending on $\sigma_{n,\varepsilon}$) such that $\sum_{n\in\mathbb{Z}} n|a_n(f)|^2\sigma_{n,\varepsilon}$ does not converge to* $\deg f$.

Corollary 5.1 is an immediate consequence (by contradiction) of Theorem 5.4. For special summation processes explicit f's satisfying the conclusion of Corollary 5.1 have been constructed by Korevaar [Ko] and Kahane [Ka]; these are:

a)
$$\sigma_{n,\varepsilon} := \begin{cases} 1 & \text{if } |n| \leq [1/\varepsilon] \text{ (the integer part of } 1/\varepsilon\text{)}, \\ 0 & \text{if } |n| > [1/\varepsilon], \end{cases}$$

b)
$$\sigma_{n,\varepsilon} := (1-\varepsilon)^{|n|},$$

c)
$$\sigma_{n,\varepsilon} := \frac{\sin n\varepsilon}{n\varepsilon} \text{ if } n \neq 0 \text{ and } \sigma_{0,\varepsilon} = 1.$$

A stronger version of Corollary 5.1 is open:

Open Problem 5.8. Does there exist some $f \in C(\mathbb{S}^1;\mathbb{S}^1)$ (resp. $f \in VMO(\mathbb{S}^1;\mathbb{S}^1)$) such that for any summation process $(\sigma_{n,\varepsilon})$, $\sum_{n\in\mathbb{Z}} n|a_n(f)|^2\sigma_{n,\varepsilon}$ does not converge to $\deg f$?

Note that a negative answer to Open Problem 5.8 amounts to a positive answer to Open Problem 5.7

An interesting direction of research concerns the distance between the homotopy classes of $W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1), 1 < p < \infty$ which are given by

$$\mathcal{E}_d = \{f \in W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1);\ \deg f = d\}, d \in \mathbb{Z}.$$

There are two natural notions of distance:

$$\text{dist}_{W^{1/p,p}}(\mathcal{E}_{d_1}, \mathcal{E}_{d_2}) = \underset{f\in\mathcal{E}_{d_1}}{\text{Inf}}\ \underset{g\in\mathcal{E}_{d_2}}{\text{Inf}}\ |f-g|_{W^{1/p,p}}$$

and

$$\text{Dist}_{W^{1/p,p}}(\mathcal{E}_{d_1}, \mathcal{E}_{d_2}) = \underset{f\in\mathcal{E}_{d_1}}{\text{Sup}}\ \underset{g\in\mathcal{E}_{d_2}}{\text{Inf}}\ |f-g|_{W^{1/p,p}}.$$

It turns out (see [BrNi2], [BMS]) that

$$\text{dist}_{W^{1/p,p}}(\mathcal{E}_{d_1}, \mathcal{E}_{d_2}) = 0, \quad \forall d_1, d_2 \in \mathbb{Z}.$$

On the other hand $\text{Dist}_{W^{1/p,p}}(\mathcal{E}_{d_1}, \mathcal{E}_{d_2})$ has an interesting interpretation:

Given $f \in \mathcal{E}_{d_1}$, $\underset{g\in\mathcal{E}_{d_2}}{\text{Inf}} |f-g|_{W^{1/p,p}}$ represents the least amount of energy required to pass from the given $f \in \mathcal{E}_{d_1}$ to a configuration in $\mathcal{E}_{d_2}$, and $\text{Dist}_{W^{1/p,p}}(\mathcal{E}_{d_1}, \mathcal{E}_{d_2})$ is the "highest price" one may have to pay as f runs in $\mathcal{E}_{d_1}$. A remarkable result of Shafrir asserts that this quantity depends only on $|d_1 - d_2|$.

Theorem 5.7 ([Sh, Confluentes]). *We have*

$$Dist^p_{W^{1/p,p}}(\mathcal{E}_{d_1}, \mathcal{E}_{d_2}) = m_{p,|d_1-d_2|}, \quad \forall p \in (1,\infty), \quad \forall d_1, d_2 \in \mathbb{Z}, \tag{5.26}$$

where $m_{p,d}$ is defined in (5.13).

