

# A SURPRISING FORMULA FOR SOBOLEV NORMS AND RELATED TOPICS

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ABSTRACT. We establish the equivalence between the Sobolev semi-norm  $\|\nabla u\|_{L^p}$  and a quantity obtained when replacing strong  $L^p$  by weak  $L^p$  in the Gagliardo semi-norm  $|u|_{W^{s,p}}$  computed at  $s = 1$ . As corollaries we derive alternative estimates in some exceptional cases (involving  $W^{1,1}$ ) where the “anticipated” fractional Sobolev and Gagliardo-Nirenberg inequalities fail.

## 1. INTRODUCTION

Fractional Sobolev spaces  $W^{s,p}$  (also called Slobodeskii spaces) play a major role in many questions involving partial differential equations. On  $\mathbb{R}^N$ ,  $N \geq 1$ , they are associated with the Gagliardo semi-norm

$$(1.1) \quad |u|_{W^{s,p}}^p := \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy$$

where  $0 < s < 1$  and  $1 \leq p < \infty$ . A well-known “drawback” of the Gagliardo semi-norm is that one does not recover the Sobolev semi-norm  $\|\nabla u\|_{L^p}^p$  if one takes  $s = 1$  in (1.1). In fact, for every  $1 \leq p < \infty$  and every measurable function  $u$

$$(1.2) \quad \left\| \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p}+1}} \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)}^p = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+p}} dx dy = \infty$$

unless  $u$  is a constant; see [5] and also [8, 19, 35].

One way to recover  $\|\nabla u\|_{L^p}^p$  out of the Gagliardo semi-norms is to consider the quantity  $(1 - s)|u|_{W^{s,p}}^p$  with  $0 < s < 1$  and show that it converges as  $s \nearrow 1$  to a multiple of  $\|\nabla u\|_{L^p}^p$ . This is a special case of the BBM formula in Bourgain-Brezis-Mironescu [6] (see also [8, 18, 39]), which furthermore enters (when  $p = 1$  and  $u$  is a characteristic function) in the study of “nonlocal minimal surfaces” and “ $s$ -perimeters” (see e.g. [3, 15, 21]).

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2010 *Mathematics Subject Classification.* 26D10 (26A33, 35A23, 46E30, 46E35).

*Key words and phrases.* Fractional Sobolev space; Marcinkiewicz space; fractional Gagliardo-Nirenberg interpolation inequalities.

The first goal of this paper is to propose an alternative route to repair this “defect”, simply replacing the  $L^p$  norm  $\|\cdot\|_{L^p}$  in (1.2) by the Marcinkiewicz  $M^p$  (i.e. weak  $L^p$ ) quasi-norm  $[\cdot]_{M^p}$ . The central result of the paper is Theorem 1.1 described below.

In a seemingly different direction, it is well-known that in some exceptional cases the “anticipated” fractional Sobolev-type and Gagliardo–Nirenberg-type estimates may fail (in particular when they involve  $\|\nabla u\|_{L^1}$ ). A second goal of this paper is to discuss a partial list of such failures (for a complete list see [10] and [11]) and to present alternative (weaker) estimates where strong  $L^p$  is replaced by weak  $L^p$ . As we are going to see they can all be derived as immediate consequences of Theorem 1.1 applied with  $p = 1$ .

Here are precise statements.

1.1. *Fixing a “defect” of the Gagliardo semi-norm  $|\cdot|_{W^{s,p}}$  when  $s = 1$*

**Theorem 1.1.** *For every  $N \geq 1$ , there exist constants  $c = c(N) > 0$  and  $C = C(N)$  such that*

$$(1.3) \quad c^p \|\nabla u\|_{L^p(\mathbb{R}^N)}^p \leq \left[ \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p} + 1}} \right]_{M^p(\mathbb{R}^N \times \mathbb{R}^N)}^p \leq C \|\nabla u\|_{L^p(\mathbb{R}^N)}^p$$

for all  $u \in C_c^\infty(\mathbb{R}^N)$  and all  $1 \leq p < \infty$ .

