A surprising formula for Sobolev norms

Haïm Brezis[a,b,c,d,1,2], **Jean Van Schaftingen**[e,1], and **Po-Lam Yung**[f,g,1]

[a]Department of Mathematics, Rutgers University, Piscataway, NJ 08854; [b]Department of Mathematics, Technion, Israel Institute of Technology, 32000 Haifa, Israel; [c]Department of Computer Science, Technion, Israel Institute of Technology, 32000 Haifa, Israel; [d]Laboratoire Jacques-Louis Lions, Sorbonne Université, 75005 Paris, France; [e]Institut de Recherche en Mathématique et Physique, Université Catholique de Louvain, 1348 Louvain-la-Neuve, Belgium; [f]Mathematical Sciences Institute, Australian National University, Canberra ACT 2601, Australia; and [g]Department of Mathematics, The Chinese University of Hong Kong, Ma Liu Shui, Hong Kong

Contributed by Haïm Brezis, December 31, 2020 (sent for review December 8, 2020; reviewed by Lawrence C. Evans, Alessio Figalli, and Fang-Hua Lin)

We establish the equivalence between the Sobolev seminorm $\|\nabla u\|_{L^p}$ and a quantity obtained when replacing strong L^p by weak L^p in the Gagliardo seminorm $|u|_{W^{s,p}}$ computed at $s=1$. As corollaries we derive alternative estimates in some exceptional cases (involving $W^{1,1}$) where the "anticipated" fractional Sobolev and Gagliardo–Nirenberg inequalities fail.

fractional Sobolev space | Marcinkiewicz space | fractional Gagliardo–Nirenberg interpolation inequalities

Fractional Sobolev spaces $W^{s,p}$ (also called Slobodeskii spaces) play a major role in many questions involving partial differential equations. On $\mathbb{R}^N$, $N\geq 1$, they are associated with the Gagliardo seminorm

$$|u|^p_{W^{s,p}} := \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y \qquad [1.1]$$

where $0<s<1$ and $1\leq p<\infty$. A well-known "drawback" of the Gagliardo seminorm is that one does not recover the Sobolev seminorm $\|\nabla u\|^p_{L^p}$ if one takes $s=1$ in [1.1]. In fact, for every $1\leq p<\infty$ and every measurable function u,

$$\left\|\frac{u(x)-u(y)}{|x-y|^{\frac{N}{p}+1}}\right\|^p_{L^p(\mathbb{R}^N\times\mathbb{R}^N)} = \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+p}}\,\mathrm{d}x\,\mathrm{d}y = \infty \qquad [1.2]$$

unless u is a constant (see refs. 1 and 2).

One way to recover $\|\nabla u\|^p_{L^p}$ out of the Gagliardo seminorms is to consider the quantity $(1-s)|u|^p_{W^{s,p}}$ with $0<s<1$ and show that it converges as $s\nearrow 1$ to a multiple of $\|\nabla u\|^p_{L^p}$. This is a special case of the Bourgain–Brezis–Mironescu (BBM) formula in ref. 3 (see also refs. 2, 4, 5), which furthermore enters (when $p=1$ and u is a characteristic function) in the study of "nonlocal minimal surfaces" and "s perimeters" (see refs. 6–8).

The primary goal of this paper is to propose an alternative route to repair this "defect," simply replacing the L^p norm $\|\cdot\|_{L^p}$ in [1.2] by the Marcinkiewicz M^p (i.e., weak L^p) quasi-norm $[\cdot]_{M^p}$. The central result of this paper is *Theorem 1.1*.

In a seemingly different direction, it is well known that in some exceptional cases the "anticipated" fractional Sobolev-type and Gagliardo–Nirenberg-type estimates may fail (in particular when they involve $\|\nabla u\|_{L^1}$). A second goal of this paper is to discuss a partial list of such failures (for a complete list see refs. 9 and 10) and to present alternative (weaker) estimates where strong L^p is replaced by weak L^p; see *Corollaries 4.1*, *5.1*, and *5.2*. They can all be derived as immediate consequences of *Theorem 1.1* applied with $p=1$.

Our main result is the following:

Theorem 1.1. *For every $N\geq 1$, there exist constants $c=c(N)>0$ and $C=C(N)$ such that*

$$c^p\|\nabla u\|^p_{L^p(\mathbb{R}^N)} \leq \left[\frac{u(x)-u(y)}{|x-y|^{\frac{N}{p}+1}}\right]^p_{M^p(\mathbb{R}^N\times\mathbb{R}^N)} \leq C\|\nabla u\|^p_{L^p(\mathbb{R}^N)} \qquad [1.3]$$

for all $u\in C^\infty_c(\mathbb{R}^N)$ and all $1\leq p<\infty$.