In particular (when $p = 2$), (5.26) becomes

$$Dist^2_{H^{1/2}}(\mathcal{E}_{d_1}, \mathcal{E}_{d_2}) = 4\pi^2|d_1 - d_2|. \tag{5.27}$$

A similar conclusion when $\mathbb{S}^1$ is replaced by $\mathbb{S}^N$, with $N \geq 2$ or just $N = 2$, is widely open. Consider e.g. $H^1(\mathbb{S}^2;\mathbb{S}^2)$ and its homotopy classes

$$\mathcal{E}_d = \left\{f \in H^1(\mathbb{S}^2;\mathbb{S}^2); \deg f = d\right\},\ d \in \mathbb{Z}.$$

The quantity of interest is

$$\text{Dist}_{H^1}(\mathcal{E}_{d_1}, \mathcal{E}_{d_2}) = \underset{f\in\mathcal{E}_{d_1}}{\text{Sup}}\ \underset{g\in\mathcal{E}_{d_2}}{\text{Inf}}\ ||\nabla(f-g)||_{L^2}.$$

Open Problem 5.9 ([BMS]). Is true that

$$\text{Dist}^2_{H^1}(\mathcal{E}_{d_1}, \mathcal{E}_{d_2}) = 8\pi|d_1 - d_2| \quad \forall d_1, d_2 \in \mathbb{Z}? \tag{5.28}$$

The inequality $\leq$ is known, (see [BMS] and the references therein) but the reverse inequality ($\geq$) has been established only when $d_2 > d_1 \geq 0$ (see [BMS, Proposition 7.3]) . Even a much easier problem is open:

Is it true that $\mathrm{Dist}_{H^1}(\mathcal{E}_{d_1}, \mathcal{E}_{d_2}) = \mathrm{Dist}_{H^1}(\mathcal{E}_{d_2}, \mathcal{E}_{d_1}) \quad \forall d_1, d_2 \in \mathbb{Z}$?

In a totally different (but somewhat related) direction one may ask whether a version of the Brouwer fixed point theorem holds for VMO maps:

Open Problem 5.10. Let B be the closed unit ball in $\mathbb{R}^N$ and let f be a VMO map from B into itself. Is it true that for every $\varepsilon > 0$, the set

$$\{x \in B; \, |f(x) - x| < \varepsilon\}$$

has positive measure?

6. Unbounded extremal solutions.

Consider the nonlinear elliptic equation

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \subset \mathbb{R}^N, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N, \lambda > 0$ is a constant, and $f : [0, +\infty) \to (0, +\infty)$ is a smooth function satisfying

$$f(0) > 0, f \text{ is increasing and convex,} \tag{6.2}$$

and

$$\lim_{t \to \infty} \frac{f(t)}{t} = +\infty. \tag{6.3}$$

Some basic general results concerning problem (6.1) are summarized in the following (see [BrVa], [Br6] and the references therein):

Theorem 6.1. *There exists a constant $\lambda^* \in (0, +\infty)$ such that*

a) For every $\lambda \in (0, \lambda^*)$ problem (6.1) admits a minimal smooth solution denoted $\underline{u}(\lambda)$; moreover $\underline{u}(\lambda)$ increases with λ (for every $x \in \Omega$).

b) For $\lambda > \lambda^*$ there is no solution of (6.1).

c) $u^* = \lim_{\lambda \uparrow \lambda^*} \underline{u}(\lambda)$ is a weak solution of (6.1) in the sense that $u^* \in L^1(\Omega), f(u^*)\delta \in L^1(\Omega)$, where $\delta(x) =$ dist $(x, \partial\Omega)$ and

$$-\int_\Omega u^*\Delta\zeta = \lambda^*\int_\Omega f(u^*)\zeta \quad \forall\zeta\in C^2(\bar\Omega), \zeta=0 \textit{ on } \partial\Omega. \tag{6.4}$$

The function u^* is called the *extremal* solution of (6.1). A notable result of Martel asserts that u^* is the unique weak solution of (6.4) when $\lambda=\lambda^*$.

