# FAMILIES OF FUNCTIONALS REPRESENTING SOBOLEV NORMS

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Abstract. We obtain new characterizations of the Sobolev spaces $\dot{W}^{1,p}(\mathbb{R}^N)$ and the bounded variation space $\dot{BV}(\mathbb{R}^N)$. The characterizations are in terms of the functionals $\nu_\gamma(E_{\lambda,\gamma/p}[u])$ where

$$E_{\lambda,\gamma/p}[u] = \Big\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y,\ \frac{|u(x)-u(y)|}{|x-y|^{1+\gamma/p}} > \lambda\Big\}$$

and the measure $\nu_\gamma$ is given by $\mathrm{d}\nu_\gamma(x,y) = |x-y|^{\gamma-N}\mathrm{d}x\mathrm{d}y$. We provide characterizations which involve the $L^{p,\infty}$-quasi-norms $\sup_{\lambda>0} \lambda\, \nu_\gamma(E_{\lambda,\gamma/p}[u])^{1/p}$ and also exact formulas via corresponding limit functionals, with the limit for $\lambda \to \infty$ when $\gamma > 0$ and the limit for $\lambda \to 0^+$ when $\gamma < 0$. The results unify and substantially extend previous work by Nguyen and by Brezis, Van Schaftingen and Yung. For $p > 1$ the characterizations hold for all $\gamma \neq 0$. For $p = 1$ the upper bounds for the $L^{1,\infty}$ quasi-norms fail in the range $\gamma \in [-1, 0)$; moreover in this case the limit functionals represent the $L^1$ norm of the gradient for $C_c^\infty$-functions but not for generic $\dot{W}^{1,1}$-functions. For this situation we provide new counterexamples which are built on self-similar sets of dimension $\gamma + 1$. For $\gamma = 0$ the characterizations of Sobolev spaces fail; however we obtain a new formula for the Lipschitz norm via the expressions $\nu_0(E_{\lambda,0}[u])$.

## 1. Introduction

In this article, we are concerned with various ways in which we can recover the Sobolev semi-norm $\|\nabla u\|_{L^p(\mathbb{R}^N)}$ via positive non-convex functionals involving differences $u(x) - u(y)$.

We begin by mentioning two relevant results already in the literature. A theorem of H.-M. Nguyen [15] (see also [5,6]) states that for $1 < p < \infty$ and $u$ in the inhomogeneous Sobolev space $W^{1,p}(\mathbb{R}^N)$,

$$\lim_{\lambda \searrow 0} \lambda^p \iint_{|u(x)-u(y)|>\lambda} |x-y|^{-p-N}\,\mathrm{d}x\,\mathrm{d}y = \frac{\kappa(p,N)}{p}\|\nabla u\|^p_{L^p(\mathbb{R}^N)} \tag{1.1}$$

with

$$\kappa(p,N) := \int_{\mathbb{S}^{N-1}} |e\cdot\omega|^p\,\mathrm{d}\omega = \frac{2\Gamma(\frac{p+1}{2})\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N+p}{2})}, \tag{1.2}$$

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and $e$ is any unit vector in $\mathbb{R}^N$. As shown in [5], (1.1) still holds for all $u \in C_c^1(\mathbb{R}^N)$ when $p = 1$ but fails for general $u \in W^{1,1}(\mathbb{R}^N)$. The limit formula (1.1) may be compared to a theorem of three of the authors [7], which states that for all $u \in C_c^\infty(\mathbb{R}^N)$ and $1 \le p < \infty$, one has

$$\lim_{\lambda\to\infty} \lambda^p \mathcal{L}^{2N}\big(\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : |u(x)-u(y)| > \lambda |x-y|^{1+\frac{N}{p}}\}\big) = \frac{\kappa(p,N)}{N} \|\nabla u\|^p_{L^p(\mathbb{R}^N)} \tag{1.3}$$

where $\mathcal{L}^{2N}$ denotes the Lebesgue measure on $\mathbb{R}^N \times \mathbb{R}^N$. Our first result, namely Theorem 1.1 below, provides an extension of (1.1) and (1.3) that unifies the two statements. Before we state the theorem, we introduce some notations that will be used throughout the paper.

First, for Lebesgue measurable subsets $E$ of $\mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N$ and $\gamma \in \mathbb{R}$, we define

$$\nu_\gamma(E) := \iint_{\substack{(x,y)\in E\\ x\neq y}} |x-y|^{\gamma-N} \,\mathrm{d}x\,\mathrm{d}y. \tag{1.4}$$

In particular, when $\gamma = N$, $\nu_N$ is just the Lebesgue measure on $\mathbb{R}^{2N}$. If $u$ is a measurable function on $\mathbb{R}^N$ and $b \in \mathbb{R}$, we define, for $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$ with $x \neq y$, a difference quotient

$$Q_b u(x,y) := \frac{u(x)-u(y)}{|x-y|^{1+b}}; \tag{1.5}$$

moreover, we define, for $\lambda > 0$, the superlevel set of $Q_b u$ at height $\lambda$ by

$$E_{\lambda,b}[u] := \Big\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y,\ |Q_b u(x,y)| > \lambda\Big\}. \tag{1.6}$$

We will denote by $\dot{W}^{1,p}(\mathbb{R}^N)$, $p \ge 1$ the homogeneous Sobolev space, i.e. the space of $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ functions for which the distributional gradient $\nabla u$ belongs to $L^p(\mathbb{R}^N)$, with the semi-norm $\|u\|_{\dot{W}^{1,p}} := \|\nabla u\|_{L^p(\mathbb{R}^N)}$. The inhomogeneous Sobolev space $W^{1,p}$ is the subspace of $\dot{W}^{1,p}$-functions $u$ for which $u \in L^p$, and we set $\|u\|_{W^{1,p}} := \|u\|_{L^p} + \|\nabla u\|_{L^p}$. For $p = 1$ we will also consider the space $\dot{BV}(\mathbb{R}^N)$ of functions of bounded variations, i.e. locally integrable functions $u$ for which the gradient $\nabla u \in \mathcal{M}$ belongs to the space $\mathcal{M}$ of $\mathbb{R}^N$-valued bounded Borel measures and we put $\|u\|_{\dot{BV}} := \|\nabla u\|_{\mathcal{M}}$; furthermore, let $BV := \dot{BV} \cap L^1$. In the dual formulation, with $C_c^1$ denoting the space of $C^1$ functions with compact support,

$$\|u\|_{\dot{BV}} := \sup\Big\{\Big|\int_{\mathbb{R}^N} u \operatorname{div}(\phi)\Big| : \phi \in C_c^1(\mathbb{R}^N, \mathbb{R}^N),\ \|\phi\|_\infty \le 1\Big\}.$$

For general background material on Sobolev spaces see [4], [21].

**Theorem 1.1.** *Suppose $N \ge 1$, $1 \le p < \infty$, $\gamma \in \mathbb{R} \setminus \{0\}$.*

*(a) If $\gamma > 0$, then for all $u \in \dot{W}^{1,p}(\mathbb{R}^N)$*

$$\lim_{\lambda\to+\infty} \lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u]) = \frac{\kappa(p,N)}{|\gamma|} \|\nabla u\|^p_{L^p(\mathbb{R}^N)}. \tag{1.7}$$

*(b) If either $\gamma < 0$, $p > 1$ or $\gamma < -1$, $p = 1$ then for all $u \in \dot{W}^{1,p}(\mathbb{R}^N)$*

$$\lim_{\lambda \searrow 0} \lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u]) = \frac{\kappa(p,N)}{|\gamma|} \|\nabla u\|^p_{L^p(\mathbb{R}^N)}. \tag{1.8}$$

*(c) If $p = 1$ and $-1 \leq \gamma < 0$ then (1.8) remains true for all $u \in C^1_c(\mathbb{R}^N)$ but fails for generic $u \in \dot{W}^{1,1}(\mathbb{R}^N)$. However we still have for all $u \in \dot{W}^{1,1}(\mathbb{R}^N)$*

$$\liminf_{\lambda \searrow 0} \lambda \nu_\gamma(E_{\lambda,\gamma}[u]) \geq \frac{\kappa(1,N)}{|\gamma|} \|\nabla u\|_{L^1(\mathbb{R}^N)}. \tag{1.9}$$

Formula (1.1) is the special case of (1.8) with $\gamma = -p$, and formula (1.3) is the special case of (1.7) with $\gamma = N$. Note that our result concerns functions in the homogeneous Sobolev space $\dot{W}^{1,p}$; we do not require $u$ to be in $L^p$.

*Remarks.* (i) The reader will note the resemblance of (1.8) and (1.7) and may wonder why in (1.8), for $\gamma < 0$, one is concerned with the limit as $\lambda \searrow 0$ and in (1.7), for $\gamma > 0$, one takes the limit as $\lambda \to \infty$. In the proofs of these formulas one relates limits involving $\lambda \nu_\gamma(E_{\lambda,\gamma/p}[u])^{1/p}$ to (the absolute value of) limits of directional difference quotients $\delta^{-1}(u(x + \delta\theta) - u(x))$ with increment $\delta = \lambda^{-p/\gamma}$, and in order to recover the directional derivative $\langle \theta, \nabla u(x) \rangle$ we need to let $\delta \to 0$, which suggests that we need to take $\lambda \to \infty$ or $\lambda \searrow 0$ depending on the sign of $\gamma$. For the calculations see the proofs of Lemma 3.2 and Lemma 3.3 below.

(ii) The failure of (1.8) for $p = 1$, $\gamma \in [-1, 0)$ and $u \in \dot{W}^{1,1}(\mathbb{R}^N)$ is generic in the sense of Baire category. It may happen that $\lim_{\lambda \searrow 0} \lambda \nu_\gamma(E_{\lambda,\gamma}[u]) = \infty$. This phenomenon was originally revealed when $\gamma = -1$ by A. Ponce and is presented in [15], see also [5, Pathology 1]. For stronger statements and more information see Theorem 1.8. For $\gamma \in (-1, 0)$ we provide new examples based on self-similarity considerations. For discussion of failure in the case $\gamma = 0$ see Theorem 1.5 below. The special case of (1.9) for $\gamma = -1$ was already established in [5, Proposition 1].

When $p = 1$ we can also consider what happens if one allows functions in $\dot{BV}(\mathbb{R}^N)$ in (1.7) and (1.8). For $\gamma = N$ in particular Poliakovsky [19] asked whether the limit formulas remain valid in this generality (with $\|\nabla u\|_{L^1}$ replaced by $\|\nabla u\|_{\mathcal{M}}$). We provide a negative answer:

**Proposition 1.2.** *(i) The analogues of the limiting formulas* (1.7) *for $\gamma > 0$, $p = 1$ and* (1.8) *for $\gamma < 0$, $p = 1$, with $\|\nabla u\|_{\mathcal{M}}$ on the right hand side, fail for suitable $u \in \dot{BV}$.*
*(ii) Specifically, let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and let $u$ be the characteristic function of $\Omega$. The limits* $\lim_{\lambda\to\infty} \lambda\nu_\gamma(E_{\lambda,\gamma}[u])$ *for $\gamma > 0$ and* $\lim_{\lambda\to 0+} \lambda\nu_\gamma(E_{\lambda,\gamma}[u])$ *for $\gamma < -1$ exist, but they are not equal to $|\gamma|^{-1}\kappa(1, N)\|\nabla u\|_{\mathcal{M}}$.*

For a more detailed discussion we refer to Section 3.6. See also Section 7.2 for a discussion about some related open problems.

Motivated by [7], we will also be interested in what happens to the larger quantity obtained by replacing the limits on the left hand side of (1.7) and (1.8) by $\sup_{\lambda>0}$. This will be formulated in terms of the Marcinkiewicz space $L^{p,\infty}(\mathbb{R}^{2N},\nu_\gamma)$ (a.k.a. weak type $L^p$) defined by the condition

$$[F]^p_{L^{p,\infty}(\mathbb{R}^{2N},\nu_\gamma)} \coloneqq \sup_{\lambda>0} \lambda^p \nu_\gamma(\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N \;:\; |F(x,y)|>\lambda\}) < \infty. \tag{1.10}$$

As an immediate consequence of Theorem 1.1 we have for $N\geq 1$, $1\leq p<\infty$, $\gamma\neq 0$ and all $u\in C_c^\infty(\mathbb{R}^N)$,

$$[Q_{\gamma/p}u]^p_{L^{p,\infty}(\mathbb{R}^{2N},\nu_\gamma)} \geq C(N,p,\gamma)\|\nabla u\|^p_{L^p(\mathbb{R}^N)} \tag{1.11}$$

where $C(N,p,\gamma)$ is a positive constant depending only on $N$, $p$ and $\gamma$. Moreover, the same conclusion holds for all $u\in\dot W^{1,p}(\mathbb{R}^N)$ when $p>1$ with any $\gamma\neq 0$, and when $p=1$ with any $\gamma\notin[-1,0]$. We shall show that the conditions in the last statement can in fact be relaxed, see the inequalities (1.14) and (1.16) below. In addition we have the important upper bounds for $Q_{\gamma/p}u$, extending the case $\gamma=N$ already dealt with in [7] for $u\in C_c^\infty(\mathbb{R}^N)$. The result in [7] states that for every $N\geq 1$, there exists a constant $C(N)$ such that

$$[Q_{N/p}u]^p_{L^{p,\infty}(\mathbb{R}^{2N},\nu_N)} \leq C(N)\|\nabla u\|^p_{L^p(\mathbb{R}^N)} \tag{1.12}$$

for all $u\in C_c^\infty(\mathbb{R}^N)$ and all $1\leq p<\infty$. In light of Theorem 1.1, it is natural to ask whether one can replace the limits on the left hand sides of (1.7) and (1.8) by $\sup_{\lambda>0}$ and still obtain a quantity that is comparable to $\|\nabla u\|^p_{L^p(\mathbb{R}^N)}$. As suggested by Theorem 1.1 the answer to our question is sensitive to the values of $\gamma$ and $p$.

**Theorem 1.3.** *Suppose that $N\geq 1$, $1<p<\infty$ and $\gamma\in\mathbb{R}$. Then the following hold.*

*(i) The inequality*

$$[Q_{\gamma/p}u]_{L^{p,\infty}(\mathbb{R}^{2N},\nu_\gamma)} \leq C(N,p,\gamma)\|\nabla u\|_{L^p(\mathbb{R}^N)} \tag{1.13}$$

*holds for all $u\in C_c^\infty(\mathbb{R}^N)$ if and only if $\gamma\neq 0$. In this case* (1.13) *extends to all $u\in\dot W^{1,p}(\mathbb{R}^N)$.*

*(ii) Suppose that $u\in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and $Q_{\gamma/p}u\in L^{p,\infty}(\mathbb{R}^{2N},\nu_\gamma)$. Then $u\in\dot W^{1,p}(\mathbb{R}^N)$ and we have the inequality*

$$\|\nabla u\|_{L^p(\mathbb{R}^N)} \leq C_{N,p,\gamma}[Q_{\gamma/p}u]_{L^{p,\infty}(\mathbb{R}^{2N},\nu_\gamma)}. \tag{1.14}$$

There is a new phenomenon for $p=1$, namely the upper bounds for $Q_\gamma u$ only hold for the more restrictive range $\gamma\in(-\infty,-1)\cup(0,\infty)$. Here it is also natural to replace $\dot W^{1,1}$ with $\dot{BV}$.

**Theorem 1.4.** *Suppose that $N\geq 1$ and $\gamma\in\mathbb{R}$. Then the following hold.*

*(i) The inequality*

$$[Q_\gamma u]_{L^{1,\infty}(\mathbb{R}^{2N},\nu_\gamma)} \leq C(N,\gamma)\|\nabla u\|_{L^1(\mathbb{R}^N)} \tag{1.15}$$

*holds for all $u \in C_c^\infty(\mathbb{R}^N)$ if and only if $\gamma \notin [-1,0]$. In this case* (1.15) *extends to all $u \in \dot{W}^{1,1}(\mathbb{R}^N)$, and, if $\|\nabla u\|_{L^1(\mathbb{R}^N)}$ is replaced by $\|\nabla u\|_{\mathcal{M}}$, to all $u \in \dot{BV}(\mathbb{R}^N)$.*

*(ii) Suppose that $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and $Q_\gamma u \in L^{1,\infty}(\mathbb{R}^{2N}, \nu_\gamma)$. Then $u \in \dot{BV}(\mathbb{R}^N)$ and we have the inequality*

$$\|\nabla u\|_{\mathcal{M}} \le C_{N,\gamma} [Q_\gamma u]_{L^{1,\infty}(\mathbb{R}^{2N},\nu_\gamma)}. \tag{1.16}$$

We note that the quantitative bounds (1.13) and (1.15) in Theorems 1.3 and 1.4 are crucial tools for establishing the limiting relations for all $\dot{W}^{1,p}$ functions in Theorem 1.1. Note that there is no restriction on $\gamma$ in (1.14) and (1.16). The constants in the inequalities will be quantified further later in the paper. In particular, $C(N,p,\gamma)$ in (1.13) remains bounded as $p \searrow 1$ only in the range $\gamma \in (0,\infty) \cup (-\infty,-1)$ (cf. Theorem 2.2 and Proposition 6.1).

*Historical comments.* Some special cases of the above quantitative estimates have been known. Estimate (1.13) for $\gamma = -p$ and $1 < p < \infty$ was discovered independently by H.M. Nguyen [15], and by A. Ponce and J. Van Schaftingen (unpublished communication to H. Brezis and H.M. Nguyen), both relying on the Hardy-Littlewood maximal inequality. A. Poliakovsky [19] recently proved generalizations of results in [7] to Sobolev spaces on domains; moreover he obtained Theorems 1.3 and 1.4 in the special case $\gamma = N$ under the additional assumption that $u \in L^p$. Other far-reaching generalizations to one-parameter families of operators were obtained by Ó. Domínguez and M. Milman [10].

*The case $\gamma = 0$.* We shall now return to the necessity of the assumption $\gamma \notin [-1,0]$ in parts of Theorems 1.1, 1.3 and 1.4. When $\gamma = 0$, the bounds for $[Q_{\gamma/p} u]_{L^{p,\infty}(\mathbb{R}^{2N},\nu_\gamma)}$ fail in a striking way. We begin by formulating a result illustrating this failure, which also gives a characterization of the semi-norm in the Lipschitz space $\dot{W}^{1,\infty}$.

**Theorem 1.5.** *Suppose $N \ge 1$, $u$ is locally integrable on $\mathbb{R}^N$ and $\nabla u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Then*

$$\|\nabla u\|_{L^\infty(\mathbb{R}^N)} = \inf\{\lambda > 0 : \nu_0(E_{\lambda,0}[u]) < \infty\}. \tag{1.17}$$

Indeed in Proposition 5.1 we shall prove the stronger statement that $\nu_0(E_{\lambda,0}[u]) = 0$ for $\lambda > \|\nabla u\|_\infty$, and $\nu_0(E_{\lambda,0}[u]) = \infty$ for $\lambda < \|\nabla u\|_\infty$. As an immediate consequence of Theorem 1.5 we get

**Corollary 1.6.** *Let $u$ be locally integrable on $\mathbb{R}^N$. If $\nabla u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and if $\nu_0(E_{\lambda,0}[u])$ is finite for all $\lambda > 0$, then $u$ is almost everywhere equal to a constant function.*

In view of other known results [3], [8] on how to recognize constant functions, a natural question arises whether the hypothesis on the local integrability of $\nabla u$ in the corollary could be relaxed; one can ask whether the constancy conclusion holds for all locally integrable functions satisfying $\nu_0(E_{\lambda,0}[u]) < \infty$ for all $\lambda > 0$. However the following example shows that such an extension fails (for details see Lemma 5.2).