Here  $M^p(X, \mu) = L_w^p(X, \mu) = L^{p,\infty}(X, \mu)$ ,  $1 \leq p < \infty$ , is the Marcinkiewicz (=weak  $L^p$ ) space modelled on  $L^p(X, \mu)$ , defined by the condition

$$(1.4) \quad [f]_{M^p(X, \mu)}^p := \sup_{\lambda > 0} \lambda^p \mu(\{x \in X : |f(x)| \geq \lambda\}) < \infty$$

(see for example [16, Chapter 5; 25, Section 1.1]). Throughout the paper,  $\mu$  is taken to be the Lebesgue measure  $\mathcal{L}^{2N}$  on  $X = \mathbb{R}^N \times \mathbb{R}^N$ , except in Section 4.

In fact, one can sharpen substantially the lower bound in (1.3).

**Theorem 1.2.** *Let  $N \geq 1$ ,  $1 \leq p < \infty$  and  $u \in C_c^\infty(\mathbb{R}^N)$ . For  $\lambda > 0$ , let*

$$(1.5) \quad E_\lambda = \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y, \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p} + 1}} \geq \lambda \right\}.$$

Then

$$(1.6) \quad \lim_{\lambda \rightarrow \infty} \lambda^p \mathcal{L}^{2N}(E_\lambda) = \frac{1}{N} k(p, N) \|\nabla u\|_{L^p(\mathbb{R}^N)}^p.$$

Here

$$(1.7) \quad k(p, N) := \int_{\mathbb{S}^{N-1}} |e \cdot \omega|^p d\omega,$$

and  $e$  is any unit vector in  $\mathbb{R}^N$ .

The proof of the upper bound in (1.3) is presented in Section 2. It relies on the Vitali covering lemma. The lower bound in (1.3) is a consequence of Theorem 1.2 whose proof is presented in Section 3. Some generalizations to weighted integrals are given in Section 4.

The assertions in Theorems 1.1 and 1.2, which are stated for convenience when  $u \in C_c^\infty(\mathbb{R}^N)$ , suggest that similar conclusions hold under minimal regularity assumptions on  $u$ , and that the Sobolev space  $\dot{W}^{1,p}$ ,  $1 < p < \infty$  (respectively  $BV$  when  $p = 1$ ), can be identified with the space of measurable functions  $u$  satisfying  $\sup_{\lambda>0} \lambda^p \mathcal{L}^{2N}(E_\lambda) < \infty$ , or just  $\limsup_{\lambda \rightarrow \infty} \lambda^p \mathcal{L}^{2N}(E_\lambda) < \infty$ . Related issues are discussed in Section 6.3. One should also be able to replace  $\mathbb{R}^N$  by domains  $\Omega \subset \mathbb{R}^N$ , etc. In another direction it would also be interesting to try to fix the “defect” of the Gagliardo semi-norm  $|\cdot|_{W^{s,p}}$  when  $s = 0$ , using weak  $L^p$  (the other strategy, à la BBM, was successfully implemented in [29, 30]). We will return to this circle of ideas in a forthcoming paper. See also [1] for another possible characterization of Sobolev norms in  $\mathbb{R}^N$ .

## 1.2. Failure of a fractional Sobolev-type estimate

A typical fractional Sobolev-type estimate would assert that

$$(1.8) \quad \dot{W}^{1,1}(\mathbb{R}^N) \subset W^{s,p}(\mathbb{R}^N), \quad \text{with continuous injection,}$$

for every  $N \geq 1$  and every  $0 < s < 1$ , where  $1 < p < \infty$  is defined by

$$(1.9) \quad \frac{1}{p} = 1 - \frac{1-s}{N}.$$

This amounts to

$$(1.10) \quad \left\| \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p} + s}} \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} \leq C \|\nabla u\|_{L^1(\mathbb{R}^N)}, \quad \forall u \in C_c^\infty(\mathbb{R}^N).$$