Here $M^p=L^p_w=L^{p,\infty}$, $1\leq p<\infty$, is the Marcinkiewicz (= weak L^p) space modeled on L^p, and

$$[f]^p_{M^p(\mathbb{R}^N\times\mathbb{R}^N)} := \sup_{\lambda>0}\lambda^p\mathcal{L}^{2N}\left(\{x\in\mathbb{R}^N\times\mathbb{R}^N : |f(x)|\geq\lambda\}\right)$$

where $\mathcal{L}^{2N}$ is the Lebesgue measure on $\mathbb{R}^N\times\mathbb{R}^N$.

In fact, one can sharpen substantially the lower bound in [1.3].

Theorem 1.2. *Let $N\geq 1$, $1\leq p<\infty$, and $u\in C^\infty_c(\mathbb{R}^N)$. For $\lambda>0$, let*

$$E_\lambda := \left\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N : x\neq y,\ \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{p}+1}}\geq\lambda\right\}. \qquad [1.4]$$

Then

$$\lim_{\lambda\to\infty}\lambda^p\mathcal{L}^{2N}(E_\lambda) = \frac{k(p,N)}{N}\|\nabla u\|^p_{L^p(\mathbb{R}^N)}. \qquad [1.5]$$

Here

Significance

The Sobolev spaces, introduced in the 1930s, have become ubiquitous in analysis and applied mathematics. They involve L^p norms of the gradient of a function u. We present an alternative point of view where derivatives are replaced by appropriate finite differences and the Lebesgue space L^p is replaced by the slightly larger Marcinkiewicz space M^p (aka weak L^p space)—a popular tool in harmonic analysis. Surprisingly, these spaces coincide with the standard Sobolev spaces, a fact which sheds additional light onto these classical objects and should have numerous applications. In particular, it rectifies some well-known irregularities occurring in the theory of fractional Sobolev spaces. The proof relies on original calculus inequalities which might be useful in other situations.

Author contributions: H.B., J.V.S., and P.-L.Y. performed research and wrote the paper.

Reviewers: L.C.E., University of California, Berkeley; A.F., ETH Zurich; and F.-H.L., New York University.

The authors declare no competing interest.

[1]H.B., J.V.S., and P.-L.Y. contributed equally to this work.

[2]To whom correspondence may be addressed. Email: brezis@math.rutgers.edu.

Published February 15, 2021.

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$$k(p,N) := \int_{\mathbb{S}^{N-1}} |e\cdot\omega|^p \,\mathrm{d}\omega, \qquad \textbf{[1.6]}$$

where $\mathbb{S}^{N-1}$ denotes the unit sphere in $\mathbb{R}^N$, and e is any unit vector in $\mathbb{R}^N$.

Some comments concerning the above results are in order.

First, the validity of the upper bound in [**1.3**] when $p=1$ is quite remarkable and somewhat unexpected. In fact, a natural strategy to establish this upper bound (such as the one presented in *Remark 2.3* below) requires a strong type estimate for the maximal function (of the gradient of u), which holds when $p>1$, but notoriously fails at the end-point $p=1$. We overcome this difficulty by applying the Vitali covering lemma in a rather unconventional way which allows us to bypass the obstruction commonly arising at $p=1$ in this kind of situation. Thus, the hard core of the proof of the upper bound in [**1.3**] concerns the case $p=1$. As it turns out, we can furthermore derive the case $p>1$ from the case $p=1$, at a crucial step of the argument. When $p=1$ and $N=1$, the upper bound in [**1.3**] amounts to the following "innocuous-looking" calculus inequality

$$\begin{aligned}&\mathcal{L}^2\left\{(x,y)\in\mathbb{R}^2 : |u(x)-u(y)| \geq |x-y|^2\right\}\\ &\quad\leq C\int_{\mathbb{R}} |u'(t)|dt \qquad \textbf{[1.7]}\end{aligned}$$

for all $u\in C_c^\infty(\mathbb{R})$, where C is a universal constant. Surprisingly, this estimate seems to have gone unnoticed in the literature and our proof is more involved than expected!

Next, the lower bound in [**1.3**] is a consequence of *Theorem 1.2*. The proof of *Theorem 1.2* involves original ideas, partially inspired from techniques developed in ref. 3; actually, the constant $k(p,N)$ in [**1.6**] already appeared in the BBM formula (ref. 3, theorem 1.2).

The proof of the upper bound in [**1.3**] is presented in *Section 2*. The proof of *Theorem 1.2* is presented in *Section 3*.

The assertions in *Theorems 1.1* and *1.2*, which are stated for convenience when $u\in C_c^\infty(\mathbb{R}^N)$, suggest that similar conclusions hold under minimal regularity assumptions on u and that the Sobolev space $\dot{W}^{1,p}$, $1<p<\infty$ (respectively $\dot{BV}$ when $p=1$), can be identified with the space of measurable functions u satisfying $\sup_{\lambda>0}\lambda^p\mathcal{L}^{2N}(E_\lambda)<\infty$, or just $\limsup_{\lambda\to\infty}\lambda^p\mathcal{L}^{2N}(E_\lambda)<\infty$. One should also be able to replace $\mathbb{R}^N$ by domains $\Omega\subset\mathbb{R}^N$, etc. We will return to this circle of ideas in a forthcoming paper.