**Example 1.7.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and let $u$ be the characteristic function of $\Omega$. Then $u \in BV(\mathbb{R}^N) \setminus \dot{W}^{1,1}(\mathbb{R}^N)$ and $\sup_{\lambda>0} \lambda\, \nu_0(E_{\lambda,0}[u]) < \infty$.*

*More on counterexamples.* We now make more explicit the exclusion of the parameters $\gamma \in [-1, 0)$ in part (c) of Theorem 1.1 and in (1.15). We shall show in Section 6.2 that for $\gamma \in (-1, 0)$ these negative results can be related to self-similar Cantor subsets of $\mathbb{R}$, of dimension $1 + \gamma$.

**Theorem 1.8.** *Suppose $N \geq 1$. Then the following hold.*

*(i) Let $-1 \leq \gamma < 0$. There exists a $C^\infty$ function $u \in \dot{W}^{1,1}(\mathbb{R}^N)$, rapidly decreasing as $|x| \to \infty$ and such that*

$$\lim_{\lambda \searrow 0} \lambda \nu_\gamma(E_{\lambda,\gamma}[u]) = \infty. \tag{1.18}$$

*(ii) Let $-1 \leq \gamma < 0$. There exists a compactly supported $u \in W^{1,1}(\mathbb{R}^N)$ for which* (1.18) *holds. The set*

$$\{u \in W^{1,1}(\mathbb{R}^N) : \limsup_{\lambda \searrow 0} \lambda \nu_\gamma(E_{\lambda,\gamma}[u]) < \infty\}$$

*is meager in $W^{1,1}(\mathbb{R}^N)$, i.e. of first category in the sense of Baire.*

*(iii) Let $-1 \leq \gamma < 0$, $N \geq 2$ or $-1 < \gamma < 0$, $N = 1$. There exists a compactly supported $u \in W^{1,1}(\mathbb{R}^N)$ such that $\nu_\gamma(E_{\lambda,\gamma}[u]) = \infty$ for all $\lambda > 0$; moreover, the set*

$$\{u \in W^{1,1}(\mathbb{R}^N) : \nu_\gamma(E_{\lambda,\gamma}[u]) < \infty \text{ for some } \lambda \in (0, \infty)\}$$

*is meager in $W^{1,1}(\mathbb{R}^N)$.*

The case $N = 1 = -\gamma$ plays a special role and is excluded in the strongest statement (iii) since for all compactly supported $u \in \dot{W}^{1,1}(\mathbb{R})$ one has $\nu_{-1}(E_{\lambda,-1}[u]) < \infty$ for all $\lambda > 0$ (*cf.* Lemma 6.5 below). The proofs of existence of counterexamples are constructive and the Baire category statements will be obtained as rather straightforward consequences of the constructions.

*Outline of the paper.*

In Section 2 we provide the upper bounds for $[Q_{\gamma/p}u]_{L^{p,\infty}(\mathbb{R}^{2N},\nu_\gamma)}$, i.e. the proof of inequalities (1.13) and (1.15) in Theorems 1.3 and 1.4. We first derive these for a dense subclass, relying on covering lemmas, and then extend in Sections 2.3 and 2.4 to general $\dot{W}^{1,p}$ and $\dot{BV}$-functions. In Section 3 we derive the limit formulas of Theorem 1.1; specifically in Section 3.2 we prove the sharp lower bounds involving a $\liminf \lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u])$ for general functions in $\dot{W}^{1,p}$ and in Section 3.3 we obtain the sharp upper bounds for $\limsup \lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u])$, under the assumption that $u \in C^1$ is compactly supported. Then in Section 3.4 we extend these limits to general $\dot{W}^{1,p}$ functions. In Section 3.6 we show that the limit formulas for $\dot{W}^{1,1}$ do not extend to general $\dot{BV}$ functions and prove Proposition 1.2.

In Section 4 we prove the reverse inequalities (1.14) and (1.16) in Theorems 1.3 and 1.4. In Section 5 we prove Theorem 1.5 on a characterization of the Lipschitz norm and also discuss Example 1.7. In Section 6 we provide various constructions of counterexamples and in particular prove Theorem 1.8. We discuss some further perspectives and open problems in Section 7.

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## 2. Bounding $[Q_{\gamma/p}u]_{L^{p,\infty}(\mathbb{R}^{2N},\nu_\gamma)}$ by the Sobolev norm.

In this section we prove inequalities (1.13) and (1.15) in Theorems 1.3 and 1.4.

### 2.1. *The bound* (1.13) *via the Hardy-Littlewood maximal operator*

Following [7], one can prove the result of Theorem 1.3 for $p > 1$ by an elementary argument involving the Hardy-Littlewood maximal function $M|\nabla u|$ of $|\nabla u|$; however the behavior of the constants as $p \searrow 1$ will only be sharp in the range $-1 \le \gamma < 0$.

**Proposition 2.1.** *Let $N \ge 1$ and $1 < p < \infty$. There exists a constant $C_N$ such that for all $\gamma \ne 0$ and all $u \in \dot{W}^{1,p}(\mathbb{R}^N)$,*

$$\sup_{\lambda>0} \lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u]) \le \frac{C_N}{|\gamma|}\Big(\frac{p}{p-1}\Big)^p \|\nabla u\|^p_{L^p(\mathbb{R}^N)}. \tag{2.1}$$

*Proof.* We assume first that $u \in C^1$ and that $\nabla u$ is compactly supported. As in [7, Remark 2.3], one uses the Lusin-Lipschitz inequality

$$\frac{|u(x)-u(y)|}{|x-y|} \le C[M(|\nabla u|)(x) + M(|\nabla u|)(y)] \tag{2.2}$$

and observes that (2.2) implies

$$E_{\lambda,\gamma/p}[u] \subseteq \{|x-y|^{\gamma/p} < 2C\lambda^{-1}M(|\nabla u|)(x)\} \cup \{|x-y|^{\gamma/p} < 2C\lambda^{-1}M(|\nabla u|)(y)\}.$$

As a consequence

$$\nu_\gamma(E_{\lambda,\gamma/p}[u]) \le 2\int_x \int_{|h|^\gamma < 2C[\lambda^{-1}M(|\nabla u|)(x)]^p} |h|^{\gamma-N}\,\mathrm{d}h\,\mathrm{d}x.$$

Direct computation of the inner integral (distinguishing the cases $\gamma > 0$ and $\gamma < 0$) yields

$$\nu_\gamma(E_{\lambda,\gamma/p}[u]) \lesssim_N C^p|\gamma|^{-1}\lambda^{-p}\int_{\mathbb{R}^N}[M(|\nabla u|)(x)]^p\,\mathrm{d}x.$$

Inequality (2.1) follows then from the standard maximal inequality $\|Mf\|_p^p \leq [C(N)p']^p \|f\|_p^p$ for $p > 1$, see [21] (here $p' = p/(p-1)$). The extension to general $\dot{W}^{1,p}$ functions will be taken up in Section 2.3. $\square$

2.2. *The case $\gamma \in \mathbb{R} \setminus [-1, 0]$*

We shall prove the following more precise versions of the estimates (1.13) and (1.15) when $\gamma \notin [-1, 0]$, with constants that stay bounded as $p \searrow 1$, indeed we cover all $p \in [1, \infty)$. We denote by $\sigma_{N-1}$ the surface area of the sphere $\mathbb{S}^{N-1}$. In the proof of the following theorem we will first establish the estimates for functions $u \in C^1(\mathbb{R}^N)$ whose gradient is compactly supported. The extension to $\dot{W}^{1,p}$ and $\dot{BV}$ will be taken up in Section 2.3 and Section 2.4.

**Theorem 2.2.** *There exists an absolute constant $C > 0$ such that for every $N \geq 1$, every $1 \leq p < \infty$, and every $u \in \dot{W}^{1,p}(\mathbb{R}^N)$*

*(a) if $\gamma > 0$, then*

$$\sup_{\lambda > 0} \lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u]) \leq C\sigma_{N-1} \frac{5^\gamma}{\gamma} \|\nabla u\|^p_{L^p(\mathbb{R}^N)}; \tag{2.3}$$

*(b) if $\gamma < -1$, then*

$$\sup_{\lambda > 0} \lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u]) \leq \frac{C\sigma_{N-1}}{|\gamma|} \Big(1 + \frac{1}{|\gamma + 1|}\Big) \|\nabla u\|^p_{L^p(\mathbb{R}^N)}. \tag{2.4}$$

*When $p = 1$ the above assertions hold for all $u \in \dot{BV}(\mathbb{R}^N)$ provided that $\|\nabla u\|_{L^1(\mathbb{R}^N)}$ is replaced by $\|\nabla u\|_{\mathcal{M}}$.*

The proof of Theorem 2.2 relies on the following proposition, in which $[x, y] \subset \mathbb{R}^N$ denotes the closed line segment connecting two points $x, y \in \mathbb{R}^N$.

**Proposition 2.3.** *Let*

$$E(f, \gamma) := \Big\{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y,\ \int_{[x,y]} |f| \,\mathrm{d}s > |x - y|^{\gamma+1}\Big\} \tag{2.5}$$

*for $f \in C_c(\mathbb{R}^N)$. There exists an absolute constant $C > 0$ such that for every $N \geq 1$ and every $f \in C_c(\mathbb{R}^N)$,*

*(i) if $\gamma > 0$, then*

$$\iint_{E(f,\gamma)} |x - y|^{\gamma - N} \,\mathrm{d}x \,\mathrm{d}y \leq C\sigma_{N-1} \frac{5^\gamma}{\gamma} \|f\|_{L^1(\mathbb{R}^N)}; \tag{2.6}$$

*(ii) if $\gamma < -1$, then*

$$\iint_{E(f,\gamma)} |x - y|^{\gamma - N} \,\mathrm{d}x \,\mathrm{d}y \leq \frac{C\sigma_{N-1}}{|\gamma|} \Big(1 + \frac{1}{|\gamma + 1|}\Big) \|f\|_{L^1(\mathbb{R}^N)}. \tag{2.7}$$

Indeed, to deduce Theorem 2.2 from Proposition 2.3 one argues as in the proof of (1.12) in [7]; for $u \in C^1(\mathbb{R}^N)$ and $1 \le p < \infty$, one has

$$|u(x)-u(y)|^p \le \Big[ \int_{[x,y]} |\nabla u| \, \mathrm{d}s \Big]^p \le \int_{[x,y]} |\nabla u|^p \, \mathrm{d}s \, |x-y|^{p-1}$$

for all $x, y \in \mathbb{R}^N$, which implies that

$$E_{\lambda,\gamma/p}[u] \subseteq E(\lambda^{-p}|\nabla u|^p, \gamma).$$

Hence for $u \in C^1(\mathbb{R}^N)$ whose gradient is compactly supported, one establishes Theorem 2.2 by applying Proposition 2.3 with $f := \lambda^{-p}|\nabla u|^p$. The extension to $u \in \dot{W}^{1,p}$ will be taken up in Section 2.3.

*Proof of Proposition 2.3.* As in the proof of [7, Proposition 2.2], using the method of rotation, we only need to prove Proposition 2.3 for $N = 1$. Indeed,

$$\iint_{E(f,\gamma)} |x-y|^{\gamma-N} \, \mathrm{d}x \, \mathrm{d}y = \frac{1}{2} \int_{\mathbb{S}^{N-1}} \int_{\omega^\perp} \iint_{E(f_{\omega,x'},\gamma)} |r-s|^{\gamma-1} \, \mathrm{d}r \, \mathrm{d}s \, \mathrm{d}x' \, \mathrm{d}\omega$$

where for every $\omega \in \mathbb{S}^{N-1}$ and every $x' \in \omega^\perp$, $f_{\omega,x'}$ is a function of one real variable defined by

$$f_{\omega,x'}(t) := f(x' + t\omega).$$

The innermost double integral can be estimated by the case $N = 1$ of Proposition 2.3, and

$$\int_{\mathbb{S}^{N-1}} \int_{\omega^\perp} \int_{\mathbb{R}} |f_{\omega,x'}(t)| \, \mathrm{d}t \, \mathrm{d}x' \, \mathrm{d}\omega = \sigma_{N-1} \|f\|_{L^1(\mathbb{R}^N)}.$$

Thus from now on, we assume $N = 1$ and $f \in C_c(\mathbb{R})$.

If $\gamma > 0$, the desired estimate (2.6) is the content of [7, Proposition 2.1]. On the other hand, suppose now $\gamma < -1$. Without loss of generality, assume $f \ge 0$ on $\mathbb{R}$. In addition, we may assume that $f$ is not identically zero, for otherwise there is nothing to prove.

Let

$$E_+(f,\gamma) := \{(x,y) \in E(f,\gamma) \colon y < x\}.$$

Then by symmetry,

$$\iint_{E(f,\gamma)} |x-y|^{\gamma-1} \, \mathrm{d}x \, \mathrm{d}y = 2 \iint_{E_+(f,\gamma)} |x-y|^{\gamma-1} \, \mathrm{d}x \, \mathrm{d}y,$$

and it suffices to estimate the latter integral.

In what follows we will need to always keep in mind that in view of our assumption $\gamma < -1$ we have $-(\gamma+1) = |\gamma| - 1 > 0$. We will now use a simple stopping time argument based on the fact that for all $c \in \mathbb{R}$ the continuous function

$$x \mapsto (x-c)^{-(\gamma+1)} \int_c^x f(s) \, \mathrm{d}s, \quad x \ge c$$

increases from 0 to $\infty$ on $[c, \infty)$.

Assume that $\operatorname{supp} f \subseteq [a,b]$. We construct a finite sequence of intervals $I_1, \dots, I_K$, that are disjoint up to end-points, that cover $\operatorname{supp} f = [a,b]$, and that satisfy

$$|I_i|^{-(\gamma+1)} \int_{I_i} f = \frac{1}{2} \quad \text{for } 1 \le i \le K. \tag{2.8}$$

Indeed, we may take $a_1 := a$, and $a_2 > a_1$ to be the unique number for which

$$(a_2 - a_1)^{-(\gamma+1)} \int_{a_1}^{a_2} f = 1/2,$$

and set $I_1 := [a_1, a_2]$. If $a_2 < b$, we may now repeat, and take $I_2 := [a_2, a_3]$ where $a_3 > a_2$ is the unique number for which $(a_3 - a_2)^{-(\gamma+1)} \int_{a_2}^{a_3} f = 1/2$. Note that the $a_i$'s chosen as such satisfy

$$(a_{i+1} - a_i)^{-(\gamma+1)} \ge \frac{1}{2} \|f\|_{L^1(\mathbb{R})}^{-1},$$

so that $a_{i+1} - a_i \ge (2\|f\|_{L^1(\mathbb{R})})^{1/(\gamma+1)}$. This shows that in finitely many steps, we would reach $a_{K+1} \ge b$ for some $K \ge 1$, with $a_K < b$ if $1 \le K$. Then we have our sequence of disjoint (up to endpoints) intervals $I_1, \dots, I_K$ that cover $[a,b]$ and satisfy (2.8). We also write $I_0 := (-\infty, a_1]$ and $I_{K+1} := [a_{K+1}, +\infty)$.

We now claim that $I_i \times I_i \cap E_+(f,\gamma) = \emptyset$ for every $0 \le i \le K+1$. This being trivially the case when $i \in \{0, K+1\}$, we consider the case $i \in \{1, \dots, K\}$: any $x, y \in I_i$ satisfy

$$|x-y|^{-(\gamma+1)} \left| \int_y^x f \right| \le |I_i|^{-(\gamma+1)} \int_{I_i} f = \frac{1}{2} < 1.$$

It follows thus that

$$E_+(f,\gamma) = \bigcup_{i=1}^{K+1} E_+(f,\gamma) \cap ((a_i, +\infty) \times (-\infty, a_i)) \tag{2.9}$$

Furthermore, for $i \in \{2, \dots, K\}$, if $y < a_i < x$ and $x - y < \min\{|I_i|, |I_{i-1}|\}$, then

$$\begin{aligned} |x-y|^{-(\gamma+1)} \left| \int_y^x f \right| &< \min\{|I_i|, |I_{i-1}|\}^{-(\gamma+1)} \Big( \int_{I_{i-1}} f + \int_{I_i} f \Big) \\ &\le |I_{i-1}|^{-(\gamma+1)} \int_{I_{i-1}} f + |I_i|^{-(\gamma+1)} \int_{I_i} f \le \frac{1}{2} + \frac{1}{2} = 1, \end{aligned}$$

(again we used $\gamma < -1$ so that $-(\gamma+1) > 0$ here), from which it follows that $(x,y) \notin E_+(f,\gamma)$. Combining this with a similar argument for $i \in \{1, K+1\}$, we get that if

$(x,y) \in E_+(f,\gamma) \cap (a_i,+\infty) \times (-\infty,a_i)$, then $|x-y| \geq \min\{|I_i|,|I_{i-1}|\}$, and thus

$$\begin{aligned}
&\int_{E_+(f,\gamma)\cap(a_i,+\infty)\times(-\infty,a_i)} |x-y|^{\gamma-1}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad\leq \int_{a_i}^{\infty}\int_{-\infty}^{\min\{a_i,x-\min\{|I_i|,|I_{i-1}|\}\}} |x-y|^{\gamma-1}\,\mathrm{d}y\,\mathrm{d}x \\
&\quad= \frac{1}{|\gamma|}\int_{a_i}^{\infty}(\max\{x-a_i,\min\{|I_i|,|I_{i-1}|\}\})^{\gamma}\,\mathrm{d}x \\
&\quad= \frac{1}{|\gamma|}\Big(1+\frac{1}{|\gamma+1|}\Big)\min\{|I_i|,|I_{i-1}|\}^{\gamma+1} \leq \frac{2}{|\gamma|}\Big(1+\frac{1}{|\gamma+1|}\Big)\int_{I_{i-1}\cup I_i} f.
\end{aligned}$$

(The computation of these integrals uses our assumption $\gamma+1<0$.) Summing the estimates, we get in view of (2.9)

$$\int_{E_+(f,\gamma)} |x-y|^{\gamma-1}\,\mathrm{d}x\,\mathrm{d}y \leq \frac{4}{|\gamma|}\Big(1+\frac{1}{|\gamma+1|}\Big)\int_{\mathbb{R}} f.$$

We have thus completed the proof of (2.7) under the assumption $\gamma<-1$ and $N=1$. $\square$

## 2.3. *Proof of Proposition 2.1 and Theorem 2.2 for general $\dot{W}^{1,p}$ functions*

We use a limiting argument, together with the following fact: if $u \in \dot{W}^{1,p}(\mathbb{R}^N)$, $N \geq 1$, and $1 \leq p < \infty$, then there exists a Lebesgue measurable set $X \subset \mathbb{R}^{2N}$ with $\mathcal{L}^{2N}(X)=0$, so that for every $(x,h) \in \mathbb{R}^{2N}\setminus X$, we have

$$u(x+h)-u(x) = \int_0^1 \langle h, \nabla u(x+th)\rangle\,\mathrm{d}t. \tag{2.10}$$