It turns out that (1.10) holds when  $N \geq 2$  but fails when  $N = 1$ . (Estimate (1.10) when  $N \geq 2$  is due to Solonnikov [36]; see also [7, Appendix D] for a proof when  $N = 2$  which can be adapted to any  $N \geq 2$  and [37, Corollary 8.2] for a proof based on cancellation properties of gradients in endpoint estimates [4].) When  $N = 1$ , (1.10) reads as

$$(1.11) \quad \left\| \frac{u(x) - u(y)}{|x - y|^{\frac{2}{p}}} \right\|_{L^p(\mathbb{R} \times \mathbb{R})} \leq C \|u'\|_{L^1(\mathbb{R})}, \quad \forall u \in C_c^\infty(\mathbb{R}),$$

which clearly fails for any  $p \in [1, \infty)$ . Indeed, take  $u = u_n$ , a sequence of smooth functions converging to the characteristic function  $\mathbf{1}_I$  of a bounded interval  $I \subset \mathbb{R}$ ; note that the right-hand side of (1.11) remains bounded while its left-hand side

tends to infinity. When  $p = 1$ , the failure of (1.11) is even more dramatic: the left-hand side is infinite for any measurable function  $u$  unless  $u$  is a constant, see Bourgain–Brezis–Mironescu [5] (see also [8, 19, 35]).

One way to repair the defect in (1.10) when  $N = 1$  consists of using again weak  $L^p$  instead of strong  $L^p$ .

**Corollary 1.3.** *There exists a constant  $C$  such that for every  $1 < p < \infty$ ,*

$$(1.12) \quad \left[ \frac{u(x) - u(y)}{|x - y|^{\frac{2}{p}}} \right]_{M^p(\mathbb{R} \times \mathbb{R})} \leq C \|u'\|_{L^1(\mathbb{R})}, \quad \forall u \in C_c^\infty(\mathbb{R}).$$

Corollary 1.3 is an obvious consequence of Corollary 1.5 below (applied with  $N = 1$ ) since  $\|u\|_{L^\infty(\mathbb{R})} \leq \|u'\|_{L^1(\mathbb{R})}$ . The proof of Corollary 1.5 is presented in Section 5.

*Remark 1.4.* When  $p = 2$  estimate (1.12) is originally due to Greco and Schiattarella [26]. The conclusion of Corollary 1.3 is also valid when  $p = 1$ ; this corresponds to the upper bound in Theorem 1.1 with  $N = 1$  and  $p = 1$ .

### 1.3. Failure of some fractional Gagliardo–Nirenberg-type estimates

We first consider a Gagliardo–Nirenberg-type inequality involving  $\dot{W}^{1,1}(\mathbb{R}^N)$  and  $L^{p_1}(\mathbb{R}^N)$  with  $N \geq 1$  and  $1 \leq p_1 \leq \infty$ .

Let  $\theta \in (0, 1)$  and set

$$(1.13) \quad s = \theta \cdot 0 + (1 - \theta) \cdot 1 = 1 - \theta \quad \text{and} \quad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1 - \theta}{1} = \frac{\theta}{p_1} + (1 - \theta).$$

It is known that the estimate

$$(1.14) \quad \|u\|_{W^{s,p}(\mathbb{R}^N)} = \left\| \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p} + s}} \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} \leq C \|u\|_{L^{p_1}(\mathbb{R}^N)}^\theta \|\nabla u\|_{L^1(\mathbb{R}^N)}^{1-\theta}$$

- *holds* for every  $\theta \in (0, 1)$  when  $1 \leq p_1 < \infty$ ,
- *fails* for every  $\theta \in (0, 1)$  when  $p_1 = \infty$ ,

see e.g. Brezis–Mironescu [10] and the references therein.