2. Proof of *Theorem 1.1*

As already mentioned the lower-bound part is a consequence of *Theorem 1.2* whose proof is presented in *Section 3*: Indeed, if E_λ is as in [**1.4**], then

$$\begin{aligned}\left[\frac{u(x)-u(y)}{|x-y|^{\frac{N}{p}+1}}\right]_{M^p(\mathbb{R}^N\times\mathbb{R}^N)}^p &= \sup_{\lambda>0}\lambda^p\mathcal{L}^{2N}(E_\lambda)\\ &\geq \lim_{\lambda\to\infty}\lambda^p\mathcal{L}^{2N}(E_\lambda)\end{aligned}$$

and Hölder's inequality gives

$$\begin{aligned}\left(\frac{k(p,N)}{N}\right)^{1/p} &\geq \frac{k(1,N)}{\sigma_{N-1}^{1-\frac{1}{p}}N^{\frac{1}{p}}} = \frac{k(1,N)}{\sigma_{N-1}}\left(\frac{\sigma_{N-1}}{N}\right)^{1/p}\\ &\geq k(1,N)\min\left\{\frac{1}{N},\frac{1}{\sigma_{N-1}}\right\} := c(N),\end{aligned}$$

where σ_{N-1} denotes the surface area of $\mathbb{S}^{N-1}$. Therefore, we concentrate here on the upper bound.

The key to our proof is the following proposition, which when $\gamma=1$ and $f=u'$ gives [**1.7**] and thus yields the desired upper bound for the $p=1$ case of *Theorem 1.1* in dimension $N=1$.

Proposition 2.1. *There exists a universal constant C such that for all $\gamma>0$ and all $f\in C_c(\mathbb{R})$, we have*

$$\iint_{E(f,\gamma)} |x-y|^{\gamma-1}\,\mathrm{d}x\,\mathrm{d}y \leq C\frac{5^\gamma}{\gamma}\|f\|_{L^1(\mathbb{R})},$$

where

$$E(f,\gamma) := \left\{(x,y)\in\mathbb{R}\times\mathbb{R}\colon x\neq y,\ \left|\int_y^x f\right| \geq |x-y|^{\gamma+1}\right\}.$$

Proof: Since $E(f,\gamma)\subseteq E(|f|,\gamma)$, without loss of generality assume f is nonnegative. Let X be the collection of all nontrivial closed intervals $I\subset\mathbb{R}$ such that

$$\int_I f \geq |I|^{\gamma+1}. \qquad \textbf{[2.1]}$$

(Here an interval is said to be nontrivial if it has positive length, and we used $|I|$ to denote the length of the interval.) Then

$$E(f,\gamma)\subseteq\bigcup_{I\in X} I\times I. \qquad \textbf{[2.2]}$$

The lengths of all intervals in X are bounded by $\|f\|_{L^1}^{1/(\gamma+1)}<\infty$. Hence we may apply the Vitali covering lemma and choose a subcollection Y of X, so that Y consists of a family of pairwise disjoint intervals J from X, and every $I\in X$ is contained in $5J$ for some $J\in Y$ (see, e.g., ref. 11, claim in the proof of theorem 1, section 1.5). It follows that

$$E(f,\gamma)\subseteq\bigcup_{J\in Y}(5J)\times(5J), \qquad \textbf{[2.3]}$$

where $5J$ is the interval with the same center as J but five times the length. As a result, we see that

$$\begin{aligned}\iint_{E(f,\gamma)}|x-y|^{\gamma-1}\,\mathrm{d}x\,\mathrm{d}y &\leq \sum_{J\in Y}\iint_{5J\times 5J}|x-y|^{\gamma-1}\,\mathrm{d}x\,\mathrm{d}y\\ &= \frac{10\cdot 5^\gamma}{\gamma(\gamma+1)}\sum_{J\in Y}|J|^{\gamma+1}. \qquad \textbf{[2.4]}\end{aligned}$$

(Here we used $\gamma>0$ to integrate in x and y.) But for each $J\in Y$, we have $J\in X$, so

$$|J|^{\gamma+1}\leq\int_J f.$$

Plugging this back into [**2.4**], we obtain

$$\begin{aligned}\iint_{E(f,\gamma)}|x-y|^{\gamma-1}\,\mathrm{d}x\,\mathrm{d}y &\leq C\frac{5^\gamma}{\gamma}\sum_{J\in Y}\int_J f\\ &\leq C\frac{5^\gamma}{\gamma}\|f\|_{L^1(\mathbb{R})},\end{aligned} \qquad \textbf{[2.5]}$$

the last inequality following from the disjointness of the different $J\in Y$. This completes the proof of *Proposition 2.1*. □

To prove the upper bound in *Theorem 1.1* when $N>1$ or $p>1$, *Proposition 2.1* still proves to be useful. Via the method of rotation, it implies the following proposition:

Proposition 2.2. *For any positive integer N, there exists a constant $C = C(N)$ such that for all $F \in C_c(\mathbb{R}^N)$, we have*

$$\mathcal{L}^{2N}(E(F)) \le C\|F\|_{L^1(\mathbb{R}^N)}$$

where

$$E(F) := \left\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : x \ne y, \left|\int_y^x F\right| \ge |x-y|^{N+1}\right\}.$$

Here $\int_x^y F$ is the integral of F along the line segment in $\mathbb{R}^N$ connecting x to y.