It was originally established by Nedev [Ne] that $u^*\in L^\infty(\Omega)$ when $N\le 3$. The question whether the same conclusion holds when $4\le N\le 9$ for *every* f satisfying (6.2) - (6.3) was a long-standing open problem raised in Brezis [Br6]. It was recently solved in a splendid piece of work by Cabré-Figalli-Ros-Oton-Serra:

Theorem 6.2 ([CFRS]). *Assume (6.2)-(6.3) and*

$$N\le 9, \tag{6.5}$$

then $u^*\in L^\infty(\Omega)$.

Assumption $N\le 9$ is optimal since a celebrated result by Joseph-Lundgren (1973) provides an explicit solution of (6.4) when $N\ge 10, \Omega=B_1$ is the unit ball in $\mathbb{R}^N$, and $f(u)=e^u$. Namely $\lambda^*=2(N-2)$ and $u^*(x)=\log(1/|x|^2)$, so that $u^*\notin L^\infty(\Omega)$.

This completes the case $N\le 9$. On the other hand many interesting questions remain open when $N\ge 10$. Here are some of them:

Open Problem 6.1 ([BrVa], [Br6]). Assume Ω is a bounded smooth convex set in $\mathbb{R}^N, N\ge 10$. Let $f(u)=e^u$. Is u^* unbounded? If the answer is negative for some domains Ω, can one find other functions f satisfying (6.2)-(6.3) (possibly depending on Ω) such that $u^*\notin L^\infty(\Omega)$?

Open Problem 6.2 ([BrVa], [Br6]). Assume $u^*\notin L^\infty(\Omega)$. What can be said about the blow-up set of u^*? Does it consist of a single point when Ω is convex?

7. Estimates à la Bourgain-Brezis.

The starting point in the following (non-trivial) estimate for the phase of $\mathbb{S}^1$-valued maps. For simplicity we work on $\mathbb{T}^N, N\ge 2$.

Theorem 7.1 ([BoBr, Corollary 1]). *Let* $\varphi:\mathbb{T}^N\to\mathbb{R}$ *be a smooth function, and set* $u:=e^{i\varphi}$. *Then*

$$\|\varphi-\int\varphi\|_{L^{N/(N-1)}}\le C(|u|_{H^{1/2}}+(u|^2_{H^{1/2}}). \tag{7.1}$$

A basic ingredient in the proof of (7.1) is the following:

Theorem 7.2 ([BoBr, Theorem 2]). *Given any $f \in L^N(\mathbb{T}^N)$ such that $\int f = 0$, there exists $Y \in W^{1,N} \cap L^\infty(\mathbb{T}^N, \mathbb{R}^N)$ satisfying*

$$\operatorname{div} Y = f \text{ on } \mathbb{T}^N, \tag{7.2}$$

and

$$\|Y\|_{W^{1,N}} + \|Y\|_{L^\infty} \leq C_N \|f\|_{L^N}. \tag{7.3}$$

It is easy to check that $W^{1,N} \subset H^{1/2}$ and therefore $Y \in H^{1/2} \cap L^\infty$ with the corresponding estimate

$$\|Y\|_{H^{1/2}} + \|Y\|_{L^\infty} \leq C_N \|f\|_{L^N}. \tag{7.4}$$

The assertion of Theorem 7.2 is equivalent via duality to the estmate:

$$\|\psi - \int \psi\|_{L^{N/(N-1)}} \leq C_N \|\nabla \psi\|_{W^{-1,N/(N-1)} + L^1} \quad \forall \psi, \tag{7.5}$$

while the weaker assertion (7.4) corresponds to the weaker estimate

$$\|\psi - \int \psi\|_{L^{N/(N-1)}} \leq C_N \|\nabla \psi\|_{H^{-1/2} + L^1}. \tag{7.6}$$

Theorem 7.1 can be deduced easily from estimate (7.4) . Indeed, we have

$$\nabla \varphi = -i \overline{u} \nabla u. \tag{7.7}$$

Multiplying (7.7) by Y and applying (7.4) yields

$$\begin{aligned} \left| \int \varphi f \right| &= \left| \int \varphi \operatorname{div} Y \right| = \left| \int \nabla \varphi \cdot Y \right| = \left| \int \overline{u} \nabla u \cdot Y \right| \\ &\leq C |u|_{H^{1/2}} |\overline{u} Y|_{H^{1/2}} \leq C |u|_{H^{1/2}} (|u|_{H^{1/2}} \|Y\|_{L^\infty} + |Y|_{H^{1/2}}) \\ &\leq C |u|_{H^{1/2}} (1 + |u|_{H^{1/2}}) \|f\|_{L^N}, \end{aligned}$$

which implies (7.1).

