A surprising formula for Sobolev norms

Haim Brezis1,a,c,d,1,2, Jean Van Schaftingen5,c, and Po-Lam Yung6,f,g,1

*Department of Mathematics, Rutgers University, Piscataway, NJ 08854; 1Department of Mathematics, Technion, Israel Institute of Technology, 32000 Haifa, Israel; 1Department of Computer Science, Technion, Israel Institute of Technology, 32000 Haifa, Israel; 1Laboratoire Jacques-Louis Lions, Sorbonne Université, 75005 Paris, France; 1Institut de Recherche en Mathématique et Physique, Université Catholique de Louvain, 1348 Louvain-la-Neuve, Belgium; 1Mathematical Sciences Institute, Australian National University, Canberra ACT 2601, Australia; and 1Department of Mathematics, The Chinese University of Hong Kong, Ma Liu Shui, Hong Kong

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We establish the equivalence between the Sobolev seminorm $\|\nabla u\|_p$ and a quantity obtained when replacing strong $L^p$ by weak $L^p$ in the Gagliardo seminorm $|u|_{W^{s,p}}$ computed at $s=1$. As corollaries we derive alternative estimates in some exceptional cases (involving $W^{1,1}$) where the “anticipated” fractional Sobolev and Gagliardo–Nirenberg inequalities fail.

The primary goal of this paper is to propose an alternative point of view where derivatives are replaced by appropriate finite differences and the Lebesgue space $L^p$ replaced by the slightly larger Marcinkiewicz space $M^p$ (aka weak $L^p$ space)—a popular tool in harmonic analysis. Surprisingly, these spaces coincide with the standard Sobolev spaces, a fact which sheds additional light onto these classical objects and should have numerous applications. In particular, it rectifies some well-known irregularities occurring in the theory of fractional Sobolev spaces. The proof relies on original calculus inequalities which might be useful in other situations.

The Sobolev spaces, introduced in the 1930s, have become ubiquitous in analysis and applied mathematics. They involve $L^p$ norms of the gradient of a function $u$. We present an alternative point of view where derivatives are replaced by appropriate finite differences and the Lebesgue space $L^p$ is replaced by the slightly larger Marcinkiewicz space $M^p$ (aka weak $L^p$ space)—a popular tool in harmonic analysis. Surprisingly, these spaces coincide with the standard Sobolev spaces, a fact which sheds additional light onto these classical objects and should have numerous applications. In particular, it rectifies some well-known irregularities occurring in the theory of fractional Sobolev spaces. The proof relies on original calculus inequalities which might be useful in other situations.

**Significance**

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1H.B., J.V.S., and P.-L.Y. contributed equally to this work.

2To whom correspondence may be addressed. Email: brezis@math.rutgers.edu.

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where $S^{N-1}$ denotes the unit sphere in $\mathbb{R}^N$, and $e$ is any unit vector in $\mathbb{R}^N$.

Some comments concerning the above results are in order.

First, the validity of the upper bound in [1.3] when $p = 1$ is quite remarkable and somewhat unexpected. In fact, a natural strategy to establish this upper bound (such as the one presented in Remark 2.3 below) requires a strong type estimate for the maximal function (of the gradient of $u$), which holds when $p > 1$, but notoriusly fails at the end-point $p = 1$. We overcome this difficulty by applying the Vitali covering lemma in a rather unconventional way which allows us to bypass the obstruction commonly arising at $p = 1$ in this kind of situation. Thus, the hard core of the proof of the upper bound in [1.3] concerns the case $p = 1$. As it turns out, we can furthermore derive the case $p > 1$ from the case $p = 1$, at a crucial step of the argument. When $p = 1$ and $N = 1$, the upper bound in [1.3] amounts to the following “innocuous-looking” calculus inequality

$$L^2 \{ (x, y) \in \mathbb{R}^2 : |u(x) - u(y)| \geq |x - y|^2 \} \leq C \int |u'(t)| dt,$$

for all $u \in C^\infty_c(\mathbb{R})$, where $C$ is a universal constant. Surprisingly, this estimate seems to have gone unnoticed in the literature and our proof is more involved than expected!