Proof: Again without loss of generality, we may assume that F is nonnegative. By a change of variable,

$$\begin{aligned}
&\mathcal{L}^{2N}\left(E(F)\right)\\
&= \mathcal{L}^{2N}\left(\left\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : y \ne 0,\right.\right.\\
&\qquad\qquad \left.\left.\int_0^{|y|} F\left(x + t\frac{y}{|y|}\right) \mathrm{d}t \ge |y|^{N+1}\right\}\right)\\
&= \int_{\mathbb{R}^N} \mathcal{L}^N\left(\left\{y \in \mathbb{R}^N \setminus \{0\} :\right.\right.\\
&\qquad\qquad \left.\left.\int_0^{|y|} F\left(x + t\frac{y}{|y|}\right) \mathrm{d}t \ge |y|^{N+1}\right\}\right) \mathrm{d}x.
\end{aligned}$$

Using polar coordinates to evaluate the integrand, we get

$$\mathcal{L}^{2N}(E(F)) = \int_{\mathbb{R}^N}\int_{\mathbb{S}^{N-1}}\int_{E(F,x,\omega)} r^{N-1}\,\mathrm{d}r\,\mathrm{d}\omega\,\mathrm{d}x,$$

where

$$E(F,x,\omega) := \left\{r \in (0,\infty) : \int_0^r F(x+t\omega)\,\mathrm{d}t \ge r^{N+1}\right\}.$$

We now use Fubini to interchange the integral over $\mathbb{R}^N$ and $\mathbb{S}^{N-1}$. Then for each $\omega \in \mathbb{S}^{N-1}$, we foliate $\mathbb{R}^N$ as an orthogonal sum $\omega^\perp \oplus \mathbb{R}\omega$, where $\omega^\perp$ is the subspace of all $x \in \mathbb{R}^N$ that is orthogonal to ω. Hence

$$\mathcal{L}^{2N}(E(F)) = \int_{\mathbb{S}^{N-1}}\int_{\omega^\perp}\int_{\mathbb{R}}\int_{E(F,x'+s\omega,\omega)} r^{N-1}\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}x'\,\mathrm{d}\omega. \qquad \textbf{[2.6]}$$

We now estimate the innermost double integral. For each fixed $\omega \in \mathbb{S}^{N-1}$ and each $x' \in \omega^\perp$, let $f_{\omega,x'} \in C_c^\infty(\mathbb{R})$ be a function of one variable defined by

$$f_{\omega,x'}(t) = F(x' + t\omega), \quad t \in \mathbb{R}.$$

Then

$$E(F, x'+s\omega, \omega) = \left\{r \in (0,\infty) : \int_0^r f_{\omega,x'}(s+t)\,\mathrm{d}t \ge r^{N+1}\right\},$$

so change of variables again gives

$$\int_{\mathbb{R}}\int_{E(F,x'+s\omega,\omega)} r^{N-1}\,\mathrm{d}r\,\mathrm{d}s = \frac{1}{2}\iint_{E(f_{\omega,x'},N)} |r-s|^{N-1}\,\mathrm{d}r\,\mathrm{d}s \qquad \textbf{[2.7]}$$

where

$$\begin{aligned}
&E(f_{\omega,x'}, N)\\
&\quad := \left\{(s,r) \in \mathbb{R}\times\mathbb{R} : s \ne r, \left|\int_s^r f_{\omega,x'}\right| \ge |r-s|^{N+1}\right\}
\end{aligned}$$

as in *Proposition 2.1* (the factor $1/2$ accounts for the fact that in the integral on the left-hand side of **[2.7]** we are working only with those $(s,r) \in E(f_{\omega,x'}, N)$ with $s < r$). Appealing to *Proposition 2.1* with $\gamma = N$, we may now estimate the double integral in the (r,s) variables on the right-hand side of **[2.6]**. We obtain

$$\begin{aligned}
\mathcal{L}^{2N}(E(F)) &\le \frac{C}{2}\cdot\frac{5^N}{N}\int_{\mathbb{S}^{N-1}}\int_{\omega^\perp}\int_{\mathbb{R}} f_{\omega,x'}(t)\,\mathrm{d}t\,\mathrm{d}x'\,\mathrm{d}\omega\\
&= \frac{C}{2}\cdot\frac{5^N\sigma_{N-1}}{N}\|F\|_{L^1(\mathbb{R}^N)},
\end{aligned}$$

the last equality holding because for every $\omega \in \mathbb{S}^{N-1}$,

$$\int_{\omega^\perp}\int_{\mathbb{R}} f_{\omega,x'}(t)\,\mathrm{d}t\,\mathrm{d}x' = \int_{\omega^\perp}\int_{\mathbb{R}} F(x'+t\omega)\,\mathrm{d}t\,\mathrm{d}x' = \|F\|_{L^1(\mathbb{R}^N)}.$$

□

The upper bound in *Theorem 1.1* follows easily from *Proposition 2.2*.