Indeed, both sides are measurable functions of $(x,h) \in \mathbb{R}^{2N}$, and if $X$ is the set of all $(x,h)$ where the two sides are not equal, then $X$ is a measurable subset of $\mathbb{R}^{2N}$, and the assertion will follow from Fubini's theorem if for every fixed $h \in \mathbb{R}^N$, we have $\mathcal{L}^N(\{x \in \mathbb{R}^N\colon (x,h) \in X\})=0$, i.e. (2.10) holds for $\mathcal{L}^N$ almost every $x$. This follows since for every $\phi \in C_c^\infty(\mathbb{R}^N)$, one has

$$\begin{aligned}
&\int_{\mathbb{R}^N}[u(x+h)-u(x)]\phi(x)\,\mathrm{d}x = \int_{\mathbb{R}^N} u(x)[\phi(x-h)-\phi(x)]\,\mathrm{d}x \\
&= -\int_{\mathbb{R}^N} u(x)\int_0^1\langle h,\nabla\phi(x-th)\rangle\,\mathrm{d}t\,\mathrm{d}x = \int_{\mathbb{R}^N}\int_0^1\langle h,\nabla u(x)\rangle\phi(x-th)\,\mathrm{d}t\,\mathrm{d}x \\
&= \int_{\mathbb{R}^N}\int_0^1\langle h,\nabla u(x+th)\rangle\,\mathrm{d}t\,\phi(x)\,\mathrm{d}x.
\end{aligned}$$

Now given $u \in \dot{W}^{1,p}(\mathbb{R}^N)$, there exists a sequence $u_n \in C^\infty(\mathbb{R}^N)$ such that $\nabla u_n$ are compactly supported, and

$$\|\nabla(u_n-u)\|_{L^p(\mathbb{R}^N)} \to 0. \tag{2.11}$$

Indeed if $N > 1$ and $p \geq 1$, or if $N = 1$ and $p > 1$, then this follows from the density of $C_c^\infty(\mathbb{R}^N)$ in $\dot{W}^{1,p}(\mathbb{R}^N)$ as asserted in [12] (in this case one may choose $u_n \in C_c^\infty(\mathbb{R}^N)$). The density of $C_c^\infty(\mathbb{R}^N)$ in $\dot{W}^{1,p}$ *fails* when $N = p = 1$ (again see [12]); the issue is that if $\nabla u$ is supported in a convex set in $\mathbb{R}^N$, $N \geq 2$, then $u$ is constant in the complement of the set, but this fails for $N = 1$ since the complement of a bounded interval has two connected components. On the other hand, in the anomalous case $N = 1$ and $p = 1$, one can choose an approximation of the identity to get a sequence $v_n$ of $C_c^\infty$ functions on $\mathbb{R}$ such that $\|v_n - u'\|_{L^1(\mathbb{R})} \to 0$. One can then take $u_n(x) := \int_0^x v_n(t)\,\mathrm{d}t$, and (2.11) follows with $u_n' = v_n$ being compactly supported (even though $u_n$ may not be compactly supported).

Let, for $R > 1$,

$$K_R = \{(x, y) \in \mathbb{R}^{2N} : |x| \leq R,\, |y| \leq R \text{ and } R^{-1} \leq |x - y|\}.$$

By monotone convergence it suffices to prove

$$\nu_\gamma(E_{\lambda,\gamma/p}[u] \cap K_R) \leq C \frac{\|\nabla u\|^p_{L^p(\mathbb{R}^N)}}{\lambda^p}. \tag{2.12}$$

with $C$ independent of $R$.

Under the assumptions of Proposition 2.1 and Theorem 2.2 on $p$ and $\gamma$, since $u_n \in C_c^\infty(\mathbb{R}^N)$, we already know

$$\nu_\gamma(E_{\lambda,\gamma/p}[u_n]) \leq C \frac{\|\nabla u_n\|^p_{L^p(\mathbb{R}^N)}}{\lambda^p}.$$

Moreover, the sequence $Q_{\gamma/p}u_n$ converges to $Q_{\gamma/p}u$ in $L^p(K_R)$ as $n \to \infty$. Indeed, using (2.10) we may write

$$Q_{\gamma/p}u(x, y) = \frac{1}{|x - y|^{\gamma/p}} \int_0^1 \Big\langle \frac{x - y}{|x - y|}, \nabla u((1 - t)y + tx) \Big\rangle \,\mathrm{d}t$$

for $\mathcal{L}^{2N}$ a.e. $(x, y) \in \mathbb{R}^{2N}$, and similarly for $u_n$ in place of $u$, which allows us to estimate

$$\begin{aligned}
&\Big( \iint_{K_R} |Q_{\gamma/p}u_n(x, y) - Q_{\gamma/p}u(x, y)|^p \,\mathrm{d}x\,\mathrm{d}y \Big)^{1/p} \\
&\leq R^{\frac{\gamma}{p}} \int_0^1 \Big( \int_{|x| \leq R} \int_{|y| \leq R} |\nabla(u_n - u)((1 - s)x + sy)|^p \,\mathrm{d}x\,\mathrm{d}y \Big)^{1/p} \mathrm{d}s \\
&\leq 2^{N/p}(2R)^{N/p} R^{\gamma/p} \|\nabla(u_n - u_{n+1})\|_p \to 0.
\end{aligned}$$

By passing to a subsequence if necessary, we may assume that $Q_{\gamma/p}u_n$ converges $\mathcal{L}^{2N}$-a.e. to $Q_{\gamma/p}u$ on $K_R$ as $n \to \infty$. Thus

$$K_R \cap E_{\lambda,\gamma/p}[u] \subseteq K_R \cap \Big( \bigcup_{n \in \mathbb{N}} \bigcap_{\ell \geq n} E_{\lambda,\gamma/p}[u_\ell] \Big)$$

which implies

$$\nu_\gamma(K_R \cap E_{\lambda,\gamma/p}[u]) \le \lim_{n\to\infty} \nu_\gamma\Big(K_R \cap \bigcap_{\ell\ge n} E_{\lambda,\gamma/p}[u_\ell]\Big) \le \liminf_{n\to\infty} \nu_\gamma(K_R \cap E_{\lambda,\gamma/p}[u_n])$$
$$\le C \liminf_{n\to\infty} \frac{\|\nabla u_n\|^p_{L^p(\mathbb{R}^N)}}{\lambda^p} \le C \frac{\|\nabla u\|^p_{L^p(\mathbb{R}^N)}}{\lambda^p}.$$

### 2.4. *Proof of Theorem 2.2 for $\dot{BV}$-functions*

We choose a sequence $\rho_n \in C_c^\infty(\mathbb{R}^N)$ with $\rho_n = 2^{nN}\rho(2^n\cdot)$ and $\int_{\mathbb{R}^N} \rho \, \mathrm{d}x = 1$ and set $u_n \coloneqq u * \rho_n$. Then $u_n \in \dot W^{1,1}(\mathbb{R}^N)$ and $u_n \to u$ almost everywhere. This means if $G_L \coloneqq \{(x,h) \in \mathbb{R}^N \times \mathbb{R}^N : |x| \le L,\, L^{-1} \le |h| \le L\}$ then

$$\lim_{L\to\infty} \nu_\gamma(E_{\lambda,\gamma}[u_n] \cap G_L) = \nu_\gamma(E_{\lambda,\gamma}[u] \cap G_L),$$

by dominated convergence. Also

$$\|\nabla u_n\|_{L^1(\mathbb{R}^N)} = \sup_{\substack{\vec\phi\in C_c^\infty \\ \|\phi\|_\infty\le 1}} \Big|\int u_n(x)\,\mathrm{div}\vec\phi(x)\,\mathrm{d}x\Big| =$$
$$= \sup_{\substack{\vec\phi\in C_c^\infty \\ \|\phi\|_\infty\le 1}} \Big|\int u(x)\,\mathrm{div}(\rho_n * \vec\phi)(x)\,\mathrm{d}x\Big| \le \|\nabla u\|_{\mathcal{M}};$$

here we used $\|\rho_n * \vec\phi\|_\infty \le \|\vec\phi\|_\infty$ for the last inequality. Combining these two limiting identities with Theorem 2.2 we get the desired inequalities with $E_{\lambda,\gamma}[u]$ replaced by $E_{\lambda,\gamma}[u] \cap G_L$. By monotone convergence we may finish the proof letting $L \to \infty$.

## 3. Proof of Theorem 1.1

We extend and refine arguments from [5], [7] which are partially inspired by techniques developed in [1].

### 3.1. *A Lebesgue differentiation lemma*

Our argument uses the following standard variant of the Lebesgue differentiation theorem. For lack of a proper reference, a proof is provided for the convenience of the reader.

**Lemma 3.1.** *Let $u \in \dot W^{1,1}(\mathbb{R}^N)$ and let $\{\delta_n\}$ be a sequence of positive numbers with $\lim_{n\to\infty} \delta_n = 0$. Then*

$$\lim_{n\to\infty} \frac{u(x+\delta_n h) - u(x)}{\delta_n} = \langle h, \nabla u(x)\rangle$$

*for almost every $(x,h) \in \mathbb{R}^N \times \mathbb{R}^N$.*

*Proof.* If $u \in C^1$ with compact support the limit relation clearly holds for all $(x,h)$. We shall below consider for each $\theta \in \mathbb{S}^{N-1}$ consider the maximal function

$$\mathfrak{M}_\theta F(x) = \sup_{t>0} \frac{1}{t} \int_0^t |F(x + r\theta)| \, \mathrm{d}r$$

which is well defined for all $\theta$, a measurable function on $\mathbb{R}^N \times \mathbb{S}^{N-1}$, and satisfies a weak type $(1,1)$ inequality

$$\mathcal{L}^N(\{x \in \mathbb{R}^N : \mathfrak{M}_\theta F(x) > a\}) \leq 5a^{-1}\|F\|_1.$$

Let $u \in \dot{W}^{1,1}(\mathbb{R}^N)$ and $\mathcal{A}_M = \{h \in \mathbb{R}^N : 2^{-M} \leq |h| \leq 2^M\}$. It suffices to prove the limit relation for almost every $(x,h) \in \mathbb{R}^N \times \mathcal{A}_M$. From (2.10) we get that for every $n \geq 1$,

$$\frac{u(x+\delta_n h) - u(x)}{\delta_n} = \frac{1}{\delta_n|h|} \int_0^{\delta_n|h|} \langle h, \nabla u(x + r\tfrac{h}{|h|}) \rangle \, \mathrm{d}r$$

for $\mathcal{L}^{2N}$ almost every $(x,h) \in \mathbb{R}^N \times \mathcal{A}_M$; as a result, there exist representatives of $u$, $\nabla u$ and a null set $\mathcal{N} \in \mathbb{R}^N \times \mathcal{A}_M$ such that the identity holds for all $(x,h) \in \mathcal{N}^{\complement}$ and all $n \geq 1$. It suffices to show that for every $\alpha > 0$, $\epsilon > 0$

(3.1)

$$\mathcal{L}^{2N}\Big(\Big\{(x,h) \in \mathbb{R}^N \times \mathcal{A}_M : \limsup_{n\to\infty} \Big|\frac{1}{\delta_n|h|} \int_0^{\delta_n|h|} \langle h, \nabla u(x+rh) \rangle \, \mathrm{d}r - \langle h, \nabla u(x) \rangle\Big| > \alpha\Big\}\Big) \leq \epsilon.$$

Let $v \in C_c^1$ so that $\|\nabla(v-u)\|_1 \leq \alpha\epsilon/(12\mathcal{L}^N(\mathcal{A}_M))$. Let $g = u - v$. Since the asserted limiting relation holds for $v$, we see that the expression on the left hand side of (3.1) is dominated by

$$\begin{aligned}
&\mathcal{L}^{2N}\Big(\big\{(x,h) \in \mathbb{R}^N \times \mathcal{A}_M : |\nabla g(x)| + \sup_{n>0} \frac{1}{\delta_n|h|} \int_0^{\delta_n|h|} |\nabla g(x + r\tfrac{h}{|h|})| \, \mathrm{d}r > \alpha\big\}\Big) \\
&\leq 2\mathcal{L}^N(\mathcal{A}_M)\alpha^{-1}\|\nabla g\|_1 + \int_{\mathcal{A}_M} \mathcal{L}^N(\{x : \mathfrak{M}_{h/|h|}|\nabla g|(x) > \alpha/2\}) \, \mathrm{d}h \\
&\leq 12\mathcal{L}^N(\mathcal{A}_M)\alpha^{-1}\|\nabla g\|_1 \leq \epsilon
\end{aligned}$$

since $\|\nabla g\|_1 \leq \alpha\epsilon/(12\mathcal{L}^N(\mathcal{A}_M))$. ☐

3.2. *The lower bounds for* $\liminf \lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u])$

We use Lemma 3.1 to establish lower bounds, relying on an idea in [5] where the case $\gamma = -1$ was considered.

**Lemma 3.2.** *Let* $1 \leq p < \infty$ *and* $u \in \dot{W}^{1,p}(\mathbb{R}^N)$. *Then*

*(i) For* $\gamma > 0$

$$\liminf_{\lambda\to\infty} \lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u]) \geq \frac{\kappa(p,N)}{|\gamma|} \|\nabla u\|^p_{L^p(\mathbb{R}^N)}$$

*(ii) For $\gamma < 0$*

$$\liminf_{\lambda\searrow 0} \lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u]) \geq \frac{\kappa(p,N)}{|\gamma|} \|\nabla u\|^p_{L^p(\mathbb{R}^N)}$$

*Proof.* We write, for $\lambda > 0$ and $\delta > 0$

$$\begin{aligned}
\lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u]) &= \lambda^p \iint_{\frac{|u(x+h)-u(x)|}{|h|^{1+\gamma/p}}>\lambda} |h|^{\gamma-N}\,\mathrm{d}h\,\mathrm{d}x \\
&= \lambda^p \delta^\gamma \iint_{\left|\frac{u(x+\delta h)-u(x)}{\delta|h|}\right|^p > \lambda^p\delta^\gamma|h|^\gamma} |h|^{\gamma-N}\,\mathrm{d}h\,\mathrm{d}x,
\end{aligned}$$

here we have changed variables replacing $h$ by $\delta h$. Hence

$$\lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u]) = \iint \mathbb{1}_{(|h|^\gamma,\infty)}\big(\big|\tfrac{u(x+\delta h)-u(x)}{\delta|h|}\big|^p\big)\,|h|^{\gamma-N}\,\mathrm{d}h\,\mathrm{d}x \text{ with } \delta = \lambda^{-p/\gamma}. \tag{3.2}$$

We now take a sequence $\{\lambda_n\}$ of positive numbers, set $\delta_n = \lambda_n^{-p/\gamma}$ and note that

$$\lim_{n\to\infty} \delta_n = 0 \text{ if } \begin{cases} \lim_{n\to\infty}\lambda_n = \infty \text{ and } \gamma > 0, \\ \lim_{n\to\infty}\lambda_n = 0 \text{ and } \gamma < 0. \end{cases} \tag{3.3}$$

Also observe that

$$\liminf_{n\to\infty} \mathbb{1}_{(|h|^\gamma,\infty)}(s_n) \geq \mathbb{1}_{(|h|^{-\gamma},\infty)}(t) \text{ if } \lim_{n\to\infty} s_n = t.$$

Now assume that $\lambda_n \to \infty$ if $\gamma > 0$ and $\lambda_n \to 0^+$ if $\gamma < 0$ and stay with $\delta_n = \lambda_n^{-p/\gamma}$, a sequence which converges to 0 in both cases. Use Fatou's lemma in (3.2) and combine it with Lemma 3.1 to get

$$\begin{aligned}
\liminf_{n\to\infty} \lambda_n^p \nu_\gamma(E_{\lambda_n,\gamma/p}[u]) &\geq \iint \liminf_{n\to\infty} \mathbb{1}_{(|h|^\gamma,\infty)}\big(\big|\tfrac{u(x+\delta_n h)-u(x)}{\delta_n|h|}\big|^p\big)\,|h|^{\gamma-N}\,\mathrm{d}h\,\mathrm{d}x \\
&\geq \iint \mathbb{1}_{(|h|^\gamma,\infty)}\big(\lim_{n\to\infty}\big|\tfrac{u(x+\delta_n h)-u(x)}{\delta_n|h|}\big|^p\big)\,|h|^{\gamma-N}\,\mathrm{d}h\,\mathrm{d}x \\
&= \iint_{|h|^\gamma < \left|\langle\frac{h}{|h|},\nabla u(x)\rangle\right|^p} |h|^{\gamma-N}\,\mathrm{d}h\,\mathrm{d}x =: J_\gamma.
\end{aligned}$$

We use polar coordinates $h = r\theta$ and write the last expression as

$$\begin{aligned}
J_\gamma &= \iint\limits_{\mathbb{R}^N\times\mathbb{S}^{N-1}} \int\limits_{r^\gamma<|\langle\theta,\nabla u(x)\rangle|^p} r^{\gamma-1}\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}x \\
&= \frac{1}{|\gamma|}\iint_{\mathbb{R}^N\times S^{N-1}} |\langle\theta,\nabla u(x)\rangle|^p\,\mathrm{d}\theta\,\mathrm{d}x = \frac{\kappa(p,N)}{|\gamma|}\|\nabla u\|^p_{L^p(\mathbb{R}^N)},
\end{aligned}$$

with the calculation valid in both cases $\gamma > 0$ and $\gamma < 0$. $\square$

3.3. *Upper bounds for* $\limsup \lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u])$*, for* $C^1_c$ *functions*

We assume that $u \in C^1$ is compactly supported and obtain the sharp upper bounds for $\limsup_{\lambda\to\infty} \lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u])$ when $\gamma > 0$ and $\limsup_{\lambda\to 0} \lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u])$ when $\gamma < 0$.

**Lemma 3.3.** *Suppose $u \in C^1_c(\mathbb{R}^N)$ and $1 \le p < \infty$. Then the following hold.*

*(i) If $\gamma > 0$ then*

$$\limsup_{\lambda\to\infty} \lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u]) \le \frac{\kappa(p,N)}{|\gamma|} \|\nabla u\|^p_{L^p(\mathbb{R}^N)}.$$

*(ii) If $\gamma < 0$ then*

$$\limsup_{\lambda\searrow 0} \lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u]) \le \frac{\kappa(p,N)}{|\gamma|} \|\nabla u\|^p_{L^p(\mathbb{R}^N)}.$$

*(iii) The statement in part (i) continues to hold for $u \in C^1(\mathbb{R}^N)$ whose gradient is compactly supported.*

*Remark* 3.4. The subtlety in part (iii) above is only relevant in dimension $N = 1$, since if $N \ge 2$, then any function in $C^1(\mathbb{R}^N)$ with a compactly supported gradient is constant outside a compact set.

*Proof of Lemma 3.3.* We distinguish the cases $\gamma > 0$ and $\gamma < 0$.