Next, the lower bound in [1.3] is a consequence of Theorem 1.2. The proof of Theorem 1.2 involves original ideas, partially inspired from techniques developed in ref. 3; actually, the constant $k(p, N)$ in [1.6] already appeared in the BBM formula (ref. 3, theorem 1.2).

The proof of the upper bound in [1.3] is presented in Section 2. The proof of Theorem 1.2 is presented in Section 3.

The assertions in Theorems 1.1 and 1.2, which are stated for convenience when $u \in C^\infty_c(\mathbb{R}^N)$, suggest that similar conclusions hold under minimal regularity assumptions on $u$ and that the Sobolev space $W^{1,p}$, $1 < p < \infty$ (respectively $BV$ when $p = 1$), can be identified with the space of measurable functions $u$ satisfying $\sup_{\lambda > 0} \lambda^p \mathcal{L}^N(E_\lambda) < \infty$, or just $\limsup_{\lambda \to \infty} \lambda^p \mathcal{L}^N(E_\lambda) < \infty$. One should also be able to replace $\mathbb{R}^N$ by domains $\Omega \subset \mathbb{R}^N$, etc. We will return to this circle of ideas in a forthcoming paper.

2. Proof of Theorem 1.1

As already mentioned the lower-bound part is a consequence of Theorem 1.2 whose proof is presented in Section 3: Indeed, if $E_\lambda$ is as in [1.4], then

$$\left[ \frac{|u(x) - u(y)|}{|x - y|^{\frac{1}{p} + 1}} \right]_{M^p(\mathbb{R}^N \times \mathbb{R}^N)}^{\frac{1}{p}} = \sup_{\lambda > 0} \lambda \mathcal{L}^N(E_\lambda) \geq \lim_{\lambda \to \infty} \lambda \mathcal{L}^N(E_\lambda),$$

and Hölder’s inequality gives

$$\left( \frac{k(p, N)}{N} \right)^{1/p} \geq \frac{k(1, N)}{\sigma_{N-1} \frac{1}{p} + \frac{1}{N}} \geq k(1, N) \min \left\{ \frac{1}{N}, \frac{1}{\sigma_{N-1}} \right\} = c(N),$$

where $\sigma_{N-1}$ denotes the surface area of $S^{N-1}$. Therefore, we concentrate here on the upper bound.

The key to our proof is the following proposition, which when $\gamma = 1$ and $f = u'$ gives [1.7] and thus yields the desired upper bound for the $p = 1$ case of Theorem 1.1 in dimension $N = 1$.

Proposition 2.1. There exists a universal constant $C$ such that for all $\gamma > 0$ and all $f \in C_c(\mathbb{R})$, we have

$$\int_{E(f, \gamma)} |x - y|^{\gamma - 1} dx dy \leq C \frac{\gamma}{\gamma} \|f\|_{L^1(\mathbb{R})},$$

where

$$E(f, \gamma) := \left\{ (x, y) \in \mathbb{R} \times \mathbb{R} : x \neq y, \int_x^y f \geq |x - y|^{\gamma + 1} \right\}.$$

Proof: Since $E(f, \gamma) \subseteq E(|f|, \gamma)$, without loss of generality assume $f$ is nonnegative. Let $X$ be the collection of all nontrivial closed intervals $I \subset \mathbb{R}$ such that

$$\int_I f \geq |I|^{\gamma + 1}.$$

(Here an interval is said to be nontrivial if it has positive length, and we used $|I|$ to denote the length of the interval.) Then

$$E(f, \gamma) \subseteq \bigcup_{I \in X} I \times I.$$

The lengths of all intervals in $X$ are bounded by $\|f\|_{L^1(\mathbb{R})}^{\gamma + 1} < \infty$. Hence we may apply the Vitali covering lemma and choose a subcollection $Y$ of $X$, so that $Y$ consists of a family of pairwise disjoint intervals $J$ from $X$, and every $I \in X$ is contained in $5J$ for some $J \in Y$ (see, e.g., ref. 11, claim in the proof of theorem 1, section 1.5). It follows that