Proof of Theorem 1.1, the upper bound: Since $u \in C_c^\infty(\mathbb{R}^N)$, by Hölder's inequality, for every $1 \le p < \infty$,

$$|u(x) - u(y)| \le \left|\int_y^x |\nabla u|\right| \le |x-y|^{1-\frac{1}{p}}\left|\int_y^x |\nabla u|^p\right|^{\frac{1}{p}},$$

so for $\lambda > 0$,

$$\begin{aligned}
&\left\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : x \ne y, \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{p}+1}} \ge \lambda\right\}\\
&\quad\subseteq \left\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : x \ne y,\right.\\
&\quad\left.\left|\int_y^x \frac{|\nabla u|^p}{\lambda^p}\right| \ge |x-y|^{N+1}\right\}. \qquad \textbf{[2.8]}
\end{aligned}$$

Applying *Proposition 2.2* to $F := |\nabla u|^p/\lambda^p$, we see that

$$\begin{aligned}
&\mathcal{L}^{2N}\left(\left\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : x \ne y, \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{p}+1}} \ge \lambda\right\}\right)\\
&\qquad\qquad \le \frac{C}{\lambda^p}\|\nabla u\|^p_{L^p(\mathbb{R}^N)}
\end{aligned}$$

with $C = C(N)$, as desired.

□

Remark 2.3: When $p > 1$, the upper bound has a short proof relying on an estimate of the difference quotient by the maximal function of the gradient. The main ingredient is the following so-called Lusin–Lipschitz inequality,

$$|u(x) - u(y)| \le C|x-y|\left(\mathcal{M}|\nabla u|(x) + \mathcal{M}|\nabla u|(y)\right), \qquad \textbf{[2.9]}$$

where $\mathcal{M}f$ denotes the Hardy–Littlewood maximal function of f (see ref. 12, p. 404 for a complete proof and refs. 13 and 14 for recent developments). Inequality **[2.9]** implies that

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$$\left\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N : x\neq y,\ \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{p}+1}}\geq\lambda\right\}$$
$$\subseteq\Big\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N :$$
$$|x-y|^{\frac{N}{p}}\leq C\lambda^{-1}\left(\mathcal{M}|\nabla u|(x)+\mathcal{M}|\nabla u|(y)\right)\Big\}$$
$$\subseteq\left\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N : |x-y|^{\frac{N}{p}}\leq 2\ C\lambda^{-1}\mathcal{M}|\nabla u|(x)\right\}$$
$$\cup\left\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N : |x-y|^{\frac{N}{p}}\leq 2\ C\lambda^{-1}\mathcal{M}|\nabla u|(y)\right\} \quad \textbf{[2.10]}$$

and thus that

$$\lambda^p\mathcal{L}^{2N}\left(\left\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N : \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{p}+1}}\geq\lambda\right\}\right)$$
$$\leq C'(p,N)\int_{\mathbb{R}^N}(\mathcal{M}\,|\nabla u|)^p(x)\,dx.$$

For $1<p<\infty$, the maximal function theorem then implies

$$\left[\frac{u(x)-u(y)}{|x-y|^{\frac{N}{p}+1}}\right]_{M^p(\mathbb{R}^N\times\mathbb{R}^N)}\leq C(p,N)\|\nabla u\|_{L^p(\mathbb{R}^N)}.$$

The constant coming the maximal function theorem deteriorates as $p\searrow 1$.

3. Proof of *Theorem 1.2*

We now prove *Theorem 1.2* and hence the lower bound in *Theorem 1.1*.

We will use the inequalities

$$|u(x)-u(y)|\leq L|x-y|\quad\forall x,y\in\mathbb{R}^N \quad \textbf{[3.1]}$$

with $L:=\|\nabla u\|_{L^\infty(\mathbb{R}^N)}$ and

$$|u(x)-u(y)-\nabla u(x)\cdot(x-y)|\leq A|x-y|^2\quad\forall x,y\in\mathbb{R}^N \quad \textbf{[3.2]}$$

with $A:=\|\nabla^2 u\|_{L^\infty(\mathbb{R}^N)}$.