*The case* $\gamma > 0$. We assume that $\nabla u$ is compactly supported. To prove part (iii) (and thus part (i)) assume

$$\lambda \ge L := \Big\| \Big( \sum_{i=1}^N |\partial_i u|^2 \Big)^{1/2} \Big\|_{L^\infty(\mathbb{R}^N)}. \tag{3.4}$$

Then

$$(x,y) \in E_{\lambda,\gamma/p}[u] \implies \lambda |x-y|^{\gamma/p} \le L \implies |x-y| \le 1. \tag{3.5}$$

Furthermore, if $(x,y) \in E_{\lambda,\gamma/p}[u]$, then writing $y = x + r\omega$ with $r > 0$ and $\omega \in \mathbb{S}^{N-1}$, we have

$$\lambda r^{\gamma/p} \le |\nabla u(x)\cdot\omega| + \rho(r) \quad \text{with } \rho(r) := \sup_{x\in\mathbb{R}^N} \sup_{|h|\le r} |\nabla u(x+h) - \nabla u(x)|; \tag{3.6}$$

since $\nabla u$ is uniformly continuous on $\mathbb{R}^N$ we have $\rho(r) \searrow 0$ as $r \searrow 0$. This, together with the first implication of (3.5), shows that

$$\lambda r^{\gamma/p} \le |\nabla u(x)\cdot\omega| + \rho((\tfrac{L}{\lambda})^{p/\gamma}). \tag{3.7}$$

Let $B$ be a ball centered at the origin containing $\mathrm{supp}(\nabla u)$, and let $\widetilde{B}$ the expanded ball with radius $1+\mathrm{rad}(B)$. Then for $x \notin \widetilde{B}$, we have $Q_{\gamma/p}u(x,y) = 0$ for every $y$ with $|x-y| \le 1$, and (3.5) shows $(x,y) \notin E_{\lambda,\gamma/p}[u]$ for every $y$ with $|x-y|>1$, so $E_{\lambda,\gamma/p}[u] \subseteq \widetilde{B} \times \mathbb{R}^N$. Define, for $x \in \tilde{B}$, $\omega \in \mathbb{S}^{N-1}$, and $\lambda > 0$

$$\overline{R}(x,\omega,\lambda) := \Big(\lambda^{-1}(|\nabla u(x)\cdot\omega| + \rho((\tfrac{L}{\lambda})^{p/\gamma})\Big)^{p/\gamma}$$

Then by (3.7),

$$\begin{aligned}\lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u]) &\le \lambda^p \int_{\widetilde{B}}\int_{\mathbb{S}^{N-1}}\int_0^{\overline{R}(x,\omega,\lambda)} r^{\gamma-1}\,\mathrm{d}r\,\mathrm{d}\omega\,\mathrm{d}x \\ &= \gamma^{-1}\int_{\widetilde{B}}\int_{\mathbb{S}^{N-1}}\Big(|\nabla u(x)\cdot\omega| + \rho((\tfrac{L}{\lambda})^{p/\gamma})\Big)^p\,\mathrm{d}\omega\,\mathrm{d}x.\end{aligned}$$

Letting $\lambda \to \infty$ we get

$$\limsup_{\lambda\to\infty} \lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u]) \le \gamma^{-1}\kappa(p,N)\int_{\widetilde{B}} |\nabla u(x)|^p\,\mathrm{d}x$$

and hence the assertion.

*The case* $\gamma < 0$. We first note that if $(x,y) \in E_{\lambda,\gamma/p}[u]$, then writing $y = x + r\omega$, we have again (3.6).

Now let $\varepsilon > 0$, and let $\delta(\varepsilon) > 0$ be such that $\rho(r) \le \varepsilon$ for $0 < r \le \delta(\varepsilon)$. Let

$$r_\lambda(x,\omega,\varepsilon) = \min\Big\{\delta(\varepsilon), \Big(\frac{\lambda}{|\nabla u(x)\cdot\omega| + \varepsilon}\Big)^{\frac{p}{-\gamma}}\Big\}.$$

Note that $r_\lambda(x,\omega,\varepsilon) > 0$ for $\lambda > 0$. Also if $(x, x+r\omega) \in E_{\lambda,\gamma/p}[u]$ then $r \ge r_\lambda(x,\omega,\varepsilon)$; indeed, either $r_\lambda(x,\omega,\varepsilon) \ge \delta(\varepsilon)$ already, or else $r_\lambda(x,\omega,\varepsilon) < \delta(\varepsilon)$ in which case (3.6) shows $r_\lambda(x,\omega,\varepsilon) \ge \Big(\frac{\lambda}{|\nabla u(x)\cdot\omega|+\varepsilon}\Big)^{\frac{p}{-\gamma}}$.

Finally let $B$ be any ball in $\mathbb{R}^N$ containing the support of $u$, and let $\widetilde{B}$ be the double ball. Then

$$\begin{aligned}&\limsup_{\lambda\searrow 0} \lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u] \cap (\widetilde{B}\times\mathbb{R}^N)) \le \limsup_{\lambda\searrow 0}\lambda^p \int_{\widetilde{B}}\int_{\mathbb{S}^{N-1}}\int_{r_\lambda(x,\omega,\varepsilon)}^{\infty} r^{\gamma-1}\,\mathrm{d}r\,\mathrm{d}\omega\,\mathrm{d}x \\ &= \limsup_{\lambda\searrow 0}\lambda^p \int_{\widetilde{B}}\int_{\mathbb{S}^{N-1}} \frac{1}{|\gamma|}[r_\lambda(x,\omega,\varepsilon)]^\gamma\,\mathrm{d}\omega\,\mathrm{d}x \\ &= \limsup_{\lambda\searrow 0}\frac{1}{|\gamma|}\int_{\widetilde{B}}\int_{\mathbb{S}^{N-1}} \max\{\lambda^p\delta(\varepsilon)^\gamma, (|\nabla u(x)\cdot\omega|+\varepsilon)^p\}\,\mathrm{d}\omega\,\mathrm{d}x \\ &= \frac{1}{|\gamma|}\int_{\widetilde{B}}\int_{\mathbb{S}^{N-1}}(|\nabla u(x)\cdot\omega|+\varepsilon)^p\,\mathrm{d}\omega\,\mathrm{d}x.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary we obtain

$$\limsup_{\lambda\searrow 0}\lambda^p\nu_\gamma\big(E_{\lambda,\gamma/p}[u]\cap(\widetilde{B}\times\mathbb{R}^N)\big) \le \frac{1}{|\gamma|}\kappa(p,N)\|\nabla u\|^p_{L^p(\mathbb{R}^N)}. \tag{3.8}$$

Since $u = 0$ in $\mathbb{R}^N \setminus B$, if $(x, y) \in E_{\lambda,\gamma/p}[u] \cap ((\mathbb{R}^N \setminus \widetilde{B}) \times \mathbb{R}^N)$ then $y \in B$. Therefore

$$\limsup_{\lambda \searrow 0} \lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u] \cap ((\mathbb{R}^N \setminus \widetilde{B}) \times \mathbb{R}^N)) \leq \limsup_{\lambda \searrow 0} \lambda^p \int_B \int_{\mathbb{R}^N \setminus \widetilde{B}} |x - y|^{\gamma - N} \,\mathrm{d}x \,\mathrm{d}y = 0.$$

This finishes the proof of part (ii). □

In dimension $N = 1$, when $\gamma < -1$, one can also weaken the hypothesis $u \in C_c^1(\mathbb{R})$ in Lemma 3.3 to $u \in C^1(\mathbb{R})$ and $u'$ is compactly supported:

**Lemma 3.5.** *Suppose $u \in C^1(\mathbb{R})$, $u'$ is compactly supported, and $1 \leq p < \infty$. If $\gamma < -1$ then*

$$\limsup_{\lambda \searrow 0} \lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u]) \leq \frac{\kappa(p, N)}{|\gamma|} \|u'\|_{L^p(\mathbb{R})}^p.$$

*Proof.* Let $\operatorname{supp}(u') \subset B \coloneqq (-\beta, \beta)$. By (3.8) we have

$$\limsup_{\lambda \searrow 0} \nu_\gamma(E_{\lambda,\gamma/p}[u] \cap (-2\beta, 2\beta) \times \mathbb{R}) \leq \frac{1}{|\gamma|} \kappa(p, 1) \|u'\|_{L^p(\mathbb{R})}^p.$$

Moreover, since $u$ is constant on $(\beta, \infty)$ and constant on $(-\infty, -\beta)$, if $(x, y) \in E_{\lambda,\gamma/p}[u]$ and $x < -2\beta$ then $y > -\beta$, and if $(x, y) \in E_{\lambda,\gamma/p}[u]$ and $x > 2\beta$ then $y < \beta$. Since $\gamma < -1$,

$$\begin{aligned} &\nu_\gamma(E_{\lambda,\gamma/p}[u] \cap (\mathbb{R} \setminus (-2\beta, 2\beta)) \times \mathbb{R}) \\ &\leq \int_{2\beta}^{\infty} \int_{-\infty}^{\beta} (x - y)^{\gamma - 1} \,\mathrm{d}y \,\mathrm{d}x + \int_{-\infty}^{-2\beta} \int_{-\beta}^{\infty} (y - x)^{\gamma - 1} \,\mathrm{d}y \,\mathrm{d}x < \infty. \end{aligned}$$

We conclude

$$\limsup_{\lambda \searrow 0} \lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u] \cap (\mathbb{R} \setminus (-2\beta, 2\beta)) \times \mathbb{R}) = 0.$$ □

3.4. *Upper bounds for* $\limsup \lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u])$, *for general* $\dot{W}^{1,p}$ *functions*

Let $N \geq 1$, $1 \leq p < \infty$ and $u \in \dot{W}^{1,p}(\mathbb{R}^N)$. In light of Lemma 3.2, to prove the limiting relations (1.7) and (1.8) in Theorem 1.1, we need only show that

$$\limsup_{\lambda \to \infty} \lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u]) \leq \frac{\kappa(p, N)}{|\gamma|} \|\nabla u\|_{L^p(\mathbb{R}^N)}^p \tag{3.9}$$

if $\gamma > 0$ and

$$\limsup_{\lambda \searrow 0} \lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u]) \leq \frac{\kappa(p, N)}{|\gamma|} \|\nabla u\|_{L^p(\mathbb{R}^N)}^p \tag{3.10}$$

if $\gamma < 0$ and $p > 1$, or $\gamma < -1$ and $p = 1$. Lemma 3.3(i)(ii) asserts that these desired inequalities hold for functions in $C_c^1(\mathbb{R}^N)$. When $N \geq 2$ or $p > 1$, a general $\dot{W}^{1,p}(\mathbb{R}^N)$ function can be approximated in $\dot{W}^{1,p}(\mathbb{R}^N)$ by functions in $C_c^1(\mathbb{R}^N)$: by [12], there exists a sequence $\{u_n\}$ in $C_c^\infty(\mathbb{R}^N)$ such that $\lim_{n\to\infty} \|\nabla(u_n - u)\|_{L^p(\mathbb{R}^N)} = 0$. If further $\gamma > 0$, or

$\gamma < 0$ and $p > 1$, or $\gamma < -1$ and $p = 1$, then by parts (i) of Theorems 1.3 and 1.4 (proved in Section 2), we have

$$\sup_{\lambda>0} \lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u_n - u]) \leq C^p_{N,p,\gamma} \|\nabla(u_n - u)\|^p_{L^p(\mathbb{R}^N)}. \tag{3.11}$$

It follows that for every $n$ and every $\delta \in (0,1)$,

$$\begin{aligned}\limsup_{\lambda\to\infty} \lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u]) &\leq \limsup_{\lambda\to\infty} \lambda^p \nu_\gamma(E_{(1-\delta)\lambda,\gamma/p}[u_n]) + \sup_{\lambda>0} \lambda^p \nu_\gamma(E_{\delta\lambda,\gamma/p}[u_n - u]) \\ &\leq \frac{\kappa(p,N)}{|\gamma|(1-\delta)} \|\nabla u_n\|^p_{L^p(\mathbb{R}^N)} + \frac{C^p_{N,p,\gamma}\|\nabla(u_n - u)\|^p_{L^p(\mathbb{R}^N)}}{\delta^p}\end{aligned} \tag{3.12}$$

if $\gamma > 0$, and a similar inequality holds with $\limsup_{\lambda\to\infty}$ replaced by $\limsup_{\lambda\searrow 0}$ if $\gamma < 0$, $p > 1$ or $\gamma < -1$, $p = 1$. Letting first $n \to \infty$ and then $\delta \to 0$, we get the desired conclusions (3.9) and (3.10) under the corresponding conditions on $\gamma$ and $p$.

It remains to tackle the case $N = p = 1$, in which case we only need to prove (3.9) when $\gamma > 0$ and (3.10) when $\gamma < -1$. Using (2.11), we approximate $u$ by finding a sequence $\{u_n\}$ in $C^\infty(\mathbb{R})$ so that $u_n'$ are compactly supported for each $n$, and $\lim_{n\to\infty} \|u_n' - u'\|_{L^1(\mathbb{R})} = 0$. Since the desired inequalities hold for $u_n$ in place of $u$ by Lemma 3.3(iii) and Lemma 3.5, and since part (i) of Theorem 1.4 applies to give (3.11) when $\gamma > 0$ or $\gamma < -1$, our earlier argument in (3.12) can be repeated to yield (3.9) when $\gamma > 0$ and (3.10) when $\gamma < -1$. This completes our proof of parts (a) and (b) of Theorem 1.1.

3.5. *Conclusion of the proof of Theorem 1.1*

In Section 3.4 we proved parts (a) and (b) of Theorem 1.1. The lower bound for the $\liminf$ in part (c) has been established in Lemma 3.2(ii), and the limiting equality for $u \in C^1_c(\mathbb{R}^N)$ when $p = 1$ and $-1 \leq \gamma < 0$ follows by combining that with the upper bound for the $\limsup$ in part (ii) of Lemma 3.3. The proof of the negative result in part (c) of the theorem (generic failure for $p = 1$, $-1 \leq \gamma < 0$) will be given in Proposition 6.6 below. □

3.6. *On limit formulas for $\dot{BV}(\mathbb{R})$-functions - The proof of Proposition 1.2*

When $p = 1$, Poliakovsky [19] asked whether (1.7) still holds for $u \in \dot{BV}(\mathbb{R}^N)$ instead of $\dot{W}^{1,1}(\mathbb{R}^N)$ if $\gamma = N$. More generally, one may wonder whether it is possible that for all $u \in \dot{BV}(\mathbb{R}^N)$, one has

$$\lim_{\lambda\to\infty} \lambda \nu_\gamma(E_{\lambda,\gamma}[u]) = \frac{\kappa(1,N)}{|\gamma|} \|\nabla u\|_{\mathcal{M}} \quad \text{when } \gamma > 0, \tag{3.13}$$

and

$$\lim_{\lambda\to 0^+} \lambda \nu_\gamma(E_{\lambda,\gamma}[u]) = \frac{\kappa(1,N)}{|\gamma|} \|\nabla u\|_{\mathcal{M}} \quad \text{when } \gamma < 0. \tag{3.14}$$

We show that this is <u>not</u> the case.

First, when $-1 \leq \gamma < 0$, Theorem 1.8(i) (proved in Proposition 6.3 below) shows that even if $u \in \dot{W}^{1,1}(\mathbb{R}^N)$, it may happen that $\lim_{\lambda \to 0^+} \lambda \nu_\gamma(E_{\lambda,\gamma}[u]) = \infty$. So (3.14) cannot hold for all $u \in \dot{BV}(\mathbb{R}^N)$ for such $\gamma$.

The following lemma provides examples of failure of (3.13) and (3.14) when $\gamma \in \mathbb{R} \backslash [-1, 0]$, since $|\gamma + 1| \neq |\gamma|$ unless $\gamma = -1/2$:

**Lemma 3.6.** *Suppose $N \geq 1$ and $u = \mathbb{1}_\Omega$ where $\Omega$ is any bounded domain in $\mathbb{R}^N$ with smooth boundary. Then $u \in \dot{BV}(\mathbb{R}^N)$ and*

$$\lim_{\lambda \to \infty} \lambda \nu_\gamma(E_{\lambda,\gamma}[u]) = \frac{\kappa(1, N)}{|\gamma + 1|} \|\nabla u\|_{\mathcal{M}} \quad \textit{for all } \gamma > -1,$$

*while*

$$\lim_{\lambda \to 0^+} \lambda \nu_\gamma(E_{\lambda,\gamma}[u]) = \frac{\kappa(1, N)}{|\gamma + 1|} \|\nabla u\|_{\mathcal{M}} \quad \textit{for all } \gamma < -1.$$

*Proof.* First consider the case $N = 1$. If $u = \mathbb{1}_{[0,\infty)}$, then for every $\gamma \in \mathbb{R} \setminus \{-1\}$ and $\lambda > 0$, one has

$$\nu_\gamma(E_{\lambda,\gamma}[u]) = 2\nu_\gamma(\{(x, y) \in \mathbb{R}\colon x \geq 0, y < 0, |x - y|^{-(\gamma+1)} \geq \lambda\}) = \frac{2}{|\gamma + 1|} \frac{1}{\lambda}, \tag{3.15}$$

which follows from a change of variables $s = x - y$, $t = x + y$: when $\gamma > -1$, one has

$$\nu_\gamma(E_{\lambda,\gamma}[u]) = \int_0^{\lambda^{-\frac{1}{\gamma+1}}} \int_{-s}^{s} \mathrm{d}t \, s^{\gamma-1} \, \mathrm{d}s = 2 \int_0^{\lambda^{-\frac{1}{\gamma+1}}} s^\gamma \, \mathrm{d}s = \frac{2}{\gamma + 1} \frac{1}{\lambda}$$

while when $\gamma < -1$, one has

$$\nu_\gamma(E_{\lambda,\gamma}[u]) = \int_{\lambda^{-\frac{1}{\gamma+1}}}^{\infty} \int_{-s}^{s} \mathrm{d}t \, s^{\gamma-1} \, \mathrm{d}s = 2 \int_{\lambda^{-\frac{1}{\gamma+1}}}^{\infty} s^\gamma \, \mathrm{d}s = \frac{2}{|\gamma + 1|} \frac{1}{\lambda}.$$

A similar calculation shows that if $u = \mathbb{1}_{I_1} + \cdots + \mathbb{1}_{I_j}$ is a sum of characteristic functions of finitely many open intervals whose closures are pairwise disjoint, then

$$\lim_{\lambda \to \infty} \lambda \nu_\gamma(E_{\lambda,\gamma}[u]) = \frac{2}{|\gamma + 1|} \|u'\|_{\mathcal{M}(\mathbb{R})} \quad \text{for all } \gamma > -1, \tag{3.16}$$

while

$$\lim_{\lambda \to 0^+} \lambda \nu_\gamma(E_{\lambda,\gamma}[u]) = \frac{2}{|\gamma + 1|} \|u'\|_{\mathcal{M}(\mathbb{R})} \quad \text{for all } \gamma < -1; \tag{3.17}$$

we also have

$$\sup_{\lambda > 0} \lambda \nu_\gamma(E_{\lambda,\gamma}[u]) \leq \frac{2}{|\gamma + 1|} \|u'\|_{\mathcal{M}(\mathbb{R})} \quad \text{for all } \gamma \in \mathbb{R} \setminus \{-1\}. \tag{3.18}$$