$$E(f, \gamma) \subseteq \bigcup_{J \in Y} (5J) \times (5J),$$

where $5J$ is the interval with the same center as $J$ but five times the length. As a result, we see that

$$\int_{E(f, \gamma)} |x - y|^{\gamma - 1} dx dy \leq \int_{J \in Y} \int_{5J \times 5J} |x - y|^{\gamma - 1} dx dy = \frac{10 \cdot 5^7}{\gamma (\gamma + 1)} \sum_{J \in Y} |J|^{\gamma + 1}.$$

(Here we used $\gamma > 0$ to integrate in $x$ and $y$.) But for each $J \in Y$, we have $J \in X$, so

$$|J|^{\gamma + 1} \leq \int_J f.$$

Plugging this back into [2.4], we obtain

$$\int_{E(f, \gamma)} |x - y|^{\gamma - 1} dx dy \leq C \frac{\gamma}{\gamma} \sum_{J \in Y} \int_J f \leq C \frac{\gamma}{\gamma} \|f\|_{L^1(\mathbb{R})},$$

the last inequality following from the disjointness of the different $J \in Y$. This completes the proof of Proposition 2.1.

To prove the upper bound in Theorem 1.1 when $N > 1$ or $p > 1$, Proposition 2.1 still proves to be useful. Via the method of rotation, it implies the following proposition:

□
**Proposition 2.2.** For any positive integer $N$, there exists a constant $C = C(N)$ such that for all $F \in C_c(\mathbb{R}^N)$, we have

$$\mathcal{L}^{2N}(E(F)) \leq C \|F\|_{L^1(\mathbb{R}^N)}$$

where

$$E(F) := \left\{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y, \int_y^x F \geq |x - y|^{N+1}\right\}.$$

Here $\int_y^x F$ is the integral of $F$ along the line segment in $\mathbb{R}^N$ connecting $x$ to $y$.

**Proof:** Again without loss of generality, we may assume that $F$ is nonnegative. By a change of variable,

$$\mathcal{L}^{2N}(E(F)) = \mathcal{L}^{2N} \left( \left\{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : y \neq 0, \int_0^{|y|} F \left(x + t \frac{y}{|y|}\right) dt \geq |y|^{N+1}\right\} \right) = \int_{\mathbb{R}^N} \mathcal{L}^N \left( \left\{y \in \mathbb{R}^N \setminus \{0\} : \int_0^{|y|} F \left(x + t \frac{y}{|y|}\right) dt \geq |y|^{N+1}\right\} \right) \omega(x) dx.$$

Using polar coordinates to evaluate the integrand, we get

$$\mathcal{L}^{2N}(E(F)) = \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} R \int_{E(F, x, \omega)} R^{N-1} \ dr \ d\omega \ dx,$$

where

$$E(F, x, \omega) := \left\{ r \in (0, \infty) : \int_0^r F \left(x + t \omega\right) dt \geq r^{N+1}\right\}.$$ 

We now use Fubini to interchange the integral over $\mathbb{R}^N$ and $\mathbb{S}^{N-1}$. Then for each $\omega \in \mathbb{S}^{N-1}$, we foliate $\mathbb{R}^N$ as an orthogonal sum $\omega^\perp \oplus \mathbb{R} \omega$, where $\omega^\perp$ is the subspace of all $x \in \mathbb{R}^N$ that is orthogonal to $\omega$. Hence

$$\mathcal{L}^{2N}(E(F)) = \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} \int_{E(F, x', s \omega)} R^{N-1} \ dr \ ds \ d\omega.$$

[2.6]