Fix $x\in\mathbb{R}^N$ and a direction $\omega\in\mathbb{S}^{N-1}$. For a large positive number λ, consider the set $E_\lambda(x,\omega)$ consisting of all $y\in\mathbb{R}^N$ such that $y-x$ is a positive multiple of ω and $(x,y)\in E_\lambda$. We will determine two numbers $\underline{R}=\underline{R}(x,\omega,\lambda)$ and $\overline{R}=\overline{R}(x,\omega,\lambda)$ such that

$$\{x+r\omega\colon r\in(0,\underline{R}]\}\subseteq E_\lambda(x,\omega)\subseteq\{x+r\omega\colon r\in(0,\overline{R}]\}.$$

Using polar coordinates, we then deduce that

$$\frac{1}{N}\int_{\mathbb{S}^{N-1}}\underline{R}(x,\omega,\lambda)^N\,\mathrm{d}\omega\leq\mathcal{L}^N\left(\{y\in\mathbb{R}^N\colon (x,y)\in E_\lambda\}\right)$$
$$\leq\frac{1}{N}\int_{\mathbb{S}^{N-1}}\overline{R}(x,\omega,\lambda)^N\,\mathrm{d}\omega. \quad \textbf{[3.3]}$$

From **[3.2]** we have

$$|u(x)-u(y)|\geq|\nabla u(x)\cdot(x-y)|-A|x-y|^2\geq\lambda|x-y|^{1+\frac{N}{p}}$$

provided

$$Ar+\lambda r^{N/p}\leq|\nabla u(x)\cdot\omega| \quad \textbf{[3.4]}$$

where $r:=|y-x|$ and $\omega=\frac{y-x}{|y-x|}\in\mathbb{S}^{N-1}$.

Fix $\delta>0$ arbitrarily small. Then by **[3.4]**, the conditions

$$Ar\leq\delta|\nabla u(x)\cdot\omega|\quad\text{and}\quad\lambda r^{N/p}\leq(1-\delta)|\nabla u(x)\cdot\omega|$$

imply that $(x,y)\in E_\lambda$. Thus, we can take $\underline{R}$ to be defined by

$$\underline{R}(x,\omega,\lambda)^N:=\min\left\{\frac{\delta^N}{A^N}|\nabla u(x)\cdot\omega|^N,\frac{(1-\delta)^p}{\lambda^p}|\nabla u(x)\cdot\omega|^p\right\}.$$

From **[3.3]** we have

$$\lambda^p\mathcal{L}^{2N}(E_\lambda)\geq\frac{1}{N}\iint\min\left\{\frac{\lambda^p\delta^N}{A^N}|\nabla u(x)\cdot\omega|^N,\right.$$
$$\left.(1-\delta)^p|\nabla u(x)\cdot\omega|^p\right\}\mathrm{d}\omega\,\mathrm{d}x,$$

where the integral is over all points $(x,\omega)\in\mathbb{R}^N\times\mathbb{S}^{N-1}$ with $\nabla u(x)\cdot\omega\neq 0$, and by monotone convergence,

$$\liminf_{\lambda\to\infty}\lambda^p\mathcal{L}^{2N}(E_\lambda)\geq\frac{(1-\delta)^p}{N}\int_{\mathbb{R}^N}\int_{\mathbb{S}^{N-1}}|\nabla u(x)\cdot\omega|^p\,\mathrm{d}\omega\,\mathrm{d}x.$$

Since $\delta>0$ is arbitrary, we conclude that

$$\liminf_{\lambda\to\infty}\lambda^p\mathcal{L}^{2N}(E_\lambda)\geq\frac{k(p,N)}{N}\int_{\mathbb{R}^N}|\nabla u(x)|^p\,\mathrm{d}x$$

where $k(p,N)$ is defined by **[1.6]**.

It remains to establish that

$$\limsup_{\lambda\to\infty}\lambda^p\mathcal{L}^{2N}(E_\lambda)\leq\frac{k(p,N)}{N}\int_{\mathbb{R}^N}|\nabla u(x)|^p\,\mathrm{d}x. \quad \textbf{[3.5]}$$

From **[3.2]** we have

$$|u(x)-u(y)|\leq|\nabla u(x)\cdot(x-y)|+A|x-y|^2$$

and thus if $(x,y)\in E_\lambda$ we obtain

$$\lambda r^{N/p}\leq|\nabla u(x)\cdot\omega|+Ar \quad \textbf{[3.6]}$$

where again $r=|y-x|$ and $\omega=\frac{y-x}{|y-x|}\in\mathbb{S}^{N-1}$. On the other hand, if $(x,y)\in E_\lambda$, we have from **[3.1]** that

$$\lambda r^{N/p}\leq L. \quad \textbf{[3.7]}$$

Inserting **[3.7]** into **[3.6]** yields

$$\lambda r^{N/p}\leq|\nabla u(x)\cdot\omega|+A\left(\frac{L}{\lambda}\right)^{p/N}. \quad \textbf{[3.8]}$$