Now consider the case $N \geq 2$. Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with smooth boundary and $u = \mathbb{1}_\Omega$. Then $u \in \dot{BV}(\mathbb{R}^N)$ with $\|\nabla u\|_{\mathcal{M}} = \mathcal{L}^{N-1}(\partial\Omega)$. The method of rotation shows

$$\lambda \nu_\gamma(E_{\lambda,\gamma}[u]) = \frac{1}{2} \int_{\mathbb{S}^{N-1}} \int_{\omega^\perp} \lambda \nu_\gamma(E_{\lambda,\gamma}[u_{\omega,x'}]) \,\mathrm{d}x' \,\mathrm{d}\omega$$

where $u_{\omega,x'}(t) := u(x' + t\omega)$ for $\omega \in \mathbb{S}^{N-1}$ and $x' \in \omega^\perp$. (3.16), (3.18) and the dominated convergence theorem allows one to show that

$$\lim_{\lambda \to \infty} \lambda \nu_\gamma(E_{\lambda,\gamma}[u]) = \frac{1}{|\gamma + 1|} \int_{\mathbb{S}^{N-1}} \int_{\omega^\perp} \|u'_{\omega,x'}\|_{\mathcal{M}(\mathbb{R})} \,\mathrm{d}x' \,\mathrm{d}\omega \quad \text{for all } \gamma > -1,$$

and using (3.17) in place of (3.16) we obtain the same conclusion with $\lim_{\lambda \to \infty}$ replaced by $\lim_{\lambda \to 0^+}$ if $\gamma < -1$. It remains to observe that

$$\int_{\mathbb{S}^{N-1}} \int_{\omega^\perp} \|u'_{\omega,x'}\|_{\mathcal{M}(\mathbb{R})} \,\mathrm{d}x' \,\mathrm{d}\omega = \kappa(1, N) \|\nabla u\|_{\mathcal{M}}; \tag{3.19}$$

this equality holds by Fubini's theorem if $u = \mathbb{1}_\Omega$ is replaced by $u_\varepsilon := u * \rho_\varepsilon$ where $\rho_\varepsilon$ is a suitable family of mollifiers, because the left hand side is then just

$$\int_{\mathbb{S}^{N-1}} \int_{\omega^\perp} \int_{\mathbb{R}} \Big| \frac{d}{dt} u_\varepsilon(x' + t\omega) \Big| \,\mathrm{d}t \,\mathrm{d}x' \,\mathrm{d}\omega = \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} |\omega \cdot \nabla u_\varepsilon(x)| \,\mathrm{d}x \,\mathrm{d}\omega$$

which equals $\kappa(1, N) \|\nabla u_\varepsilon\|_{L^1(\mathbb{R}^N)}$. One then just need to let $\varepsilon \to 0$ to obtain (3.19). (See also the integral-geometric formula for the surface measure in [11, Chapters 2.10.15, 3.2.13 & 3.2.26], which extends the classical Crofton formula.) $\square$

## 4. From weak type bounds on quotients to $\dot{W}^{1,p}$ and $\dot{BV}$

In this section we complete the proofs of Theorems 1.3 and 1.4 proving part (ii) of these theorems. We use as a key tool the BBM formula discovered in [1] (see also [9] for additional information for the $BV$ case), in a way that is reminiscent of the proof of [15, Theorem 2], and we apply duality for Lorentz spaces to control the double integral arising in the BBM formula. The BBM formula stated in [1] is quite flexible, involving a bounded smooth domain $\Omega$ and a sequence of non-negative radial mollifiers $\rho_n(|x|)$ with $\int_0^\infty \rho_n(r) r^{N-1} \,\mathrm{d}r = 1$ and $\lim_{n \to \infty} \int_\delta^\infty \rho_n(r) r^{N-1} \,\mathrm{d}r = 0$ for every $\delta > 0$; we will apply it in the case when $\Omega = B_R$, the ball of radius $R$ centered at 0, and $\rho_n(r) = s_n p (2R)^{-s_n p} r^{-N + s_n p} \mathbb{1}_{[0,2R]}(r)$ where $\{s_n\}$ is a sequence of positive numbers tending to 0. As a result, we conclude that if $R > 0$, $1 \leq p < \infty$, $u \in L^p(B_R)$ and

$$\liminf_{s \to 0^+} s \iint_{B_R \times B_R} \frac{|u(x) - u(y)|^p}{|x - y|^{N + p - sp}} \,\mathrm{d}x \,\mathrm{d}y < \infty,$$

then for $p = 1$ we have $u \in \dot{BV}(B_R)$ with $\|\nabla u\|_{\mathcal{M}(B_R)}$ being bounded by $\kappa(1, N)$ times the above liminf, and for $1 < p < \infty$ we have $u \in \dot{W}^{1,p}(B_R)$ and $\|\nabla u\|_{L^p(B_R)}$ being bounded by $\kappa(p, N)/p$ times the above liminf. The assumption $u \in L^p(B_R)$ can easily be relaxed to $u \in L^1(B_R)$, via an observation of Stein as explained in [3, proof of Theorem 2]: if

$u \in L^1(B_R)$ and the above liminf is finite for some $1 < p < \infty$, then for any $\delta > 0$ and any $\varepsilon \in (0, \delta)$, we may consider $u_\varepsilon := u * \phi_\varepsilon(x)$ where $\phi_\varepsilon(x) := \varepsilon^{-N}\phi(\varepsilon^{-1}x)$ and $\phi \in C_c^\infty(B_1)$ is non-negative and has integral 1. Then $u_\varepsilon$ is $C^\infty$ on the closure of the ball $B_{R-\delta}$, so the above formulation of BBM applies, and $\|\nabla u_\varepsilon\|_{L^p(B_{R-\delta})}$ is uniformly bounded independent of $\varepsilon \in (0, \delta)$; indeed Jensen's inequality implies

$$\iint_{B_{R-\delta}\times B_{R-\delta}} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^p}{|x-y|^{N+p-sp}} \,\mathrm{d}x\,\mathrm{d}y \le \iint_{B_R\times B_R} \frac{|u(x) - u(y)|^p}{|x-y|^{N+p-sp}} \,\mathrm{d}x\,\mathrm{d}y$$

for every $\varepsilon$. This shows that a subsequence of $\{\nabla u_\varepsilon\}$ converges weakly in $L^p(B_{R-\delta})$ to the distributional gradient $\nabla u$ on $B_{R-\delta}$, and a desired bound on $\|\nabla u\|_{L^p(B_{R-\delta})}$ follows for every $\delta > 0$.

Suppose now $N \ge 1$, $1 \le p < \infty$, $\gamma \in \mathbb{R}$, $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and $Q_{\gamma/p}u \in L^{p,\infty}(\mathbb{R}^{2N}, \nu_\gamma)$. Let

$$A \coloneqq \sup_{R>0} \liminf_{s\to 0^+} s \iint_{B_R\times B_R} \frac{|u(x) - u(y)|^p}{|x-y|^{N+p-sp}} \,\mathrm{d}x\,\mathrm{d}y. \tag{4.1}$$

Suppose $A$ is finite. If $p = 1$, then the BBM formula above implies $u \in \dot{BV}(B_R)$ for every $R > 0$, with $\|\nabla u\|_{\mathcal{M}(B_R)} \le \kappa(1, N)A$ independent of $R$; as a result, $u \in \dot{BV}(\mathbb{R}^N)$, with $\|\nabla u\|_{\mathcal{M}(\mathbb{R}^N)} \le \kappa(1, N)A$. Similarly, if $1 < p < \infty$, the above BBM formula (applicable for $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$) implies $u \in \dot{W}^{1,p}(\mathbb{R}^N)$, with $\|\nabla u\|_{L^p(\mathbb{R}^N)} \le (\kappa(1, N)A/p)^{1/p}$.

It remains to prove that $A < \infty$. By considering truncations of $u$ we may assume additionally that $u \in L^\infty(\mathbb{R}^N)$; the reduction is based on the pointwise bound

$$Q_{\gamma/p}u_n(x, y) \le Q_{\gamma/p}u(x, y) \text{ where } u_n(x) = \begin{cases} u(x) & \text{if } |u(x)| < n, \\ n\frac{u(x)}{|u(x)|} & \text{if } |u(x)| \ge n. \end{cases}$$

Using the definition of weak derivative we see by a limiting argument that the conclusion $\sup_n \|\nabla u_n\|_p \le C$ implies $\|\nabla u\|_p \le C$ if $p > 1$ and $\sup_n \|\nabla u_n\|_{\mathcal{M}} \le C$ implies $\|\nabla u\|_{\mathcal{M}} \le C$.

In order to establish our estimate for bounded functions we will use Lorentz duality in the following form: if $F$, $G$ are measurable functions on $\mathbb{R}^{2N}$, then for any $1 < q < \infty$, we have

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} F(x, y)G(x, y) \,\mathrm{d}\nu_\gamma \le q'[F]_{L^{q,\infty}(\mathbb{R}^{2N},\nu_\gamma)}[G]_{L^{q',1}(\mathbb{R}^{2N},\nu_\gamma)} \tag{4.2}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$,

$$[F]_{L^{q,\infty}(\mathbb{R}^{2N},\nu_\gamma)} \coloneqq \sup_{\lambda>0} \lambda\nu_\gamma(\{|F| > \lambda\})^{1/q} = \sup_{t>0} t^{1/q}F^*(t),$$

and

$$[G]_{L^{q',1}(\mathbb{R}^{2N},\nu_\gamma)} \coloneqq \int_0^\infty \nu_\gamma(\{|G| > \lambda\})^{1/q'} \,\mathrm{d}\lambda = \frac{1}{q'}\int_0^\infty t^{1/q'}G^*(t)\frac{\mathrm{d}t}{t};$$

here $F^*(t) \coloneqq \inf\{s>0\colon \nu_\gamma(\{|F|>\lambda\}) \le s\}$ is the non-increasing rearrangement of $F$, and similarly for $G^*(t)$ (see [13, 22]). Indeed, (4.2) follows by noticing that

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} F(x,y)G(x,y)\,\mathrm{d}\nu_\gamma \le \int_0^\infty F^*(t)G^*(t)\,\mathrm{d}t = \int_0^\infty [t^{1/q}F^*(t)][t^{1/q'}G^*(t)]\frac{\mathrm{d}t}{t}$$

which is clearly $\le q'[F]_{L^{q,\infty}(\mathbb{R}^{2N},\nu_\gamma)}[G]_{L^{q',1}(\mathbb{R}^{2N},\nu_\gamma)}$.

First we consider the case $\gamma>0$. For sufficiently small $s>0$, define

$$\theta \coloneqq \frac{s}{1+\frac{\gamma}{p}}$$

so that $\theta\in(0,1)$ and $p-sp = p(1-\theta)(1+\frac{\gamma}{p})-\gamma$. Then for every $R>0$,

$$\begin{aligned}
&\iint_{B_R\times B_R} \frac{|u(x)-u(y)|^p}{|x-y|^{N+p-sp}}\,\mathrm{d}x\,\mathrm{d}y\\
=&\iint_{\mathbb{R}^N\times\mathbb{R}^N} \left(Q_{\gamma/p}u(x,y)\right)^{p(1-\theta)} \left(|u(x)-u(y)|\mathbb{1}_{B_R\times B_R}(x,y)\right)^{p\theta}\,\mathrm{d}\nu_\gamma\\
\le&\frac{1}{\theta}\left[\left(Q_{\gamma/p}u\right)^{p(1-\theta)}\right]_{L^{\frac{1}{1-\theta},\infty}(\mathbb{R}^{2N},\nu_\gamma)} \left[|u(x)-u(y)|^{p\theta}\right]_{L^{\frac{1}{\theta},1}(B_R\times B_R,\nu_\gamma)}
\end{aligned}$$

by (4.2). But

$$\left[\left(Q_{\gamma/p}u\right)^{p(1-\theta)}\right]_{L^{\frac{1}{1-\theta},\infty}(\mathbb{R}^{2N},\nu_\gamma)} = \left[Q_{\gamma/p}u\right]^{p(1-\theta)}_{L^{p,\infty}(\mathbb{R}^{2N},\nu_\gamma)}$$

and

$$\begin{aligned}
\left[|u(x)-u(y)|^{p\theta}\right]_{L^{\frac{1}{\theta},1}(B_R\times B_R,\nu_\gamma)} &\le (2\|u\|_{L^\infty(\mathbb{R}^N)})^{p\theta}[\mathbb{1}_{B_R\times B_R}]_{L^{\frac{1}{\theta},1}(\mathbb{R}^N\times\mathbb{R}^N,\nu_\gamma)}\\
&= (2\|u\|_{L^\infty(\mathbb{R}^N)})^{p\theta}\nu_\gamma(B_R\times B_R)^\theta,
\end{aligned}$$

from which it follows that

$$s\iint_{B_R\times B_R} \frac{|u(x)-u(y)|^p}{|x-y|^{N+p-sp}}\,\mathrm{d}x\,\mathrm{d}y \le \frac{s}{\theta}\left[Q_{\gamma/p}u\right]^{p(1-\theta)}_{L^{p,\infty}(\mathbb{R}^{2N},\nu_\gamma)} (2\|u\|_{L^\infty(\mathbb{R}^N)})^{p\theta}\nu_\gamma(B_R\times B_R)^\theta.$$

Furthermore, since $\gamma>0$, we have

$$\nu_\gamma(B_R\times B_R) \le |B_R|\int_{B_{2R}} \frac{1}{|h|^{N-\gamma}}\,\mathrm{d}h < \infty.$$

Recall $\theta = \frac{s}{1+\frac{\gamma}{p}}$. Thus as $s\to 0^+$, we have

$$\limsup_{s\to0^+} s\iint_{B_R\times B_R} \frac{|u(x)-u(y)|^p}{|x-y|^{N+p-sp}}\,\mathrm{d}x\,\mathrm{d}y \le \left(1+\frac{\gamma}{p}\right)\left[Q_{\gamma/p}u\right]^p_{L^{p,\infty}(\mathbb{R}^{2N},\nu_\gamma)} < \infty.$$

Since this upper bound holds uniformly over all $R>0$, this concludes the argument for the case $\gamma>0$.

Next we turn to the case $\gamma \le 0$. We then observe that for $0 < s < 1$ and every $R > 0$,

$$
\begin{aligned}
&\iint_{B_R \times B_R} \frac{|u(x)-u(y)|^p}{|x-y|^{N+p-sp}} \,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_{\mathbb{R}^N \times \mathbb{R}^N} \left(Q_{\gamma/p} u(x,y)\right)^{p(1-\frac{s}{2})} \left(|u(x)-u(y)||x-y|^{1-\frac{\gamma}{p}} \mathbb{1}_{B_R \times B_R}\right)^{p\frac{s}{2}} \,\mathrm{d}\nu_\gamma \\
&\le \frac{2}{s} \left[\left(Q_{\gamma/p} u\right)^{p(1-\frac{s}{2})}\right]_{L^{\frac{1}{1-\frac{s}{2}},\infty}(\mathbb{R}^{2N},\nu_\gamma)} \left[\left(|u(x)-u(y)||x-y|^{1-\frac{\gamma}{p}}\right)^{p\frac{s}{2}}\right]_{L^{\frac{2}{s},1}(B_R \times B_R,\nu_\gamma)}.
\end{aligned}
$$

Again

$$
\left[\left(Q_{\gamma/p} u\right)^{p(1-\frac{s}{2})}\right]_{L^{\frac{1}{1-\frac{s}{2}},\infty}(\mathbb{R}^{2N},\nu_\gamma)} = \left[Q_{\gamma/p} u\right]^{p(1-\frac{s}{2})}_{L^{p,\infty}(\mathbb{R}^{2N},\nu_\gamma)}
$$

and

$$
\begin{aligned}
\text{(4.3)}\quad &\left[\left(|u(x)-u(y)||x-y|^{1-\frac{\gamma}{p}}\right)^{p\frac{s}{2}}\right]_{L^{\frac{2}{s},1}(B_R \times B_R,\nu_\gamma)} \\
&\qquad \le (2\|u\|_{L^\infty(\mathbb{R}^N)})^{p\frac{s}{2}} \left[|x-y|^{(p-\gamma)\frac{s}{2}}\right]_{L^{\frac{2}{s},1}(B_R \times B_R,\nu_\gamma)}.
\end{aligned}
$$

We will show that

$$
\limsup_{s \to 0^+} \left[|x-y|^{(p-\gamma)\frac{s}{2}}\right]_{L^{\frac{2}{s},1}(B_R \times B_R,\nu_\gamma)} \le 1 - \frac{\gamma}{p} \tag{4.4}
$$

when $\gamma \le 0$. We then see that

$$
\limsup_{s \to 0^+} s \iint_{B_R \times B_R} \frac{|u(x)-u(y)|^p}{|x-y|^{N+p-sp}} \,\mathrm{d}x\,\mathrm{d}y \le 2\left(1 - \frac{\gamma}{p}\right) \left[Q_{\gamma/p} u\right]^p_{L^{p,\infty}(\mathbb{R}^{2N},\nu_\gamma)}
$$

which concludes the argument in this case since this bound is uniform in $R > 0$.

It remains to prove (4.4) when $\gamma \le 0$. Note that in this case $p - \gamma > 0$, so $|x-y|^{(p-\gamma)\frac{s}{2}} \le (2R)^{(p-\gamma)\frac{s}{2}}$ on $B_R \times B_R$. Thus

$$
\left[|x-y|^{(p-\gamma)\frac{s}{2}}\right]_{L^{\frac{2}{s},1}(B_R \times B_R,\nu_\gamma)} = \int_0^{(2R)^{(p-\gamma)\frac{s}{2}}} \nu_\gamma\{(x,y) \in B_R \times B_R \colon |x-y|^{(p-\gamma)\frac{s}{2}} > \lambda\}^{\frac{s}{2}} \,\mathrm{d}\lambda
$$

If $\gamma < 0$, then

$$
\nu_\gamma\{(x,y) \in B_R \times B_R \colon |x-y|^{(p-\gamma)\frac{s}{2}} > \lambda\} \le |B_R| \int_{|h| > \lambda^{\frac{2}{s(p-\gamma)}}} \frac{1}{|h|^{N-\gamma}} \,\mathrm{d}h \le \sigma_{N-1} |B_R| \frac{1}{|\gamma|} \lambda^{\frac{2\gamma}{s(p-\gamma)}}
$$

where $\sigma_{N-1}$ is the surface area of $\mathbb{S}^{N-1}$. Hence in this case,

$$
\begin{aligned}
\left[|x-y|^{(p-\gamma)\frac{s}{2}}\right]_{L^{\frac{2}{s},1}(B_R \times B_R,\nu_\gamma)} &\le \left(\sigma_{N-1}|B_R|\frac{1}{|\gamma|}\right)^{\frac{s}{2}} \int_0^{(2R)^{(p-\gamma)\frac{s}{2}}} \lambda^{\frac{\gamma}{p-\gamma}} \,\mathrm{d}\lambda \\
&= \left(1 - \frac{\gamma}{p}\right) \left(\sigma_{N-1}|B_R|\frac{1}{|\gamma|}\right)^{\frac{s}{2}} (2R)^{p\frac{s}{2}}.
\end{aligned}
$$

(Here we used $\frac{\gamma}{p-\gamma} = -\frac{1}{1-\frac{\gamma}{p}} \in (-1,0)$ whenever $\gamma < 0$.) This proves (4.4) when $\gamma < 0$. Next, suppose $\gamma = 0$. Then

$$\begin{aligned}
\left[|x-y|^{(p-\gamma)\frac{s}{2}}\right]_{L^{\frac{2}{s},1}(B_R\times B_R,\nu_\gamma)} &= \int_0^{(2R)^{p\frac{s}{2}}} \nu_0\{(x,y)\in B_R\times B_R\colon |x-y|^{p\frac{s}{2}} > \lambda\}^{\frac{s}{2}}\,\mathrm{d}\lambda \\
&\leq \int_0^{(2R)^{p\frac{s}{2}}} \left(|B_R| \int_{\lambda^{\frac{2}{sp}}\leq |h|\leq 2R} \frac{1}{|h|^N} dh\right)^{\frac{s}{2}} \mathrm{d}\lambda \\
&= \int_0^{(2R)^{p\frac{s}{2}}} \left(|B_R|\omega_{N-1}\frac{2}{ps}\log\left(\frac{(2R)^{p\frac{s}{2}}}{\lambda}\right)\right)^{\frac{s}{2}} \mathrm{d}\lambda \\
&= (2R)^{p\frac{s}{2}} \int_0^1 \left(|B_R|\omega_{N-1}\frac{2}{ps}\log\left(\frac{1}{\lambda}\right)\right)^{\frac{s}{2}} \mathrm{d}\lambda
\end{aligned}$$

which shows (4.4) remains valid when $\gamma = 0$ by the dominated convergence theorem. $\square$

## 5. Finiteness of $\nu_0(E_{\lambda,0}[u])$ and the Lipschitz norm

In this section we prove Theorem 1.5 which we put in the following more precise form.