We now estimate the innermost double integral. For each fixed $\omega \in \mathbb{S}^{N-1}$ and each $x' \in \omega^\perp$, let $f_{x', \omega} \in C_c(\mathbb{R})$ be a function of one variable defined by

$$f_{x', \omega}(t) = F(x' + t \omega), \quad t \in \mathbb{R}.$$

Then

$$E(F, x' + s \omega, \omega) = \left\{ r \in (0, \infty) : \int_0^r f_{x', \omega}(s + t) dt \geq r^{N+1}\right\},$$

so change of variables again gives

$$\int_{\mathbb{R}} \int_{E(F, x' + s \omega, \omega)} R^{N-1} \ dr \ ds = \frac{1}{2} \int_{\mathbb{R}} \int_{E(F, x', N)} \left|r - s\right|^{N+1} \ dr \ ds.$$

[2.7]

where

$$E(f_{x', \omega}, N) := \left\{(s, r) \in \mathbb{R} \times \mathbb{R} : s \neq r, \int_s^r f_{x', \omega} \geq |r - s|^{N+1}\right\}$$

as in Proposition 2.1 (the factor 1/2 accounts for the fact that in the integral on the left-hand side of [2.7] we are working only with those $(s, r) \in E(f_{x', \omega}, N)$ with $s < r$). Appealing to Proposition 2.1 with $\gamma = N$, we may now estimate the double integral in the $(r, s)$ variables on the right-hand side of [2.6]. We obtain

$$\mathcal{L}^{2N}(E(F)) \leq \frac{C}{2} \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} E(f_{x', \omega}, t) dt \ dx' \ d\omega \ \text{as long as} \ \left|\frac{x'}{\lambda^p}\right| \leq \frac{1}{\lambda^p} \sum_{\mathbb{S}^{N-1}} E(f_{x', \omega}, N).$$

The last equality holding because for every $\omega \in \mathbb{S}^{N-1}$,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} E(f_{x', \omega}, t) dt \ dx' = \int_{\mathbb{R}} \int_{\mathbb{R}} F(x' + t \omega) dt \ dx' = \|F\|_{L^1(\mathbb{R}^N)}.$$
\[(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y, \quad |u(x) - u(y)| \geq \lambda \frac{1}{|x-y|^{\frac{N}{p}+1}} \geq \lambda \}
\subseteq \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : |x-y|^\frac{N}{p-1} \leq C\lambda^{-1} (M|\nabla u(x)| + M|\nabla u(y)|)\}
\subseteq \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : |x-y|^\frac{N}{p-1} \leq 2C\lambda^{-1} M|\nabla u(x)|\}
\cup \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : |x-y|^\frac{N}{p-1} \leq 2C\lambda^{-1} M|\nabla u(y)|\} \tag{2.10}
\]

and thus that
\[\lambda^p L^{2N}(\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : |u(x) - u(y)| \geq \lambda \frac{1}{|x-y|^{\frac{N}{p}+1}} \}) \leq C' p, N) \int_{\mathbb{R}^N} (M|\nabla u|)^p (x) dx.
\]

For \(1 < p < \infty\), the maximal function theorem then implies
\[\left[ \frac{|u(x) - u(y)|}{|x-y|^\frac{N}{p+1}} \right]_{p, N} \leq C(p, N) ||\nabla u||_{L^p(\mathbb{R}^N)}.
\]

The constant coming the maximal function theorem deteriorates as \(p \downarrow 1\).

3. Proof of Theorem 1.2

We now prove Theorem 1.2 and hence the lower bound in Theorem 1.1.

We will use the inequalities
\[|u(x) - u(y)| \leq L|x-y| \quad \forall x, y \in \mathbb{R}^N \tag{3.1}
\]

with \(L := ||\nabla u||_{L^\infty(\mathbb{R}^N)}\) and
\[|u(x) - u(y) - \nabla u(x) \cdot (x-y)| \leq A|x-y|^2 \quad \forall x, y \in \mathbb{R}^N \tag{3.2}
\]

with \(A := ||\nabla^2 u||_{L^\infty(\mathbb{R}^N)}\).