In what follows we will consider only

$$\lambda>L. \quad \textbf{[3.9]}$$

Observe that if $\operatorname{dist}(x,\operatorname{supp}u)>1$, then

$$\{y\in\mathbb{R}^N\colon (x,y)\in E_\lambda\}=\emptyset. \quad \textbf{[3.10]}$$

Indeed by **[3.7]** and **[3.9]** we have, for any $(x,y)\in E_\lambda$, that $|x-y|\leq 1$. So if $\operatorname{dist}(x,\operatorname{supp}u)>1$ and $y\in\mathbb{R}^N$ is such that $(x,y)\in E_\lambda$, then $y\notin\operatorname{supp}u$, from which it follows that $\lambda|x-y|^{\frac{N}{p}+1}\leq|u(x)-u(y)|=0$, i.e., $x=y$, which is a contradiction since $(x,x)\notin E_\lambda$.

Using [3.8] and [3.10] we may take $\overline{R}$ to be

$$\overline{R}(x,\omega,\lambda)^N = \begin{cases} \left(\dfrac{|\nabla u(x)\cdot\omega| + A\left(\frac{L}{\lambda}\right)^{p/N}}{\lambda}\right)^p & \text{if } \operatorname{dist}(x,\operatorname{supp} u)\le 1 \\ 0 & \text{otherwise.}\end{cases}$$

Consequently from [3.3]

$$\lambda^p \mathcal{L}^{2N}(E_\lambda) \le \frac{1}{N}\int_{\mathbb{R}^N}\int_{\mathbb{S}^{N-1}} \mathbf{1}_{\operatorname{dist}(x,\operatorname{supp} u)\le 1} \left(|\nabla u(x)\cdot\omega| + A\left(\tfrac{L}{\lambda}\right)^{p/N}\right)^p \mathrm{d}\omega\,\mathrm{d}x \qquad [3.11]$$

which yields [3.5] by dominated convergence. □

4. Fixing a "Defect" of a Fractional Sobolev-Type Estimate

A typical fractional Sobolev-type estimate would assert that

$$\dot{W}^{1,1}(\mathbb{R}^N) \subset W^{s,p}(\mathbb{R}^N), \text{ with continuous injection,}$$

for every $N\ge 1$ and every $0<s<1$, where $1<p<\infty$ is defined by

$$\frac{1}{p} = 1 - \frac{1-s}{N}\,. \qquad [4.1]$$

This amounts to

$$\left\|\frac{u(x)-u(y)}{|x-y|^{\frac{N}{p}+s}}\right\|_{L^p(\mathbb{R}^N\times\mathbb{R}^N)} \le C\|\nabla u\|_{L^1(\mathbb{R}^N)}\,, \quad \forall u\in C_c^\infty(\mathbb{R}^N)\,. \qquad [4.2]$$

It turns out that [4.2] holds when $N\ge 2$ but fails when $N=1$. [Estimate [4.2] when $N\ge 2$ is due to Solonnikov (15); see also ref. 16, appendix D for a proof when $N=2$ which can be adapted to any $N\ge 2$ and ref. 17, corollary 8.2 for a proof based on cancellation properties of gradients in endpoint estimates (developed in ref. 18).] When $N=1$, [4.2] reads as

$$\left\|\frac{u(x)-u(y)}{|x-y|^{\frac{2}{p}}}\right\|_{L^p(\mathbb{R}\times\mathbb{R})} \le C\|u'\|_{L^1(\mathbb{R})}\,, \quad \forall u\in C_c^\infty(\mathbb{R})\,, \qquad [4.3]$$

which clearly fails for any $p\in[1,\infty)$. Indeed, take $u=u_n$, a sequence of smooth functions converging to the characteristic function $\mathbf{1}_I$ of a bounded interval $I\subset\mathbb{R}$; note that the right-hand side of [4.3] remains bounded while its left-hand side tends to infinity. When $p=1$, the failure of [4.3] is even more dramatic: The left-hand side is infinite for any measurable function u unless u is a constant, as mentioned in [1.2].

One way to repair the defect in [4.2] when $N=1$ consists of using again weak L^p instead of strong L^p.

Corollary 4.1. *There exists a constant C such that for every $1<p<\infty$,*

$$\left[\frac{u(x)-u(y)}{|x-y|^{\frac{2}{p}}}\right]_{M^p(\mathbb{R}\times\mathbb{R})} \le C\,\|u'\|_{L^1(\mathbb{R})}\,, \quad \forall u\in C_c^\infty(\mathbb{R})\,. \qquad [4.4]$$

Remark 4.2: When $p=2$, estimate [4.4] is originally due to Greco and Schiattarella (19).

5. Fixing a "Defect" of Some Fractional Gagliardo–Nirenberg-Type Estimates

We first consider a Gagliardo–Nirenberg-type inequality involving $\dot{W}^{1,1}(\mathbb{R}^N)$ and $L^{p_1}(\mathbb{R}^N)$ with $N\ge 1$ and $1\le p_1\le\infty$.