It turns out that there is a substantial improvement of Theorem 7.1:

Theorem 7.3 ([BoBr, Theorem 4], [BBM4, Theorem 3], [BM, Theorem 9.7]). *With the same notations as in Theorem 7.1, we have*

$$\|\varphi - \int \varphi\|_{H^{1/2}+W^{1,1}} \leq C_N(|u|_{H^{1/2}} + |u|^2_{H^{1/2}}). \tag{7.8}$$

[Note that $H^{1/2} \subset L^{N/(N-1)}$ and $W^{1,1} \subset L^{N/(N-1)}$].

If we try to establish (7.8) by the same method as above we would need to invoke a stronger version of (7.6), namely

$$\|\psi - \int \psi\|_{H^{1/2}+W^{1,1}} \leq C\|\nabla\psi\|_{H^{-1/2}+L^1} \quad \forall\psi. \tag{7.9}$$

However such an estimate is still undecided:

Open Problem 7.1 ([BoBr]). Does (7.9) hold?

8. Regularity of minimizers for functionals involving the total variation.

Let Ω be a smooth bounded domain in $\mathbb{R}^N, N \geq 2$. Assume (for simplicity) that f is a smooth function on $\bar{\Omega}$ and consider the functional

$$\Phi(u) := \int_\Omega |\nabla u| + \frac{1}{2}\int_\Omega |u - f|^2$$

defined for $u \in BV(\Omega) \cap L^2(\Omega)$. This functional has been extensively used e.g. in Image Processing following the classical work of Rudin-Osher-Fatemi.

Standard Functional Analysis yields the existence and uniqueness of a minimizer denoted $U \in BV(\Omega) \cap L^2(\Omega)$ for the problem

$$\operatorname*{Min}_{u \in BV \cap L^2} \Phi(u). \tag{8.1}$$

So far the best regularity result is

Theorem 8.1 ([CCN], [Por]). *The minimizer U satisfies $\nabla U \in L^\infty(\Omega)$, but in general ∇U is not continuous.*

This still leaves room for improvement:

Open Problem 8.1 ([Br10, Open Problem 1]). Does ∇U belong to $BV(\Omega)$?

When $N = 1$ the answer to Open Problem 8.1 is positive (see [Br10, Theorem 1]). The proof is based on a transformation relating the solution of (8.1) to the derivative of the solution of an obstacle problem. One may then apply a result of [BK] (valid in all dimensions N). Unfortunately this transformation seems to be restricted to $N = 1$.

9. Characterizations of constant functions and beyond.

Let Ω be a smooth bounded connected domain in $\mathbb{R}^N, N \geq 1$. It is known (see [Br4], [BM, Corollary 6.4]) that any measurable function $u : \Omega \to \mathbb{R}$ satisfying

$$\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{p+N}} dxdy < \infty, \tag{9.1}$$

for some $1 \leq p < \infty$, must be a constant (i.e., there exists a constant c such that $u(x) = c$ a.e.). As was pointed out in [Br4] this fact is an immediate consequence of the BBM formula [BBM1]; alternative direct elementary proofs are presented in [BM].

A significant extension of this result goes as follows. Given $\lambda > 0, p \geq 1$, and a measurable function $u : \Omega \to \mathbb{R}$, set

$$\Phi_{\lambda,p}(u) := \text{meas} \left\{ (x,y) \in \Omega \times \Omega; \frac{|u(x)-u(y)|^p}{|x-y|^{p+N}} \geq \lambda \right\}. \tag{9.2}$$

Theorem 9.1. *Assume that a measurable function u satisfies*

$$\lim_{\lambda \to \infty} \lambda \Phi_{\lambda,p}(u) = 0 \textit{ for some } p \geq 1, \tag{9.3}$$

then u must be a constant.