**Proposition 5.1.** *Let $u$ be locally integrable on $\mathbb{R}^N$ and $\nabla u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Then*

$$\nu_0(E_{\lambda,0}[u]) = \begin{cases} 0 & \text{if } \lambda > \|\nabla u\|_\infty, \\ \infty & \text{if } \lambda < \|\nabla u\|_\infty. \end{cases}$$

*Proof.* First assume $\nabla u \in L^\infty$ and $\lambda > \|\nabla u\|_\infty$. Then for every $h \in \mathbb{R}^N$ we have $\frac{|u(x+h)-u(x)|}{|h|} \leq \lambda$ for almost every $x \in \mathbb{R}^N$. This immediately implies $\nu_0(E_{\lambda,0}[u]) = 0$.

For the more substantial part assume $\lambda < \|\nabla u\|_\infty$ where $\|\nabla u\|_\infty$ may be finite or infinite. We need to show that $\nu_0(E_{\lambda,0}[u]) = \infty$. We pick $\lambda_1, \lambda_2$ such that

$$\lambda < \lambda_1 < \lambda_2 < \|\nabla u\|_\infty.$$

Let $B_R = \{x \in \mathbb{R}^N : |x| < R\}$ and assume that $R > 1$ is so large that $\|\nabla u\|_{L^\infty(B_R)} > \lambda_2$ is not the zero distribution on $B_R$. Let $\chi \in C_c^\infty$ such that $\chi(x) = 1$ in a neighborhood of $\overline{B_{2R}}$ and set $u_\circ = \chi u$, Then $\nabla u_\circ = \nabla u$ in the sense of $L^1(B_{2R})$. There is a measurable set $F_0 \subset B_R$ of positive measure such that $|\nabla u(x)| > \lambda_2$ for all $x \in F_0$.

Fix $0 < \varepsilon \ll 1 - \frac{\lambda_1}{\lambda_2}$. We now consider the set $\mathfrak{S}_\varepsilon$ of all spherical balls $S \subset \mathbb{S}^{N-1}$ with positive radius and the property that $\langle\theta_1,\theta_2\rangle > 1-\varepsilon$ for all $\theta_1,\theta_2 \in S$. By pigeonholing there exists a spherical ball $S \in \mathfrak{S}_\varepsilon$ and a Lebesgue measurable subset $F \subset F_0$ such that $\mathcal{L}^N(F) > 0$ and $\frac{\nabla u(x)}{|\nabla u(x)|} \in S$ for all $x \in F$. For the remainder of the argument we fix this spherical ball $S$; we denote by $\sigma(S)$ its spherical measure.

We first note that for $|h| \leq 1$ and for almost every $|x| \leq R$

$$\frac{u(x+h)-u(x)}{|h|} = \frac{u_\circ(x+h)-u_\circ(x)}{|h|} = \Big\langle \frac{h}{|h|}, \int_0^1 \nabla u_\circ(x+sh)\,\mathrm{d}s \Big\rangle. \tag{5.1}$$

Secondly since the translation operator is continuous in the strong operator topology of $L^1$ we see that there exists $\delta_0 < 1$ such that

$$\|\nabla u_\circ(\cdot + w) - \nabla u_\circ\|_{L^1(\mathbb{R}^N)} < \frac{\mathcal{L}^N(F)(\lambda_1 - \lambda)}{10} \text{ for } |w| \leq \delta_0. \tag{5.2}$$

In what follows we let $\delta \ll \delta_0$ and set

$$S(\delta,\delta_0) = \Big\{ h \in \mathbb{R}^N : \delta \leq |h| \leq \delta_0, \frac{h}{|h|} \in S \Big\}.$$

Let

$$\mathcal{E}_0 = \Big\{ (x,h) : x \in F,\ h \in S(\delta,\delta_0),\ \frac{|u(x+h)-u(x)|}{|h|} > \lambda \Big\}$$

so that $(x,h) \in \mathcal{E}_0$ implies $(x,x+h) \in E_{\lambda,0}[u]$. We then have by (5.1)

$$\begin{aligned} \nu_0(E_{\lambda,0}[u]) &\geq \nu_0(\mathcal{E}_0) = \nu_0\Big(\Big\{ (x,h) : x \in F,\ h \in S(\delta,\delta_0),\ |\langle \tfrac{h}{|h|}, \int_0^1 \nabla u_\circ(x+sh)\,\mathrm{d}s \rangle| > \lambda \Big\}\Big) \\ &\geq \nu_0(\mathcal{E}_1) - \nu_0(\mathcal{E}_2) \end{aligned} \tag{5.3}$$

where

$$\begin{aligned} \mathcal{E}_1 &= \{ (x,h) : x \in F,\ h \in S(\delta,\delta_0),\ |\langle \tfrac{h}{|h|}, \nabla u_\circ(x) \rangle| > \lambda_1 \} \\ \mathcal{E}_2 &= \Big\{ (x,h) : x \in F, h \in S(\delta,\delta_0),\ \int_0^1 |\nabla u_\circ(x+sh) - \nabla u_\circ(x)|\,\mathrm{d}s > \lambda_1 - \lambda \Big\}. \end{aligned}$$

Indeed, if $(x,h) \notin \mathcal{E}_0 \cup \mathcal{E}_2$ then

$$|\langle \tfrac{h}{|h|}, \nabla u_\circ(x) \rangle| \leq |\langle \tfrac{h}{|h|}, \int_0^1 \nabla u_\circ(x+sh)\,\mathrm{d}s \rangle| + \int_0^1 |\nabla u_\circ(x+sh) - \nabla u_\circ(x)|\,\mathrm{d}s$$

which is then $\leq \lambda_1$, so $(x,h) \notin \mathcal{E}_1$, establishing $\mathcal{E}_1 \subset \mathcal{E}_0 \cup \mathcal{E}_2$ and thus (5.3).

The set $\mathcal{E}_1$ does not change if we replace $u_\circ$ by $u$ in its definition. Since

$$\langle \tfrac{h}{|h|}, \nabla u(x) \rangle \geq (1-\varepsilon)|\nabla u(x)| > (1-\varepsilon)\lambda_2 > \lambda_1 \text{ for } x \in F,\ \tfrac{h}{|h|} \in S$$

we get

$$\nu_0(\mathcal{E}_1) \geq \int_F dx \int_{S(\delta,\delta_0)} \frac{\mathrm{d}h}{|h|^N} = \mathcal{L}^N(F)\sigma(S)\log(\delta_0/\delta).$$

Moreover, using (5.2) and Chebyshev's inequality we see that

$$\begin{aligned}\nu_0(\mathcal{E}_2) &\le \int_{S(\delta,\delta_0)} \frac{\int_0^1 \|\nabla u_\circ(\cdot + sh) - \nabla u_\circ\|_{L^1(\mathbb{R}^N)}\,\mathrm{d}s}{\lambda_1-\lambda}\frac{dh}{|h|^N}\\ &\le \int_{S(\delta,\delta_0)} \frac{\mathcal{L}^N(F)(\lambda_1-\lambda)/10}{\lambda_1-\lambda}\frac{dh}{|h|^N} = \frac{\mathcal{L}^N(F)}{10}\sigma(S)\log(\delta_0/\delta)\end{aligned}$$

and hence putting pieces together we obtain for $\delta < \delta_0$

$$\nu_0(E_{\lambda,0}[u]) \ge \nu_0(\mathcal{E}_1) - \nu_0(\mathcal{E}_2) > \frac{\mathcal{L}^N(F)}{2}\sigma(S)\log(\delta_0/\delta).$$

Here $\delta < \delta_0$ was arbitrary and by letting $\delta \to 0$ we conclude that $\nu_0(E_{\lambda,0}[u]) = \infty$. □

We now give a more precise version of Example 1.7.

**Lemma 5.2.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary and let $u = \mathbb{1}_\Omega$. Then $u \in BV(\mathbb{R}^N) \setminus \dot{W}^{1,1}(\mathbb{R}^N)$ with*

$$\nu_0(E_{\lambda,0}[u]) \le C_\Omega \times \begin{cases} \log(2/\lambda) & \text{if } \lambda \le 1,\\ \lambda^{-1} & \text{if } \lambda > 1;\end{cases}$$

*in particular we have* $\sup_{\lambda>0} \lambda\, \nu_0(E_{\lambda,0}[u]) < \infty$.

*Proof.* Let

$$E(r,\lambda) = \{(x,y) \in E_{\lambda,0}[u] \colon r \le |x-y| \le 2r\}.$$

We begin with the observation that $r\lambda \le 2$ if $\nu_0(E(r,\lambda)) > 0$. Furthermore, if $(x,y) \in E(r,\lambda)$ for some $y \in \mathbb{R}^N$, then $x$ belongs to the $2r$-neighborhood of $\partial\Omega$. The Lebesgue measure of such a neighborhood is $O(r)$ if $r \le r_0$ where $r_0$ is some positive constant depending on $\Omega$ (because the boundary of a bounded Lipschitz domain can be covered by finitely many Lipschitz graphs, and the $2r$-neighborhood of such graphs can be approximated by a union of $O(r)$ neighborhoods of suitable hyperplanes). Hence for $r \le r_0$ we have $\nu_0(E(r,\lambda)) \le Cr$ if $r \le 2/\lambda$ and $\nu_0(E(r,\lambda)) = 0$ if $r > 2/\lambda$. As a result, if $2/\lambda \le r_0$ we get

$$\nu_0(E_{\lambda,0}[u]) \le \sum_{j\in\mathbb{Z}\colon 2^j \le 2/\lambda} \nu_0(E(2^j,\lambda)) \lesssim \lambda^{-1}$$

and if $2/\lambda > r_0$ we get

$$\nu_0(E_{\lambda,0}[u]) \le \sum_{j\in\mathbb{Z}\colon 2^j\le r_0} \nu_0(E(2^j,\lambda)) + 2\int_\Omega \int_{r_0 \le |x-y| \le 2/\lambda} \frac{dy}{|x-y|^N}dx \lesssim 1 + \log(\lambda^{-1}).$$

□

## 6. When the upper bound (1.15) fails

In this section we make various constructions demonstrating the failure of (1.15) in the range $-1 \le \gamma < 0$, and give the proof of Theorem 1.8. We first establish

**Proposition 6.1.** *Suppose $N \ge 1$ and $-1 \le \gamma < 0$.*

*(i) For every $m > 0$, there exists $u \in C_c^\infty(\mathbb{R}^N)$ such that*

$$\nu_\gamma(E_{1,\gamma}[u]) > m\|\nabla u\|_{L^1(\mathbb{R}^N)}. \tag{6.1}$$

*(ii) There exists $C = C(N,\gamma) > 0$ and $p_0 = p_0(N,\gamma) > 1$ such that for all $1 < p < p_0$,*

$$\sup_{\substack{u\in C_c^\infty(\mathbb{R}^N)\\ \|\nabla u\|_{L^p}\le 1}} \nu_\gamma(E_{1,\gamma/p}[u]) \ge C\frac{p}{p-1}. \tag{6.2}$$

### 6.1. *Proof of Proposition 6.1: The case $\gamma = -1$.*

Here we may choose, for $m > 1$,

$$v_m = \mathbb{1}_{B_1} * \eta_{1/m} \in C_c^\infty(\mathbb{R}^N) \tag{6.3}$$

where $\eta_{1/m}(x) := 2^{mN}\rho(2^m x)$ for some non-negative, radially decreasing $\eta \in C_c^\infty(B_1)$ with $\int_{\mathbb{R}^N}\rho = 2$. Then when $1 \le p < \infty$ and $m \le p' = p/(p-1)$ (which is no restriction on $m$ if $p = 1$) we have $\|\nabla v_m\|_p \lesssim 2^{m/p'} \lesssim 1$, while $E_{1,-1/p}[v_m] \supseteq \{|x| \le 1-2^{-m}, 1+2^{-m} \le |y| \le 2\}$, and hence

$$\begin{aligned}\nu_{-1}(E_{1,-1/p}[v_m]) \ge& \int_{|x|\le 1-2^{-m}}\int_{1+2^{-m}\le|y|\le 2} |x-y|^{-1-N}\,\mathrm{d}x\,\mathrm{d}y\\ \ge& c_N\int_{|x|\le 1-2^{-m}} (1+2^{-m}-|x|)^{-1} - (2-|x|)^{-1}\,\mathrm{d}x \ge c_N' m.\end{aligned}$$

This proves both (i) and (ii) of Proposition 6.1 in the case $\gamma = -1$. □

### 6.2. *The case $-1 < \gamma < 0$: Examples of Cantor-Lebesgue type on the real line.*

We now discuss some examples related to self-similar Cantor sets of dimension $\beta = 1 + \gamma$. Recall the definition of $\nu_\gamma$, $Q_\gamma$ in (1.4), (1.5) and observe the behavior under dilations:

$$\nu_\gamma(tE) = t^{1+\gamma}\nu_\gamma(E)\,. \tag{6.4}$$

We have

**Lemma 6.2.** *Let $-1 < \gamma < 0$. There exist constants $c_\gamma > 0$, $C_\gamma > 0$, and a sequence of functions $g_m \in C^\infty(\mathbb{R})$ with $g_m(x) = 0$ for $x \le 0$ and $g_m(x) = 1$ for $x \ge 1$, such that for all $1 \le p < \infty$,*

$$\|g_m'\|_p \le c_\gamma 2^{\frac{m|\gamma|}{1+\gamma}(1-\frac{1}{p})} \tag{6.5}$$

*and if* $m-1 \le \frac{\gamma+1}{|\gamma|}\frac{p}{p-1}$, *then*

$$\nu_\gamma(\{(x,y)\in[0,1]^2 : |Q_{\gamma/p} g_m(x,y)| > \tfrac14\}) \ge m/C_\gamma\,. \tag{6.6}$$

*Proof.* For $-1<\gamma<0$ let

$$\rho = 2^{-\frac{1}{1+\gamma}} \tag{6.7}$$

so that $0<\rho<1/2$. We construct $g_m$ such that its derivative is supported on the $m$–th step of the construction of symmetric Cantor sets of dimension $\beta = 1+\gamma = \frac{\log 2}{\log(1/\rho)}$, with an equal variation on each of its $2^m$ components [14, ch. 8.1].

Let $g_0 \in C^\infty(\mathbb{R})$ be such that $0\le g_0\le 1$, $g_0(x)=0$ for $x\le\rho$ and $g_0(x)=1$ for $x\ge 1-\rho$. Set for $m\in\mathbb{N}$,

$$g_{m+1}(x) \coloneqq \tfrac12 g_m(\tfrac{x}{\rho}) + \tfrac12 g_m(1-\tfrac{1-x}{\rho}).$$

Since $\rho<1/2$, we have for $p\in[1,\infty)$, $\|g'_{m+1}\|^p_{L^p(\mathbb{R})} = 2\times(2\rho)^{-p}\rho\|g'_m\|^p_{L^p(\mathbb{R})}$ and thus

$$\|g'_m\|_{L^p(\mathbb{R})} = (2\rho)^{(\frac1p-1)m}\|g'_0\|_{L^p(\mathbb{R})} = 2^{\frac{m|\gamma|}{\gamma+1}(1-\frac1p)}\|g'_0\|_{L^p(\mathbb{R})}.$$

Fix now $1\le p<\infty$, and for $m\in\mathbb{N}$, $\lambda>0$ define

$$A_{m,\lambda} \coloneqq \nu_\gamma(\{(x,y)\in[0,1]^2 : |Q_{\gamma/p} g_m(x,y)|>\lambda\}).$$

Our goal is to estimate $A_{m,1/4}$, which we do by deriving a recursive estimate for $A_{m,\lambda}$. We have the decomposition

$$\begin{aligned} A_{m+1,\lambda} \ge\ & \nu_\gamma(\{(x,y)\in[0,\rho]^2 : |Q_{\gamma/p} g_{m+1}(x,y)|>\lambda\}) \\ &+ \nu_\gamma(\{(x,y)\in[1-\rho,1]^2 : |Q_{\gamma/p} g_{m+1}(x,y)|>\lambda\}) \\ &+ \nu_\gamma(\{(x,y)\in[0,\rho]\times[1-\rho,1] : |Q_{\gamma/p} g_{m+1}(x,y)|>\lambda\}). \end{aligned} \tag{6.8}$$

Using the definition of $g_{m+1}$, (6.7) and (6.4), we compute the first term in the right-hand side of (6.8) as

$$\begin{aligned} &\nu_\gamma(\{(x,y)\in[0,\rho]^2 : |Q_\gamma g_{m+1}(x,y)|>\lambda\}) \\ &= \nu_\gamma(\{(\rho w,\rho z) : (w,z)\in[0,1]^2,\ |Q_\gamma g_m(w,z)| > 2\rho^{1+\frac{\gamma}{p}}\lambda\}) \\ &= \rho^{\gamma+1}\nu_\gamma(\{(w,z)\in[0,1]^2 : |Q_\gamma g_m(w,z)| > 2^{\frac{|\gamma|}{p'(\gamma+1)}}\lambda\}) = \tfrac12 A_{m,s\lambda} \end{aligned} \tag{6.9}$$

where $s\coloneqq 2\rho^{1+\frac{\gamma}{p}} = 2^{\frac{|\gamma|}{p'(\gamma+1)}}$, and similarly the second term as

$$\nu_\gamma(\{(x,y)\in[1-\rho,1]^2 : |Q_\gamma g_{m+1}(x,y)|>\tfrac12\}) = \tfrac12 A_{m,s\lambda}. \tag{6.10}$$

Thus

$$A_{m+1,\lambda} \ge A_{m,s\lambda} + \nu_\gamma(\{(x,y)\in[0,\rho]\times[1-\rho,1] : |Q_{\gamma/p} g_{m+1}(x,y)|>\lambda\}),$$

which iterates to give

$$A_{m,1/4} \geq A_{0,s^m/4} + \sum_{j=1}^{m} \nu_\gamma(\{(x,y) \in [0,\rho] \times [1-\rho,1] : |Q_{\gamma/p} g_j(x,y)| > \tfrac{1}{4}s^{m-j}\}).$$

We drop the first term, and note that as long as $m-1 \leq \frac{\gamma+1}{|\gamma|}\frac{p}{p-1}$, we have $\frac{1}{4}s^{m-j} \leq \frac{1}{2}$ for all $j = 1, \ldots, m$. Moreover, for every $x \in [0,\rho^2] \times [1-\rho^2, 1]$ and every $j \geq 1$, we have $g_j(x) \leq 1/4$ and $g_j(y) \geq 3/4$, so $|Q_{\gamma/p} g_j(x,y)| > \frac{1}{2}$. Thus we obtain the desired conclusion

$$A_{m,1/4} \geq m\nu_\gamma([0,\rho^2] \times [1-\rho^2, 1]) = m/C_\gamma.$$

□

6.3. *Conclusion of the proof of Proposition 6.1.*

We continue with the case $-1 < \gamma < 0$. Let $\eta_1 \in C_c^\infty(\mathbb{R})$ supported in $(-1,2)$ such that $\eta_1(s) = 1$ on $(-1/2, 3/2)$ and $0 \leq \eta_1(s) \leq 1$ for all $s \in \mathbb{R}$.