We investigate here what happens when  $p_1 = \infty$  and the “anticipated” inequality

$$(1.15) \quad \left\| \frac{u(x) - u(y)}{|x - y|^{\frac{N+1}{p}}} \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} \leq C \|u\|_{L^\infty(\mathbb{R}^N)}^{1-1/p} \|\nabla u\|_{L^1(\mathbb{R}^N)}^{1/p}, \quad \forall u \in C_c^\infty(\mathbb{R}^N)$$

fails for every  $1 \leq p < \infty$ . (The argument is the same as above for the failure of (1.11).)

Our main result in this direction is

**Corollary 1.5.** *For every  $N \geq 1$ , there exists a constant  $C = C(N)$  such that for all  $1 < p < \infty$ ,*

$$(1.16) \quad \left[ \frac{u(x) - u(y)}{|x - y|^{\frac{N+1}{p}}} \right]_{M^p(\mathbb{R}^N \times \mathbb{R}^N)} \leq C \|u\|_{L^\infty(\mathbb{R}^N)}^{1-1/p} \|\nabla u\|_{L^1(\mathbb{R}^N)}^{1/p}, \quad \forall u \in C_c^\infty(\mathbb{R}^N).$$

Note that the conclusion of Corollary 1.5 is also valid when  $p = 1$ ; this corresponds to the upper bound in Theorem 1.1 (applied with  $p = 1$ ).

We now turn to another situation, also involving  $\dot{W}^{1,1}$ , where the Gagliardo–Nirenberg-type inequality fails. Let  $0 < s_1 < 1$ ,  $1 < p_1 < \infty$  and  $0 < \theta < 1$ . Set

$$(1.17) \quad s = \theta s_1 + (1 - \theta) \quad \text{and} \quad \frac{1}{p} = \frac{\theta}{p_1} + (1 - \theta).$$

It is known that the estimate

$$(1.18) \quad |u|_{W^{s,p}(\mathbb{R}^N)} = \left\| \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p} + s}} \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} \leq C |u|_{W^{s_1,p_1}(\mathbb{R}^N)}^\theta \|\nabla u\|_{L^1(\mathbb{R}^N)}^{1-\theta}$$

- *holds* for every  $\theta \in (0, 1)$  when  $s_1 p_1 < 1$  (Cohen, Dahmen, Daubechies and DeVore [17]),
- *fails* for every  $\theta \in (0, 1)$  when  $s_1 p_1 \geq 1$  (Brezis and Mironescu [10]).

We investigate here what happens in the regime  $s_1 p_1 \geq 1$ . Our main result in this direction is

**Corollary 1.6.** *For every  $N \geq 1$ , there exists a constant  $C = C(N)$  such that for any  $s_1 \in (0, 1)$ ,  $p_1 \in (1, \infty)$  with  $s_1 p_1 \geq 1$  and for any  $\theta \in (0, 1)$ , we have*

$$(1.19) \quad \left[ \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p} + s}} \right]_{M^p(\mathbb{R}^N \times \mathbb{R}^N)} \leq C |u|_{W^{s_1,p_1}(\mathbb{R}^N)}^\theta \|\nabla u\|_{L^1(\mathbb{R}^N)}^{1-\theta}, \quad \forall u \in C_c^\infty(\mathbb{R}^N)$$

where  $0 < s < 1$  and  $1 < p < \infty$  are defined by (1.17).

The proof of Corollary 1.6 is presented in Section 5. It might be interesting to find out whether the space of measurable functions  $u$  for which the left hand side of (1.19) is finite corresponds to some classical space such as Besov, Triebel-Lizorkin, etc.

Another natural question is whether the above results can be improved within the scale of Lorentz spaces  $L^{p,q}$ , which refine the Marcinkiewicz spaces  $M^p = L^{p,\infty}$ . This is discussed in Sections 6 and 7.