Theorem 9.1 is due to Brezis-Van Schaftingen-Yung [BVY] when $p > 1$, (with $\Omega = \mathbb{R}^N$ but the proof is unchanged when Ω is a ball), and to Poliakovsky [Pol, Corollary 1.1] when $p = 1$. Actually, Poliakovsky derives the result from a deeper assertion (see below); as a consequence the proof of Theorem 9.1 when $p = 1$ is quite intricate and it would be desirable to find a more direct and elementary proof, possibly in the spirit of [BVY].

Here is a natural question related to Theorem 9.1:

Open Problem 9.1 ([BVY, Open Problem 1]). Assume that u satisfies

$$\liminf_{\lambda \to \infty} \lambda \Phi_{\lambda,p}(u) = 0 \text{ for some } p \geq 1. \tag{9.4}$$

Can one conclude that u is a constant?

A far-reaching version of this question is the following:

Open Problem 9.2 ([BSVY, Section 7.2]). Given any $1 < p < \infty$, is there a constant $C = C(p, N, \Omega)$ such that, for all measurable functions u,

$$\|\nabla u\|_{L^p}^p \leq C \liminf_{\lambda \to \infty} \lambda \Phi_{\lambda,p}(u)? \tag{9.5}$$

in the sense that $\|\nabla u\|_{L^p} = \infty$ if $u \notin W^{1,p}(\Omega)$.

Similarly for $p = 1$, is there a constant $C = C(N, \Omega)$ such that, for all measurable functions u,

$$\|\nabla u\|_M \leq C \liminf_{\lambda \to \infty} \lambda \Phi_{\lambda,1}(u)? \tag{9.6}$$

where $\|\nabla u\|_M$ denotes the total mass of ∇u if $u \in BV(\Omega)$, and $\|\nabla u\|_M = \infty$ if $u \notin BV(\Omega)$.

Poliakovsky [Pol] gave a positive answer to Open Problem 9.2 under the stronger assumption that lim inf is replaced by lim sup in (9.5) and (9.6).

Along the same lines one may even try to go one step further:

Open Problem 9.3 ([BSVY, Section 7.3]). Does the family of functionals $\lambda \Phi_{\lambda,p}$ converge as $\lambda \to \infty$, in the sense of Γ-convergence, to the functional Ψ defined, when $p > 1$, by

$$\Psi(u) := \begin{cases} c_p \|\nabla u\|_{L^p}^p & \text{if } u \in W^{1,p}(\Omega), \\ +\infty & \text{if } u \notin W^{1,p}(\Omega), \end{cases}$$

and when $p = 1$ by

$$\Psi(u) := \begin{cases} c_1 \|\nabla u\|_M & \text{if } u \in BV(\Omega), \\ +\infty & \text{if } u \notin BV(\Omega), \end{cases}$$

for some positive constants c_p and c_1 ?

The three problems above are already interesting when $N = 1$!

10. A sharp relative isoperimetric inequality for the cube.

Let Q be the unit cube in $\mathbb{R}^N, N \geq 2$. Given a measurable set $S \subset Q$, denote by $\mathbf{1}_S$ the characteristic function of S, by $|S| = \|\mathbf{1}_S\|_{L^1}$ the volume of S, and by $P(S)$ the relative perimeter of S, i.e., taking into account only the part of the boundary of S inside Q; in other words $P(S)$ is the total mass of the measure $\nabla \mathbf{1}_S$ (possibly infinite if S is not rectifiable). Consider the function $f_N(t)$ defined for $0 \leq t \leq 1$ by

$$f_N(t) := \inf\{P(S); S \text{ is a measurable subset of } Q \text{ such that } |S| = t\}. \tag{10.1}$$

Clearly $f_N(t) = f_N(1 - t) \quad \forall t \in [0, 1]$; just replace S by $Q \setminus S$ in (10.1).

An explicit formula for $f_N(t)$ is known when $N = 2$:

Theorem 10.1 ([BreBru]). *We have*

$$f_2(t) = \begin{cases} (\pi t)^{1/2} & \textit{if } 0 \le t \le \frac{1}{\pi} \\ 1 & \textit{if } \frac{1}{\pi} \le t \le \frac{1}{2} \end{cases} . \tag{10.2}$$

Open Problem 10.1 ([BreBru]). Is there a formula similar to (10.2) for $f_N(t)$ when $N \ge 3$? In particular, is it true that $f_N(t) \equiv 1$ near $t = 1/2$?

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