We split $x = (x_1, x')$ with $x' \in \mathbb{R}^{N-1}$, where the variable $x'$ should simply be dropped in the case $N = 1$. Set $\eta(x) = \prod_{i=1}^N \eta_1(x_i)$ and define

$$u_m(x_1, x') = 16 g_m(x_1)\eta(x) \tag{6.11}$$

where $g_m$ is as in Lemma 6.2. Then $u_m \in C_c^\infty(\mathbb{R}^N)$, and if $1 \leq p < \infty$ and $m - 1 \leq \frac{\gamma+1}{|\gamma|}\frac{p}{p-1}$ we have $\|\nabla u_m\|_p \lesssim 1$. Both parts of Proposition 6.1 will follow, if we can prove that under the same hypotheses on $p$ and $m$, we have

$$\nu_\gamma(E_{1,\gamma/p}[u_m]) \geq c(N,\gamma)m - C(N,\gamma)^p. \tag{6.12}$$

We aim to reduce to the one-dimensional situation in Lemma 6.2 and split

$$Q_{\gamma/p} u_m(x,y) = 16\eta(x)\frac{g_m(x_1) - g_m(y_1)}{|x-y|^{1+\frac{\gamma}{p}}} + 16 g_m(y_1)\frac{\eta(x) - \eta(y)}{|x-y|^{1+\frac{\gamma}{p}}} = I_m(x,y) + II_m(x,y)$$

so that

$$\begin{aligned} \nu_\gamma(E_{1,\gamma/p}[u_m]) &\geq \iint\limits_{\substack{x_1, y_1 \in [0,1] \\ |I_m(x,y) + II_m(x,y)| > 1}} |x-y|^{\gamma - N} \,\mathrm{d}x\,\mathrm{d}y \\ &\geq \iint\limits_{\substack{x \in [0,1]^N, y_1 \in [0,1] \\ |x_1 - y_1| \geq |x' - y'| \\ |I_m(x,y)| > 2}} |x-y|^{\gamma-N} \,\mathrm{d}x\,\mathrm{d}y - \iint\limits_{|II_m(x,y)| > 1} |x-y|^{\gamma-N} \,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{6.13}$$

Clearly if $B_2$ is the ball in $\mathbb{R}^N$ of radius 2 centered at the origin then

$$|II_m(x,y)| \leq c_N |x-y|^{-\frac{\gamma}{p}} (\mathbb{1}_{B_2}(x) + \mathbb{1}_{B_2}(y))$$

and it follows immediately (since $-\gamma > 0$) that

$$\iint\limits_{|II_m(x,y)|>1} |x-y|^{\gamma-N}\,\mathrm{d}x\,\mathrm{d}y \le |\gamma|^{-1}C(N)^p.$$

For the first term in (6.13), we prove a lower bound and estimate by integrating in $y'$

$$\iint\limits_{\substack{x\in[0,1]^N,y_1\in[0,1]\\|x_1-y_1|\ge|x'-y'|\\|I_m(x,y)|>2}} |x-y|^{\gamma-N}\,\mathrm{d}x\,\mathrm{d}y \ge \iint\limits_{\substack{x\in[0,1]^N,y_1\in[0,1]\\|x_1-y_1|\ge|x'-y'|\\\frac{|16g_m(x_1)-16g_m(y_1)|}{|x_1-y_1|^{1+\frac{\gamma}{p}}}>4}} |x-y|^{\gamma-N}\,\mathrm{d}x\,\mathrm{d}y$$

$$\ge c_N \iint\limits_{\substack{x_1,y_1\in[0,1]\\|Q_{\gamma/p}g_m(x_1,y_1)|>\frac{1}{4}}} |x_1-y_1|^{\gamma-1}\,\mathrm{d}x_1\,\mathrm{d}y_1,$$

but by Lemma 6.2 the last expression is bounded below for large $m$ by $c_N m/C_\gamma$ under our hypothesis on $m$. This concludes the proof of (6.12). $\square$

For later purposes, note the inequality (6.13) (with $p = 1$) and the argument that follows proved also that for all sufficiently large $m > m(N, \gamma)$, we have

$$\nu_\gamma(E_{1,\gamma}[u_m] \cap ([0,1]\times\mathbb{R}^{N-1})^2) \ge c(N,\gamma)m. \tag{6.14}$$

6.4. *Examples related to Theorems 1.1 and 1.8*

We now consider the limit (1.8) in the range $-1 \le \gamma < 0$ and provide counterexamples for cases where $u$ is no longer required to be a $C_c^\infty$ function. The following proposition covers part (i) of Theorem 1.8.

**Proposition 6.3.** *Let $-1 \le \gamma < 0$. Let $s \mapsto \omega(s)$ be any decreasing function on $[0,\infty)$ with $\omega(0) \le 1$ and $\omega(s) > 0$ for all $s \ge 0$. Then there exists a $C^\infty$ function $u \in \dot W^{1,1}(\mathbb{R}^N)$ such that*

$$|u(x)| \le C\omega(|x|) \text{ for all } x \in \mathbb{R}^N \tag{6.15}$$

*and*

$$\lim_{\lambda \searrow 0} \lambda\nu_\gamma(E_{\lambda,\gamma}[u]) = \infty. \tag{6.16}$$

*Proof.* We consider the case $-1 < \gamma < 0$. Let $u_m \in C_c^\infty(\mathbb{R}^N)$ be as in (6.11) and define

$$f_m(x) = u_m(x_1 - 2, x') \tag{6.17}$$

so that $f_m(x) = 0$ if $x_1 \notin [1,4]$. Let, for $n \in \mathbb{N}$,

$$R_n = 2^{2n}, \;\; \lambda_n = R_n^{-(N+\gamma)}\omega(R_{n+1}), \;\; m(n) \geq 4\frac{\lambda_n}{\lambda_{n+1}}\omega(R_{n+1})^{-1}n^3. \tag{6.18}$$

We also assume $m(n) > m(N,\gamma)$ so that by (6.14) in Section 6.3,

$$\nu_\gamma(\{(x,y) : x_1, y_1 \in [2,3], |Q_\gamma f_{m(n)}(x,y)| > 1\}) \geq c(N,\gamma)m(n) \tag{6.19}$$

for all $n \in \mathbb{N}$. Finally let

$$u(x) = \sum_{n=2}^{\infty} \frac{\omega(R_{n+1})}{R_n^{N-1}n^2} f_{m(n)}\Big(\frac{x}{R_n}\Big). \tag{6.20}$$

Since $\|f_m\|_{\dot{W}^{1,1}} \leq C$, and $\omega$ is bounded, it is easy to see that the sum converges in $\dot{W}^{1,1}(\mathbb{R}^N)$, and that $u$ is in $\dot{W}^{1,1}(\mathbb{R}^N)$. Also, the supports of $f_{m(n)}(R_n^{-1}\cdot)$, namely $[R_n, 4R_n] \times [-4R_n, 4R_n]^{N-1}$, are disjoint as $n$ varies, so clearly $u \in C^\infty(\mathbb{R}^N)$. Since $\|f_m\|_{L^\infty} \leq C$, we have

$$|u(x)| \leq \omega(R_{n+1})R_n^{-(N-1)}n^{-2} \quad \text{for } |x| \geq R_n,$$

so $|u(x)| \leq C'|x|^{-N+1}\omega(|x|)$ for $|x| \geq 2$. In particular $|u(x)| \leq C\omega(|x|)$.

For $\lambda \in ((n+1)^{-2}\lambda_{n+1}, n^{-2}\lambda_n]$ we estimate

$$\lambda\nu_\gamma(E_{\lambda,\gamma}[u]) \geq (n+1)^{-2}\lambda_{n+1}\nu_\gamma(E_{n^{-2}\lambda_n,\gamma}[u]) \geq \frac{\lambda_{n+1}}{4\lambda_n}\, n^{-2}\lambda_n\nu_\gamma(\mathcal{E}_n)$$

where $\mathcal{E}_n := E_{n^{-2}\lambda_n,\gamma}[u] \cap ([2R_n, 3R_n] \times \mathbb{R}^{N-1})^2$. Moreover, for $(x,y) \in ([2R_n, 3R_n] \times \mathbb{R}^{N-1})^2$, we have

$$u(x) - u(y) = R_n^{1-N}n^{-2}\omega(R_{n+1})(f_{m(n)}(R_n^{-1}x) - f_{m(n)}(R_n^{-1}y))$$

so

$$|Q_\gamma u(x,y)| > n^{-2}\lambda_n \iff \frac{|f_{m(n)}(R_n^{-1}x) - f_{m(n)}(R_n^{-1}y)|}{|R_n^{-1}x - R_n^{-1}y|^{1+\gamma}} > \frac{R_n^{N+\gamma}}{\omega(R_{n+1})}\lambda_n = 1$$

where the last equality follows from (6.18). Hence rescaling using (6.4) yields

$$\begin{aligned} n^{-2}\lambda_n\nu_\gamma(\mathcal{E}_n) &= n^{-2}\lambda_n R_n^{\gamma+N}\nu_\gamma(\{(x,y) : x_1, y_1 \in [2,3], |Q_\gamma f_{m(n)}(x,y)| > 1\}) \\ &\geq c(N,\gamma)m(n)\omega(R_{n+1})n^{-2} \end{aligned} \tag{6.21}$$

with $c(N,\gamma) > 0$, by (6.19). Thus we have shown

$$\inf_{\lambda \in ((n+1)^{-2}\lambda_{n+1}, n^{-2}\lambda_n]} \lambda\nu_\gamma(E_{\lambda,\gamma}[u]) \geq c(N,\gamma)\frac{\lambda_{n+1}}{4\lambda_n}\omega(R_{n+1})m(n)n^{-2} \geq c(N,\gamma)n$$

where for the last inequality we have used our assumption (6.18) on $m(n)$. The assertion follows for $-1 < \gamma < 0$.

Finally consider the case $\gamma = -1$. We now choose $v_m$ as in (6.3) and

$$R_n = 2^{2n}, \;\; \lambda_n = R_n^{-(N-1)}\omega(R_{n+1}), \;\; m(n) \geq 4\frac{\lambda_n}{\lambda_{n+1}}\frac{n^3}{\omega(R_{n+1})}. \tag{6.22}$$

In analogy to (6.20) we now use

$$u(x)=\sum_{n=2}^{\infty}\frac{\omega(R_{n+1})}{R_n^{N-1}n^2}v_{m(n)}(\frac{x}{R_n}) \tag{6.23}$$

Since $\omega$ is bounded it is immediate that $u\in \dot W^{1,1}(\mathbb{R}^N)$ and also that $|u(x)|\lesssim\omega(|x|)$. We need to check that $\lambda\nu_{-1}(E_{\lambda,-1}[u])\to\infty$ as $\lambda\to 0^+$. If $|x|\le R_n(1-2^{m(n)})$ and $|y|\ge R_n(1+2^{m(n)})$, then

$$u(x)-u(y)\ge\frac{\omega(R_{n+1})}{R_n^{N-1}n^2}v_{m(n)}(\frac{x}{R_n})=2\frac{\omega(R_{n+1})}{R_n^{N-1}n^2}=2n^{-2}\lambda_n>n^{-2}\lambda_n$$

so $(x,y)\in E_{n^{-2}\lambda_n,-1}[u]$. Hence we get

$$\begin{aligned}n^{-2}\lambda_n\nu_{-1}(E_{n^{-2}\lambda_n,-1}[u])&\ge n^{-2}\lambda_n\iint_{\substack{|x|\le R_n(1-2^{m(n)})\\|y|\ge R_n(1+2^{m(n)})}}|x-y|^{-1-N}\,\mathrm{d}x\,\mathrm{d}y\\&\ge n^{-2}\lambda_nR_n^{N-1}\iint_{\substack{|x|\le 1-2^{m(n)}\\|y|\ge 1+2^{m(n)}}}|x-y|^{-1-N}\,\mathrm{d}x\,\mathrm{d}y\\&\ge c_Nm(n)\omega(R_{n+1})n^{-2}\end{aligned}$$

(using (6.22) in the last inequality). This together with our assumption on $m(n)$ imply that $\inf_{\lambda\in((n+1)^{-2}\lambda_{n+1},n^{-2}\lambda_n]}\lambda\nu_{-1}(E_{\lambda,-1}[u])\ge c_Nn\to\infty$ when $n\to\infty$, as desired. □

The next proposition is relevant for part (ii) of Theorem 1.8.

**Proposition 6.4.** *Suppose $-1\le\gamma<0$. Then there exists a compactly supported $u\in W^{1,1}(\mathbb{R}^N)$ such that $u$ is $C^\infty$ for $x\ne 0$,*

$$|u(x)|\le\frac{C}{|x|^{N-1}[\log(2+|x|^{-1})]^2} \tag{6.24}$$

*and*

$$\lim_{\lambda\searrow 0}\lambda\nu_\gamma(E_{\lambda,\gamma}[u])=\infty. \tag{6.25}$$

*If in addition $N\ge 2$ or $-1<\gamma<0$ there exists $u$ with the above properties and*

$$\nu_\gamma(E_{\lambda,\gamma}[u])=\infty \text{ for all } \lambda>0. \tag{6.26}$$

*Proof.* Consider first the case $-1<\gamma<0$. We choose for $n\in\mathbb{N}$

$$R_n=2^{-2n},\ m(n)\ge 2^{2^n}. \tag{6.27}$$

and with these choices of $R_n$ and $m(n)$ and $f_m$ as in (6.17) and (6.11) we define again

$$u(x)=\sum_{n=2}^{\infty}\frac{1}{n^2R_n^{N-1}}f_{m(n)}(\frac{x}{R_n}).$$

The sum converges in $W^{1,1}$ to a function supported in $[-4,4]^N$. We have $|u(x)| \le C2^{2n(N-1)}n^{-2}$ for $0 < x_1 \le 2^{-2n}$; moreover $|x'| \lesssim |x_1|$ on the support of $u$. This implies $|u(x)| \le C'[|x|^{1-N}\log(1/|x|)]^{-2}$ for small $x$. Also, because of the choices of $R_n$ we see that $u$ is smooth away from 0.

Fix $\lambda > 0$. Since $\lim_{n\to\infty} R_n^{N+\gamma} n^2 = 0$ we may choose $n_0$ such that

$$\lambda R_n^{N+\gamma} n^2 \le 1, \quad \forall n \ge n_0. \tag{6.28}$$

Now $\nu_\gamma(E_{\lambda,\gamma}[u]) \ge \nu_\gamma(E_{\lambda,\gamma}[u] \cap ([2R_n, 3R_n] \times \mathbb{R}^{N-1})^2)$, and again $f_{m(n)}(R_n^{-1}\cdot)$ is supported in $\mathcal{R}(n) = [R_n, 4R_n] \times [-4R_n, 4R_n]^{N-1}$. Hence by the same rescaling argument as in (6.21), we obtain

$$\nu_\gamma(E_{\lambda,\gamma}[u]) \ge R_n^{N+\gamma} \nu_\gamma(\{(x,y)\colon x_1, y_1 \in [2,3], |Q_\gamma f_{m(n)}(x,y)| > \lambda R_n^{N+\gamma} n^2\}).$$

If $n \ge n_0$ then this gives

$$\nu_\gamma(E_{\lambda,\gamma}[u]) \ge R_n^{N+\gamma} \nu_\gamma(\{(x,y)\colon x_1, y_1 \in [2,3], |Q_\gamma f_{m(n)}(x,y)| > 1\}) \ge c(N,\gamma) m(n) R_n^{N+\gamma}$$

by (6.19). Since $\lim_{n\to\infty} m(n) R_n^{N+\gamma} = \infty$ by (6.27) we conclude $\nu_\gamma(E_{\lambda,\gamma}[u]) = \infty$.

For the case $\gamma = -1$ and $N \ge 2$, define $u$ as in (6.23) but with the choice of the parameters $R_n$, $m(n)$ as in (6.27) to obtain a compactly supported $u \in W^{1,1}$ satisfying (6.24). We now fix $\lambda > 0$ and note that when $N \ge 2$ we have $\lambda R_n^{N-1} n^2 \to 0$ as $n \to \infty$. The above calculation gives $\nu_{-1}(E_{\lambda,-1}[u]) \ge c(N) m(n) R_n^{N-1}$ provided that $\lambda R_n^{N-1} n^2 \le 1$ and thus the conclusion $\nu_{-1}(E_{\lambda,-1}[u]) = \infty$.

Finally, clearly (6.25) follows from (6.26), and the latter was proved if $-1 < \gamma < 0$ or $N \ge 2$. It remains to consider the case $N = 1$, $\gamma = -1$. We define $u$ as in the previous paragraph. The above calculation shows that $\nu_{-1}(E_{\lambda,-1}[u]) \ge c m(n)$ provided that $\lambda < 1/n^2$ which establishes (6.25) in this last case. $\square$

The case $N = 1$, $\gamma = -1$ plays a special role. The following lemma shows that the conclusion (6.26) in Proposition 6.4 fails in this case.

**Lemma 6.5.** *Let $u \in \dot{W}^{1,1}(\mathbb{R})$ be compactly supported. Then $\nu_{-1}(E_{\lambda,-1}[u]) < \infty$ for all $\lambda > 0$.*

*Proof.* Let $u \in \dot{W}^{1,1}(\mathbb{R})$ be compactly supported in $[-R, R]$. Then given any $\lambda \in (0,1)$, there exists $\delta(\lambda) > 0$ such that $\int_I |u'| \le \lambda/2$ for every interval $I \subset \mathbb{R}$ with length $\le \delta(\lambda)$. As a result, $u$ is uniformly continuous on $\mathbb{R}$, with $\sup_{x\in\mathbb{R}} |u(x+h) - u(x)| \le \lambda/2$ for $|h| \le \delta(\lambda)$.