## Acknowledgments

This work was completed during two visits of J. Van Schaftingen to Rutgers University. He thanks H. Brezis for the invitation and the Department of Mathematics for its hospitality. P-L. Yung was partially supported by the General Research Fund CUHK14313716 from the Hong Kong Research Grant Council. H. Brezis is grateful to C. Sbordone who communicated to him the interesting paper [26] by Greco and Schiatarella which triggered our work. We are indebted to J. Serra for calling our attention to the papers [23, 24], and to E. Tadmor for useful comments.

## 2. PROOF OF THEOREM 1.1

As already mentioned the lower bound part is a consequence of Theorem 1.2 whose proof is presented in Section 3. Therefore we concentrate here on the upper bound. We shall present two arguments. The first one, based on an estimate of the difference quotient by the maximal function of the gradient, is very short but works only for  $1 < p < \infty$  and yields a constant  $C$  which deteriorates as  $p \searrow 1$ . The second one, based on the Vitali covering lemma, holds for all  $1 \leq p < \infty$ . In fact, most of the work deals with the case  $p = 1$ ; the case  $1 < p < \infty$  can easily be reduced to the case  $p = 1$ .

### 2.1. An argument for the upper bound when $1 < p < \infty$

The main ingredient is the following so-called Lusin-Lipschitz inequality,

$$(2.1) \quad |u(x) - u(y)| \leq C|x - y| \left( \mathcal{M}|\nabla u|(x) + \mathcal{M}|\nabla u|(y) \right),$$

where  $\mathcal{M}f$  denotes the Hardy–Littlewood maximal function of  $f$ ; see [27, p. 404] for a complete proof, and [2, 14] for recent developments. Inequality (2.1) implies that

$$(2.2) \quad \begin{aligned} & \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y, \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p}+1}} \geq \lambda \right\} \\ & \subseteq \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : |x - y|^{\frac{N}{p}} \leq C\lambda^{-1} \left( \mathcal{M}|\nabla u|(x) + \mathcal{M}|\nabla u|(y) \right) \right\} \\ & \subseteq \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : |x - y|^{\frac{N}{p}} \leq 2C\lambda^{-1} \mathcal{M}|\nabla u|(x) \right\} \\ & \quad \cup \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : |x - y|^{\frac{N}{p}} \leq 2C\lambda^{-1} \mathcal{M}|\nabla u|(y) \right\}. \end{aligned}$$

and thus that

$$\lambda^p \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p}+1}} \geq \lambda \right\} \right) \leq C'(p, N) \int_{\mathbb{R}^N} (\mathcal{M}|\nabla u|)^p.$$

For  $1 < p < \infty$ , the maximal function theorem then implies

$$\left[ \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p} + 1}} \right]_{M^p(\mathbb{R}^N \times \mathbb{R}^N)} \leq C(p, N) \|\nabla u\|_{L^p(\mathbb{R}^N)}.$$

□

Note that this proof breaks down when  $p = 1$ . The argument presented in the next section relies on a different strategy, which is still valid when  $p = 1$ .

## 2.2. An argument for the upper bound valid for all $1 \leq p < \infty$

The key is the following proposition, which when  $\gamma = 1$  and  $f = u'/\lambda$  gives the desired upper bound for the  $p = 1$  case of Theorem 1.1 in dimension  $N = 1$ .

**Proposition 2.1.** *There exists a universal constant  $C$  such that for all  $\gamma > 0$  and all  $f \in C_c(\mathbb{R})$ , we have*

$$\iint_{E(f, \gamma)} |x - y|^{\gamma - 1} dx dy \leq C \frac{3^\gamma}{\gamma} \|f\|_{L^1(\mathbb{R})},$$

where

$$E(f, \gamma) := \left\{ (x, y) \in \mathbb{R} \times \mathbb{R} : x \neq y, \left| \int_y^x f \right| \geq |x - y|^{\gamma + 1} \right\}.$$