Let $\theta\in(0,1)$ and set

$$s = \theta\cdot 0 + (1-\theta)\cdot 1 = 1-\theta$$
$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{1} = \frac{\theta}{p_1} + (1-\theta)\,. \qquad [5.1]$$

It is known that the estimate

$$|u|_{W^{s,p}(\mathbb{R}^N)} = \left\|\frac{u(x)-u(y)}{|x-y|^{\frac{N}{p}+s}}\right\|_{L^p(\mathbb{R}^N\times\mathbb{R}^N)} \le C\|u\|^\theta_{L^{p_1}(\mathbb{R}^N)}\|\nabla u\|^{1-\theta}_{L^1(\mathbb{R}^N)} \qquad [5.2]$$

- holds for every $\theta\in(0,1)$ when $1\le p_1<\infty$ and
- fails for every $\theta\in(0,1)$ when $p_1=\infty$;

see, e.g., Brezis and Mironescu (9) and the references therein.

We investigate here what happens when $p_1=\infty$ and the anticipated inequality

$$\left\|\frac{u(x)-u(y)}{|x-y|^{\frac{N+1}{p}}}\right\|_{L^p(\mathbb{R}^N\times\mathbb{R}^N)} \le C\|u\|^{1-1/p}_{L^\infty(\mathbb{R}^N)}\|\nabla u\|^{1/p}_{L^1(\mathbb{R}^N)}\,, \qquad [5.3]$$

for $u\in C_c^\infty(\mathbb{R}^N)$, fails for every $1\le p<\infty$. (The argument is the same as above for the failure of [4.3].)

Our main result in this direction is the following:

Corollary 5.1. *For every $N\ge 1$, there exists a constant $C=C(N)$ such that for all $1<p<\infty$ and all $u\in C_c^\infty(\mathbb{R}^N)$,*

$$\left[\frac{u(x)-u(y)}{|x-y|^{\frac{N+1}{p}}}\right]_{M^p(\mathbb{R}^N\times\mathbb{R}^N)} \le C\|u\|^{1-1/p}_{L^\infty(\mathbb{R}^N)}\|\nabla u\|^{1/p}_{L^1(\mathbb{R}^N)}\,. \qquad [5.4]$$

Finally, we turn to another situation, also involving $\dot{W}^{1,1}$, where the Gagliardo–Nirenberg-type inequality fails. Let $0<s_1<1$, $1<p_1<\infty$, and $0<\theta<1$. Set

$$s = \theta s_1 + (1-\theta) \qquad \text{and} \qquad \frac{1}{p} = \frac{\theta}{p_1} + (1-\theta)\,. \qquad [5.5]$$

It is known that the estimate

$$|u|_{W^{s,p}(\mathbb{R}^N)} = \left\|\frac{u(x)-u(y)}{|x-y|^{\frac{N}{p}+s}}\right\|_{L^p(\mathbb{R}^N\times\mathbb{R}^N)} \le C|u|^\theta_{W^{s_1,p_1}(\mathbb{R}^N)}\|\nabla u\|^{1-\theta}_{L^1(\mathbb{R}^N)} \qquad [5.6]$$

- holds for every $\theta\in(0,1)$ when $s_1p_1<1$ [Cohen, Dahmen, Daubechies, and DeVore (20)] and
- fails for every $\theta\in(0,1)$ when $s_1p_1\ge 1$ [Brezis and Mironescu (9)].

We investigate here what happens in the regime $s_1p_1\ge 1$. Our main result in this direction is the following:

Corollary 5.2. *For every $N\ge 1$, there exists a constant $C=C(N)$ such that for any $s_1\in(0,1)$, $p_1\in(1,\infty)$ with $s_1p_1\ge 1$ and for any $\theta\in(0,1)$, we have*

$$\left[\frac{u(x)-u(y)}{|x-y|^{\frac{N}{p}+s}}\right]_{M^p(\mathbb{R}^N\times\mathbb{R}^N)} \le C|u|^\theta_{W^{s_1,p_1}(\mathbb{R}^N)}\|\nabla u\|^{1-\theta}_{L^1(\mathbb{R}^N)}\,, \qquad \forall u\in C_c^\infty(\mathbb{R}^N) \qquad [5.7]$$

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where $0 < s < 1$ and $1 < p < \infty$ are defined by [**5.5**].

The proofs of *Corollaries 4.1*, *5.1*, and *5.2* rely on our main *Theorem 1.1* and the details will appear in a forthcoming article.

Data Availability. There are no data underlying this work.

ACKNOWLEDGMENTS. This work was completed during two visits of J.V.S. to Rutgers University. J.V.S. thanks H.B. for the invitation and the Department of Mathematics for its hospitality. P.-L.Y. was partially supported by the General Research Fund CUHK14313716 from the Hong Kong Research Grant Council. H.B. is grateful to C. Sbordone who communicated to him the interesting paper by Greco and Schiattarella (listed below) which triggered our work. We are indebted to E. Tadmor for useful comments.

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