Thus

$$\nu_{-1}(E_{\lambda,-1}[u]) = 2\int_{-\infty}^{\infty} \int\limits_{\substack{h>0\\ |u(x+h)-u(x)|>\lambda}} \frac{\mathrm{d}h}{h^2}\,\mathrm{d}x$$
$$\leq \int_{-2R}^{2R}\int_{\delta(\lambda)}^{\infty} \frac{\mathrm{d}h}{h^2}\,\mathrm{d}x + \int_{\mathbb{R}\setminus[-2R,2R]}\int_{|x|-R}^{|x|+R} \frac{\mathrm{d}h}{h^2}\,\mathrm{d}x \leq 4R(\delta(\lambda))^{-1} + 4. \qquad \square$$

6.5. *Generic failure in $W^{1,1}$, for the case $-1 \leq \gamma < 0$.*

**Proposition 6.6.** *Let $-1 \leq \gamma < 0$, $N \geq 2$ or $-1 < \gamma < 0$, $N \geq 1$. Let*

$$\mathcal{V} = \{f \in W^{1,1}(\mathbb{R}^N) : \ \nu_\gamma(E_{\lambda,\gamma}[f]) < \infty \text{ for some } \lambda > 0.\} \tag{6.29}$$

*Then $\mathcal{V}$ is of first category in $W^{1,1}(\mathbb{R}^N)$, in the sense of Baire.*

Let

$$\begin{aligned} U_k &= \{(x,y) \in \mathbb{R}^{2N} : 2^{k-1} \leq |x-y| \leq 2^k\}, \\ \Omega_\ell &= \textstyle\bigcup_{k=1-\ell}^{\ell} U_k. \end{aligned} \tag{6.30}$$

For the proof of Proposition 6.6 we use an elementary estimate for the intersections $E_{\lambda,\gamma}[u] \cap \Omega_\ell$.

**Lemma 6.7.** *For all $\gamma \in \mathbb{R}$, $u \in W^{1,1}(\mathbb{R}^N)$, $\ell > 0$ and $\Omega_\ell$ as in* (6.30)*,*

$$\sup_{\lambda>0} \lambda\nu_\gamma(E_{\lambda,\gamma}[u] \cap \Omega_\ell) \leq C(N,\gamma)\ell\|\nabla u\|_1$$

*Proof.* For $u \in C^1$ we use the Lusin-Lipschitz inequality (2.2) to see that

$$\lambda \iint_{E_{\lambda,\gamma}[u]\cap U_k} |x-y|^{\gamma-N}\,\mathrm{d}x\,\mathrm{d}y$$
$$\leq C(\gamma)\lambda 2^{k\gamma}\mathcal{L}^N\{x \in \mathbb{R}^N : M(|\nabla u|)(x) > c2^{k\gamma}\lambda\} \leq C(N,\gamma)\|\nabla u\|_1$$

by the Hardy-Littlewood maximal inequality. Now sum in $1-\ell \leq k \leq \ell$. The extension to general $u \in W^{1,1}$ is obtained as in the limiting argument of Section 2.3. $\square$

*Proof of Proposition 6.6.* Let, for $m \in \mathbb{N}$ and $j \in \mathbb{Z}$

$$\mathcal{V}(m,j) = \{u \in W^{1,1}(\mathbb{R}^N) : \nu_\gamma(E_{\lambda,\gamma}[u]) \leq m \text{ for all } \lambda > 2^j\}.$$

Since $\lambda \mapsto \nu_\gamma(E_{\lambda,\gamma}[u])$ is decreasing we see that $\mathcal{V}$ is contained in $\bigcup_{m\geq1}\bigcup_{j\in\mathbb{Z}}\mathcal{V}(m,j)$. To show that $\mathcal{V}$ is of first category in $W^{1,1}(\mathbb{R}^N)$, we need to show that for every $m \in \mathbb{N}$, $j \in \mathbb{Z}$ the set $\mathcal{V}(m,j)$ is nowhere dense.

We first show that $\mathcal{V}(m,j)$ is closed in $W^{1,1}(\mathbb{R}^N)$. Let $u_n \in \mathcal{V}(m,j)$ and $u \in W^{1,1}(\mathbb{R}^N)$ such that $\lim_{n\to\infty}\|u-u_n\|_{W^{1,1}(\mathbb{R}^N)} = 0$. It suffices to show that given $\varepsilon > 0$ we have

$\nu_\gamma(E_{\lambda,\gamma}[u]) \le m+\varepsilon$ for all $\lambda > 2^j$. By the monotone convergence theorem, we have $\lim_{\ell\to\infty} \nu_\gamma(E_{\lambda,\gamma}[u]\cap\Omega_\ell) = \nu_\gamma(E_{\lambda,\gamma}[u])$, and it suffices to verify that

$$\nu_\gamma(E_{\lambda,\gamma}[u]\cap\Omega_\ell) \le m+\varepsilon \text{ for } \lambda > 2^j, \tag{6.31}$$

for all $\ell\in\mathbb{N}$. Now let $\delta>0$ such that $(1-\delta)\lambda > 2^j$. Then

$$\nu_\gamma(E_{\lambda,\gamma}[u]\cap\Omega_\ell) \le \nu_\gamma(E_{(1-\delta)\lambda,\gamma}[u_n]\cap\Omega_\ell) + \nu_\gamma(E_{\delta\lambda,\gamma}[u-u_n]\cap\Omega_\ell)$$

and using that $u_n\in\mathcal{V}(m,j)$ together with $(1-\delta)\lambda > 2^j$, and Lemma 6.7, we see that for $\lambda > 2^j$

$$\nu_\gamma(E_{\lambda,\gamma}[u]\cap\Omega_\ell) \le m + C(N,\gamma)\ell\frac{1+\delta}{\delta 2^j}\|\nabla(u_n-u)\|_1.$$

Since $\delta>0$ was arbitrary and since $\|\nabla(u_n-u)\|_{L^1(\mathbb{R}^N)}\to 0$ by assumption we obtain (6.31).

To show that the closed set $\mathcal{V}(m,j)$ is nowhere dense when $-1\le\gamma<0$ we need to verify that for every $u\in\mathcal{V}(m,j)$ and $\varepsilon_1>0$ there exists $f\in W^{1,1}(\mathbb{R}^N)$ such that $\|f-u\|_{W^{1,1}(\mathbb{R}^N)}<\varepsilon_1$ and $f\notin\mathcal{V}(m,j)$. To see this we use Proposition 6.4 according to which there exists a compactly supported $W^{1,1}$ function $f_0$ for which $\nu_\gamma(E_{\lambda,\gamma}[f_0])=\infty$ for all $\lambda>0$. It is then clear that $f = u+\frac{\varepsilon_1}{2}\frac{f_0}{\|f_0\|_{W^{1,1}}}$ satisfies $\|f-u\|_{W^{1,1}}\le\varepsilon_1/2$ and also,

$$\nu_\gamma(E_{\lambda,\gamma}[f]) \ge \nu_\gamma(E_{2\lambda,\gamma}[\tfrac{\varepsilon_1}{2}\tfrac{f_0}{\|f_0\|_{W^{1,1}}}]) - \nu_\gamma(E_{\lambda,\gamma}[u]) = \infty$$

for every $\lambda>2^j$, for all $j\in\mathbb{Z}$. The proposition is proved. □

To include a result of generic failure of the limiting relation in the case $N=1$, $\gamma=-1$ we give

**Proposition 6.8.** *Let $-1\le\gamma<0$. Let*

$$\mathcal{W} = \{f\in W^{1,1}(\mathbb{R}):\ \limsup_{R\to0}\sup_{\lambda>R} R\nu_\gamma(E_{\lambda,\gamma}[f])<\infty\}.$$

*Then $\mathcal{W}$ is of first category in $W^{1,1}$, in the sense of Baire.*

*Proof.* Clearly $\mathcal{W}\subset\mathcal{V}$ where $\mathcal{V}$ is defined in (6.29). We define

$$\mathcal{W}(m,j) = \{u\in W^{1,1}(\mathbb{R}):\ \sup_{0<R\le 2^{-j}}\sup_{\lambda>R} R\nu_\gamma(E_{\lambda,\gamma}[u])\le m\}$$

and note that

$$\mathcal{W}\subset \cup_{j\ge1}\cup_{m\ge1}\mathcal{W}(m,j) \tag{6.32}$$

The arguments in the proof of Proposition 6.6 that was used to show that the sets $\mathcal{V}(m,j)$ are closed in $W^{1,1}(\mathbb{R}^N)$ also show that the sets $\mathcal{W}(m,j)$ are closed in $W^{1,1}(\mathbb{R})$.

Let $u\in\mathcal{W}(m,j)$, and let $\varepsilon_1>0$. By Proposition 6.4 there is $f_0\in W^{1,1}(\mathbb{R})$ such that $\lim_{\lambda\searrow0}\lambda\nu_\gamma(E_{\lambda,\gamma}[f_0])=\infty$. We may normalize so that $\|f_0\|_{W^{1,1}(\mathbb{R})}=1$. Pick $R\in(0,2^{-j}]$ so that $\lambda\nu_\gamma(E_{\lambda,\gamma}[f_0]) > 16m/\varepsilon_1$ for $\lambda \le 8R/\varepsilon_1$. Let $f = u + (\varepsilon_1/2)f_0$ so that $\|f-u\|_{W^{1,1}(\mathbb{R})} \le \varepsilon_1/2$. Moreover if $\lambda = 2R$, then $\lambda > R$ and

$$
\begin{aligned}
&R\nu_\gamma(E_{\lambda,\gamma}[f]) \ge R\nu_\gamma(E_{2\lambda,\gamma}[\tfrac{\varepsilon_1}{2}f_0]) - R\nu_\gamma(E_{\lambda,\gamma}[u])\\
&= \tfrac{\varepsilon_1}{8}\tfrac{8R}{\varepsilon_1}\nu_\gamma(E_{8R/\varepsilon_1,\gamma}[f_0]) - R\nu_\gamma(E_{\lambda,\gamma}[u]) > \tfrac{\varepsilon_1}{8}\tfrac{16m}{\varepsilon_1} - m = m
\end{aligned}
$$

and we see that $f \notin \mathcal{W}(m,j)$. Thus we have shown that $\mathcal{W}(m,j)$ is nowhere dense in $W^{1,1}(\mathbb{R})$. By (6.32) the proof is concluded. $\square$

## 7. Perspectives and open problems

### 7.1. *Subspaces of $\dot{W}^{1,1}$ and $\dot{BV}$ and related spaces*

The failure of the upper bounds for $[Q_\gamma u]_{L^{1,\infty}(\mathbb{R}^{2N},\nu_\gamma)}$ for $\gamma \in [-1,0)$ raises a number of interesting questions. Consider the space $\dot{BV}(\gamma)$ consisting of all $\dot{BV}$ functions satisfying

$$
\|u\|_{\dot{BV}(\gamma)} := \|\nabla u\|_{\mathcal{M}} + \sup_{\lambda>0} \lambda\nu_\gamma(E_{\lambda,\gamma}[u]) < \infty \tag{7.1}
$$

and the corresponding subspace $\dot{W}^{1,1}(\gamma)$ of $\dot{W}^{1,1}$.

*Embeddings.* We proved in this paper that for $\gamma \notin [-1,0]$ we have $\dot{BV}(\gamma) = \dot{BV}$ and $\dot{W}^{1,1}(\gamma) = \dot{W}^{1,1}$. It is natural to ask how in the range $-1 \le \gamma < 0$ the proper subspaces $\dot{BV}(\gamma)$ and $\dot{W}^{1,1}(\gamma)$ relate to other families of function spaces, in particular to the Hardy-Sobolev space $\dot{F}^1_{1,2}$, another subspace of $\dot{W}^{1,1}$.

*Triangle inequalities.* The spaces $\dot{W}^{1,1}(\gamma)$ and $\dot{BV}(\gamma)$ are defined via $L^{1,\infty}$ -quasi-norms, and the space $L^{1,\infty}$ is not normable (unlike $L^{p,\infty}$ for $1 < p < \infty$ which is normable [13]). However Theorem 1.4 tells us that $\dot{W}^{1,1}(\gamma)$ and $\dot{BV}(\gamma)$ are normable for $\gamma \notin [-1,0]$. Are these spaces normable in the range $\gamma \in [-1,0)$?

*Related quasi-norms.* Consider for $0 < s \le 1$

$$
\|u\|_{(p,s,\gamma)} = \Big[\frac{u(x)-u(y)}{|x-y|^{\frac{\gamma}{p}+s}}\Big]_{L^{p,\infty}(\mathbb{R}^{2N},\nu_\gamma)}.
$$

It is an obvious consequence of Theorem 1.3 that for $s = 1$ and fixed $p > 1$, these expressions define equivalent (semi/quasi)-norms on $C_c^\infty$ as $\gamma$ varies over $\mathbb{R}\setminus\{0\}$. It would be interesting to find a more direct proof of this observation which does not involve the relation with $\dot{W}^{1,p}$. We note that the equivalence for varying $\gamma$ breaks down for $0 < s < 1$. This result, and more about the spaces for which $\|u\|_{(p,s,\gamma)} < \infty$ with $0 < s < 1$, such as their connection to Besov spaces and interpolation, can be found in [20].

7.2. *Other limit functionals*

Our results, combined with the various developments presented in [5, 6, 16, 18], suggest several possible directions of research.

Can one prove a generalization of (1.14), (1.16) where the supremum is replaced by the $\liminf_{\lambda\to\infty}$ when $\gamma > 0$ and by a $\liminf_{\lambda\to 0^+}$ when $\gamma < 0$. More precisely, for $1 < p < \infty$ is there a positive constant $C(N, \gamma, p)$ such that for all $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$

$$\|\nabla u\|^p_{L^p} \le C(N,\gamma,p) \liminf_{\lambda\to\infty} \lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u]) \text{ if } \gamma > 0, \tag{7.2a}$$

$$\|\nabla u\|^p_{L^p} \le C(N,\gamma,p) \liminf_{\lambda\searrow 0} \lambda^p \nu_\gamma(E_{\lambda,\gamma.p}[u]) \text{ if } \gamma < 0, \tag{7.2b}$$

in the sense that $\|\nabla u\|_{L^p} = \infty$ if $u \in L^1_{\mathrm{loc}} \setminus \dot W^{1,p}$?

For $p = 1$ we can also ask: Is there a positive constant $C(N, \gamma)$ such that for all $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$,

$$\|\nabla u\|_{\mathcal{M}} \le C(N,\gamma) \liminf_{\lambda\to\infty} \lambda \nu_\gamma(E_{\lambda,\gamma}[u]) \text{ if } \gamma > 0, \tag{7.3a}$$

$$\|\nabla u\|_{\mathcal{M}} \le C(N,\gamma) \liminf_{\lambda\searrow 0} \lambda \nu_\gamma(E_{\lambda,\gamma}[u]) \text{ if } \gamma < 0, \tag{7.3b}$$

in the sense that $\|\nabla u\|_{\mathcal{M}} = \infty$ if $u \in L^1_{\mathrm{loc}} \setminus \dot{BV}$?

Theorem 1.1 gives (7.2a) and (7.2b) if we additionally assume $u \in \dot W^{1,p}(\mathbb{R}^N)$. It also gives (7.3a) and (7.3b) if we additionally assume that $u \in \dot W^{1,1}(\mathbb{R}^N)$. It would already be interesting to establish (7.3a), (7.3b) for all $\dot{BV}$ functions.

When $\gamma = -1$, $p = 1$, (7.3b) holds for all $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ as established in Nguyen [17, Theorem 2] and Brezis–Nguyen [5, Section 3.4]. For $\gamma = -p$, $1 < p < \infty$ inequality (7.2b) was proved in Bourgain–Nguyen [2]. For $\gamma = N$, Poliakovsky [19] proved weaker versions of (7.2a) and (7.3a) where the lim inf is replaced by a lim sup.

7.3. $\Gamma$*–convergence*

This is a far-reaching generalization of the questions raised in Section 7.2. For fixed $p \ge 1$ and $\gamma \in \mathbb{R} \setminus \{0\}$ consider the functionals

$$\Phi_\lambda[u] := \lambda^p \nu_\gamma(E_{\lambda,\gamma/p}[u]), \quad \lambda \in (0,\infty)$$

defined for all $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. It would be very interesting to study the $\Gamma$-limit of $\Phi_\lambda$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$, in the sense of De Giorgi, as $\lambda \to \infty$ when $\gamma > 0$, resp. as $\lambda \searrow 0$ when $\gamma < 0$. More specifically, if $p > 1$ define on $L^1_{\mathrm{loc}}(\mathbb{R}^N)$,

$$\Phi_{*,c}[u] = \begin{cases} c\|\nabla u\|^p_{L^p} & \text{if } u \in \dot W^{1,p}(\mathbb{R}^N) \\ \infty & \text{otherwise,} \end{cases}$$

and for $p = 1$ define

$$\Phi_{*,c}[u] = \begin{cases} c\|\nabla u\|_{\mathcal{M}} & \text{if } u \in \dot{BV}(\mathbb{R}^N) \\ \infty & \text{otherwise.} \end{cases}$$

A challenging question is whether there exists a constant $c = c(p, \gamma, N) > 0$ such that $\Phi_\lambda \to \Phi_{*,c}$ in the sense of $\Gamma$-convergence, meaning

(1) whenever $u_\lambda \to u$ in $L^1_{\mathrm{loc}}$ then $\liminf \Phi_\lambda[u_\lambda] \geq \Phi_{*,c}[u]$, and
(2) for each $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ there exist $(v_\lambda)$ with $v_\lambda \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, $v_\lambda \to u$ in $L^1_{\mathrm{loc}}$ and $\limsup \Phi_\lambda[v_\lambda] \leq \Phi_{*,c}[u]$.

This question is especially meaningful in the case $p = 1$ where the pointwise limit behaves somewhat pathologically. Indeed, recall that for $p = 1$, $-1 \leq \gamma < 0$ there is no universal upper bound for $\Phi_\lambda[u]$ in terms of $\|\nabla u\|_{L^1}$. Also when $p = 1$ and $\gamma \in \mathbb{R} \setminus [-1, 0]$ the examples in Section 3.6 show that the pointwise limit in $\dot{W}^{1,1}$ and on $\dot{BV} \setminus \dot{W}^{1,1}$ may differ (by a multiplicative constant). A remarkable result of Nguyen [16,18] states that $\Phi_\lambda \to \Phi_{*,c}$ as $\lambda \to 0$, in the sense of $\Gamma$–convergence, when $p \geq 1$, and $\gamma = -p$ for some appropriate constant $c = c(p, N)$; see also Brezis–Nguyen [6] (note however that $\dot{W}^{1,p}$ and $\dot{BV}$ are replaced in these papers by $W^{1,p}$ and $BV$).

### 7.4. *More general families of functionals*

Consider a monotone nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ and set (inspired by [5,6])

$$\Psi_\lambda[u] := \lambda^p \iint_{\mathbb{R}^N \times \mathbb{R}^N} \varphi\Big(\frac{|u(x) - u(y)|}{\lambda |x - y|^{1+\gamma/p}}\Big) |x - y|^{\gamma - N} \,\mathrm{d}x\,\mathrm{d}y.$$

The family $\Phi_\lambda$ in Section 7.3 corresponds to $\varphi = \mathbb{1}_{(1,\infty)}$. It is an interesting generalization of the above problems to study the limit of $\Psi_\lambda$ as $\lambda \searrow 0$ when $\gamma < 0$ and the limit of $\Psi_\lambda$ as $\lambda \to \infty$ when $\gamma > 0$, both in the sense of pointwise convergence or in the sense of $\Gamma$-convergence. A formal computation suggests that our Theorem 1.1 should go over modulo a factor $\int_0^\infty \frac{\varphi(s)}{s^{p+1}}\,\mathrm{d}s$ (see [6]). We refer to [5] for a further discussion of applications.

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