Fix \(x \in \mathbb{R}^N\) and a direction \(\omega \in \mathbb{S}^{N-1}\). For a large positive number \(\lambda\), consider the set \(E_\lambda(x, \omega)\) consisting of all \(y \in \mathbb{R}^N\) such that \(y - x\) is a positive multiple of \(\omega\) and \((x, y) \in E_\lambda\). We will determine two numbers \(\overline{R} = \overline{R}(x, \omega, \lambda)\) and \(\overline{R} = \overline{R}(x, \omega, \lambda)\) such that
\[\{ x + rw : r \in (0, \overline{R}) \} \subseteq E_\lambda(x, \omega) \subseteq \{ x + rw : r \in (0, \overline{R}) \} \tag{3.3}
\]

Using polar coordinates, we then deduce that
\[\frac{1}{N} \int_{\mathbb{S}^{N-1}} \overline{R}(x, \omega, \lambda)^N d\omega \leq \lambda^N \left( \{ y \in \mathbb{R}^N : (x, y) \in E_\lambda \} \right) \leq \frac{1}{N} \int_{\mathbb{S}^{N-1}} \overline{R}(x, \omega, \lambda)^N d\omega. \tag{3.4}
\]

From [3.2] we have
\[|u(x) - u(y)| \geq |\nabla u(x) \cdot (x-y)| - A|x-y|^2 \geq \lambda|x-y|^1 + \frac{2}{p} \tag{3.5}
\]

and thus if \((x, y) \in E_\lambda\) we obtain
\[\lambda r N/p \leq |\nabla u(x) \cdot \omega| - Ar \tag{3.6}
\]

where \(r := |y-x|\) and \(\omega = \frac{y-x}{|y-x|^N} \in \mathbb{S}^{N-1}\).

Fix \(\delta > 0\) arbitrarily small. Then by [3.4], the conditions
\[Ar \leq \delta|\nabla u(x) \cdot \omega| \quad \text{and} \quad \lambda r N/p \leq (1-\delta)|\nabla u(x) \cdot \omega| \tag{3.7}
\]

imply that \((x, y) \in E_\lambda\). Thus, we can take \(R\) to be defined by
\[R(x, \omega, \lambda)^N := \min \left\{ \delta^N \lambda |\nabla u(x) \cdot \omega|^N, \frac{(1-\delta)^p}{\lambda^p} |\nabla u(x) \cdot \omega|^p \right\} \tag{3.8}
\]

From [3.3] we have
\[\lambda^p L^{2N}(E_\lambda) \geq \frac{1}{N} \int_{\mathbb{R}^N} \min \left\{ \frac{\delta^N}{\lambda^N} |\nabla u(x) \cdot \omega|^N, \frac{(1-\delta)^p}{\lambda^p} |\nabla u(x) \cdot \omega|^p \right\} d\omega dx,
\]

where the integral is over all points \((x, \omega) \in \mathbb{R}^N \times \mathbb{S}^{N-1}\) with \(|\nabla u(x) \cdot \omega| \neq 0\), and by monotone convergence,
\[\lim_{\lambda \to \infty} \lambda^p L^{2N}(E_\lambda) \geq \frac{1-\delta}{N} \int_{\mathbb{R}^N} |\nabla u(x) \cdot \omega|^p d\omega dx.\]

Since \(\delta > 0\) is arbitrary, we conclude that
\[\liminf_{\lambda \to \infty} \lambda^p L^{2N}(E_\lambda) \geq \frac{k(p, N)}{N} \int_{\mathbb{R}^N} |\nabla u(x)\cdot \omega|^p dx \tag{3.9}
\]

where \(k(p, N)\) is defined by [1.6]. It remains to establish that
\[\limsup_{\lambda \to \infty} \lambda^p L^{2N}(E_\lambda) \leq \frac{k(p, N)}{N} \int_{\mathbb{R}^N} |\nabla u(x)\cdot \omega|^p dx \tag{3.10}
\]