*Proof.* Without loss of generality assume  $f$  is non-negative. Let  $X$  be the collection of all non-trivial closed intervals  $I \subset \mathbb{R}$  such that

$$(2.3) \quad \int_I f \geq \left( \frac{|I|}{3} \right)^{\gamma + 1}.$$

(Here an interval is said to be non-trivial if it has positive length, and we used  $|I|$  to denote the length of the interval.) Then  $X$  is partially ordered by set inclusion. Let  $Y$  be the set of all  $J \in X$  that are maximal with respect to this partial ordering, i.e. the set of all  $J \in X$  that are not properly contained in any other interval in  $X$ . The lengths of all intervals in  $X$  (and hence in  $Y$ ) are bounded by  $3\|f\|_{L^1}^{1/(\gamma+1)} < \infty$ . Hence we may apply the Vitali covering lemma, and choose a subcollection  $Z$  of  $Y$ , so that  $Z$  consists of a family of pairwise disjoint intervals  $\{K\}$  from  $Y$ , and every  $J \in Y$  is contained in  $5K$  for some  $K \in Z$ . We claim that

$$(2.4) \quad E(f, \gamma) \subseteq \bigcup_{K \in Z} (5K) \times (10K),$$

where  $\lambda K$  is the interval with the same center as  $K$  but  $\lambda$  times the length. Assuming this claim for the moment, we see that

$$(2.5) \quad \iint_{E(f, \gamma)} |x - y|^{\gamma - 1} dx dy \leq \sum_{K \in Z} \iint_{5K \times 10K} |x - y|^{\gamma - 1} dx dy \leq \frac{C}{\gamma} \sum_{K \in Z} |K|^{\gamma + 1}.$$

(Here we used  $\gamma > 0$  to integrate in  $x$  and  $y$ .) But for each  $K \in Z$ , we have  $K \in X$ , so

$$|K|^{\gamma+1} \leq 3^{\gamma+1} \int_K f.$$

Plugging this back into (2.5), we obtain

$$(2.6) \quad \iint_{E(f,\gamma)} |x-y|^{\gamma-1} dx dy \leq \frac{3^{\gamma+1}C}{\gamma} \sum_{K \in Z} \int_K f \leq \frac{3^{\gamma+1}C}{\gamma} \|f\|_{L^1(\mathbb{R})},$$

the last inequality following from the disjointness of the different  $K \in Z$ . This completes the proof of the proposition, modulo the proof of the claim (2.4).

Before proving claim (2.4), we first show that every  $I \in X$  is contained in some  $J \in Y$ . Let  $I \in X$ . Let  $X_I$  be the set of all intervals in  $X$  that contains  $I$ . All intervals in  $X_I$  are contained in a fixed compact subset of  $\mathbb{R}$ , because they all have lengths bounded by  $3\|f\|_{L^1}^{1/(\gamma+1)}$ . Hence the supremum of lengths of intervals from  $X_I$  is finite. Let  $[a_n, b_n]$  be a sequence of intervals in  $X_I$  such that  $b_n - a_n$  converges to this supremum as  $n \rightarrow \infty$ . By passing to a subsequence, we may assume that  $a_n$  converges to some  $a \in \mathbb{R}$  and  $b_n$  converges to some  $b \in \mathbb{R}$ . Then dominated convergence shows that  $[a, b] \in X_I$ ; also  $b - a = \lim_{n \rightarrow \infty} (b_n - a_n) = \sup_{I' \in X_I} |I'|$ . Hence  $[a, b]$  is a maximal interval in  $X$ , i.e.  $[a, b] \in J$ , and of course  $[a, b]$  contains  $I$ . So  $[a, b]$  is the desired interval in  $J$  that contains  $I$ .

Going back to the proof of claim (2.4), let  $(x, y) \notin \bigcup_{K \in Z} (5K) \times (10K)$ . We need to show that  $(x, y) \notin E(f, \gamma)$ . We may assume  $x \neq y$ . We consider two cases.