From [3.2] we have
\[|u(x) - u(y)| \leq |\nabla u(x) \cdot (x-y)| + A|x-y|^2 \tag{3.11}
\]

and thus if \((x, y) \in E_\lambda\) we obtain
\[\lambda r N/p \leq |\nabla u(x) \cdot \omega| + Ar \tag{3.12}
\]

where again \(r := |y-x|\) and \(\omega = \frac{y-x}{|y-x|^N} \in \mathbb{S}^{N-1}\). On the other hand, if \((x, y) \in E_\lambda\), we have from [3.1] that
\[\lambda r N/p \leq L \tag{3.13}
\]

\[\lambda r N/p \leq |\nabla u(x) \cdot \omega| + A \left( \frac{L}{\lambda} \right)^{p/N} \tag{3.14}
\]

In what follows we will consider only
\[\lambda > L \tag{3.15}
\]

Observe that if dist\((x, \text{supp } u) > 1\), then
\[\{ y \in \mathbb{R}^N : (x, y) \in E_\lambda \} = \emptyset. \tag{3.16}
\]

Indeed by [3.7] and [3.9] we have, for any \((x, y) \in E_\lambda\), that \(|x-y| \leq 1\). So if dist\((x, \text{supp } u) > 1\) and \(y \in \mathbb{R}^N\) is such that \((x, y) \in E_\lambda\), then \(y \notin \text{supp } u\), from which it follows that
\[\lambda|x-y|^\frac{N}{p+1} \leq |u(x) - u(y)| = 0, i.e., x = y, which is a contradiction since \((x, x) \notin E_\lambda\).
Using [3.8] and [3.10] we may take $\mathcal{R}$ to be

$$
\mathcal{R}(x, \omega, \lambda)^\mathcal{N} = \begin{cases} 
\left( \frac{\|
abla u(x) \cdot \omega\| + A \left( \frac{\lambda}{\omega} \right)^{p/N} \right)^p & \text{if } \text{dist}(x, \text{supp } u) \leq 1 \\
0 & \text{otherwise}
\end{cases}
$$

Consequently from [3.3]

$$
\lambda^p L^{2N}(E_\lambda) \leq \frac{1}{N^2} \int_{\mathbb{R}^N} \int_{S^{N-1}} 1_{\text{dist}(x, \text{supp } u) \leq 1} \left( \|
abla u(x) \cdot \omega\| + A \left( \frac{\lambda}{\omega} \right)^{p/N} \right)^p d\omega dx
$$

[3.11]

which yields [3.5] by dominated convergence. □

4. Fixing a “Defect” of a Fractional Sobolev-Type Estimate

A typical fractional Sobolev-type estimate would assert that

$$
W^{1,1}(\mathbb{R}^N) \subset W^{s,p}(\mathbb{R}^N),
$$

with continuous injection, for every $N \geq 1$ and every $0 < s < 1$, where $1 < p < \infty$ is defined by

$$
\frac{1}{p} = 1 - \frac{1}{N} - s. \tag{4.1}
$$

This amounts to

$$
\left\| \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p}} + s} \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} \leq C \| \nabla u \|_{L^1(\mathbb{R}^N)}, \quad \forall u \in C_c^\infty(\mathbb{R}^N). \tag{4.2}
$$

It turns out that [4.2] holds when $N \geq 2$ but fails when $N = 1$. [Estimate [4.2] when $N \geq 2$ is due to Solonnikov (15); see also ref. 16, appendix D for a proof when $N = 2$ which can be adapted to any $N \geq 2$ and ref. 17, corollary 8.2 for a proof based on cancellation properties of gradients in endpoint estimates (developed in ref. 18).] When $N = 1$, [4.2] reads as

$$
\left\| \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p}} + s} \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} \leq C \| \nabla u \|_{L^i(\mathbb{R}^N)}, \quad \forall u \in C_c^\infty(\mathbb{R}^N). \tag{4.3}
$$

which clearly fails for any $p \in [1, \infty)$. Indeed, take $u = u_0$, a sequence of smooth functions converging to the characteristic function $1_I$ of a bounded interval $I \subset \mathbb{R}$; note that the right-hand side of [4.3] remains bounded while its left-hand side tends to infinity. When $p = 1$, the failure of [4.3] is even more dramatic: The left-hand side is infinite for any measurable function $u$ unless $u$ is a constant, as mentioned in [1.2].

One way to repair the defect in [4.2] when $N = 1$ consists of using again weak $L^p$ instead of strong $L^p$.

Corollary 4.1. There exists a constant $C$ such that for every $1 < p < \infty$,

$$
\left\| \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p}} + s} \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} \leq C \| \nabla u \|_{L^i(\mathbb{R}^N)}, \quad \forall u \in C_c^\infty(\mathbb{R}^N). \tag{4.4}
$$

Remark 4.2: When $p = 2$, estimate [4.4] is originally due to Greco and Schiattarella (19).

5. Fixing a “Defect” of Some Fractional Gagliardo–Nirenberg-Type Estimates

We first consider a Gagliardo–Nirenberg-type inequality involving $W^{1,1}(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$ with $N \geq 1$ and $1 \leq p_1 \leq \infty$.

Let $\theta \in (0, 1)$ and set

$$
\frac{1}{p} = \frac{\theta}{p_1} + \frac{1 - \theta}{1} = \frac{\theta}{p_1} + (1 - \theta) \tag{5.1}
$$

It is known that the estimate

$$
\left\| \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p}} + s} \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} \leq C \| \nabla u \|_{L^1(\mathbb{R}^N)}^{1 - \theta}, \tag{5.2}
$$

holds for every $\theta \in (0, 1)$ when $1 \leq p_1 < \infty$ and

• fails for every $\theta \in (0, 1)$ when $p_1 = \infty$; see, e.g., Brezis and Mironescu (9) and the references therein.

We investigate here what happens when $p_1 = \infty$ and the anticipated inequality

$$
\left\| \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p}} + s} \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} \leq C \| \nabla u \|_{L^1(\mathbb{R}^N)}^{1 - \theta}, \tag{5.3}
$$

for $u \in C_c^\infty(\mathbb{R}^N)$, fails for every $1 \leq p < \infty$. (The argument is the same as above for the failure of [4.3].)

Our main result in this direction is the following:

Corollary 5.1. For every $N \geq 1$, there exists a constant $C = C(N)$ such that for all $1 < p < \infty$ and all $u \in C_c^\infty(\mathbb{R}^N)$,

$$
\left\| \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p}} + s} \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} \leq C \| \nabla u \|_{L^1(\mathbb{R}^N)}^{1 - \theta}, \tag{5.4}
$$

Finally, we turn to another situation, also involving $W^{1,1}$, where the Gagliardo–Nirenberg-type inequality fails. Let $0 < s_1 < 1$, $1 < p_1 < \infty$, and $0 < \theta < 1$. Set

$$
\frac{s}{p} + \frac{1 - \theta}{1} = \frac{s_1}{p_1} + (1 - \theta) \tag{5.5}
$$

It is known that the estimate

$$
\left\| \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p}} + s} \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} \leq C \| \nabla u \|_{L^1(\mathbb{R}^N)}^{1 - \theta}, \tag{5.6}
$$

holds for every $\theta \in (0, 1)$ when $s_1 p_1 < 1$ [Cohen, Dahmen, Daubechies, and DeVore (20)] and

• fails for every $\theta \in (0, 1)$ when $s_1 p_1 \geq 1$ [Brezis and Mironescu (9)].

We investigate here what happens in the regime $s_1 p_1 >= 1$. Our main result in this direction is the following:

Corollary 5.2. For every $N \geq 1$, there exists a constant $C = C(N)$ such that for any $s_1 \in (0, 1)$, $p_1 \in (1, \infty)$ with $s_1 p_1 \geq 1$ and for any $\theta \in (0, 1)$, we have

$$
\left\| \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p}} + s} \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} \leq C \| \nabla u \|_{L^1(\mathbb{R}^N)}^{1 - \theta}, \quad \forall u \in C_c^\infty(\mathbb{R}^N) \tag{5.7}
$$
where $0 < s < 1$ and $1 < p < \infty$ are defined by [5,5].

The proofs of Corollaries 4.1, 5.1, and 5.2 rely on our main Theorem 1.1 and the details will appear in a forthcoming article.

Data Availability. There are no data underlying this work.

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