**Case 1.**  $x \notin 5K$  for all  $K \in Z$ . Then  $x \notin J$  for any  $J \in Y$ . Hence from the above,  $x \notin I$  for any  $I \in X$ . In particular,

$$\left| \int_z^x f \right| < \frac{1}{3^{\gamma+1}} |x - z|^{\gamma+1}$$

for any  $z \in \mathbb{R}$ ,  $z \neq x$ . This shows that  $(x, z) \notin E(f, \gamma)$  for any  $z \in \mathbb{R}$ ,  $z \neq x$ . In particular,  $(x, y) \notin E(f, \gamma)$ .

**Case 2.**  $x \in 5K$  for some  $K \in Z$ . Then  $y \notin 10K$ . Let's write  $5K = [a, b]$ , so that  $x \in [a, b]$ . If  $y < x$ , then

$$(b - y) = (b - x) + (x - y) \leq 5|K| + (x - y) \leq 3(x - y),$$

so

$$\int_y^x f \leq \int_y^b f < \left( \frac{|b - y|}{3} \right)^{\gamma+1} \leq |x - y|^{\gamma+1},$$

the second inequality following by maximality of  $K$  and that  $K \subsetneq [y, b]$ . On the other hand, if  $y > x$ , then

$$(y - a) = (y - x) + (x - a) \leq (y - x) + 5|K| \leq 3(y - x),$$

so

$$\int_x^y f \leq \int_a^y f < \left(\frac{|y-a|}{3}\right)^{\gamma+1} \leq |y-x|^{\gamma+1},$$

again using the maximality of  $K$  and that  $K \subsetneq [a, y]$ . In either case,  $(x, y) \notin E(f, \gamma)$ . This proves (2.4).  $\square$

To prove the upper bound in Theorem 1.1 when  $N > 1$  or  $p > 1$ , Proposition 2.1 still proves to be useful. Via the method of rotation, it implies the following proposition:

**Proposition 2.2.** *For any positive integer  $N$ , there exists a constant  $C = C(N)$  such that for all  $F \in C_c(\mathbb{R}^N)$ , we have*

$$\mathcal{L}^{2N}(E(F)) \leq C \|F\|_{L^1(\mathbb{R}^N)}$$

where

$$E(F) := \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y, \left| \int_y^x F \right| \geq |x-y|^{N+1} \right\}.$$

Here  $\int_x^y F$  is the integral of  $F$  along the line segment in  $\mathbb{R}^N$  connecting  $x$  to  $y$ .

*Proof.* Again without loss of generality, we may assume that  $F$  is non-negative. By a change of variable,

$$\begin{aligned} \mathcal{L}^{2N}(E(F)) &= \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : y \neq 0, \int_0^{|y|} F \left( x + t \frac{y}{|y|} \right) dt \geq |y|^{N+1} \right\} \right) \\ &= \int_{\mathbb{R}^N} \mathcal{L}^N \left( \left\{ y \in \mathbb{R}^N \setminus \{0\} : \int_0^{|y|} F \left( x + t \frac{y}{|y|} \right) dt \geq |y|^{N+1} \right\} \right) dx. \end{aligned}$$

Using polar coordinates to evaluate the integrand, we get

$$\mathcal{L}^{2N}(E(F)) = \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} \int_{E(F, x, \omega)} r^{N-1} dr d\omega dx,$$

where

$$E(F, x, \omega) := \left\{ r \in (0, \infty) : \int_0^r F(x + t\omega) dt \geq r^{N+1} \right\}.$$

We now use Fubini to interchange the integral over  $\mathbb{R}^N$  and  $\mathbb{S}^{N-1}$ . Then for each  $\omega \in \mathbb{S}^{N-1}$ , we foliate  $\mathbb{R}^N$  as an orthogonal sum  $\omega^\perp \oplus \mathbb{R}\omega$ , where  $\omega^\perp$  is the subspace of all  $x \in \mathbb{R}^N$  that is orthogonal to  $\omega$ . Hence

$$(2.7) \quad \mathcal{L}^{2N}(E(F)) = \int_{\mathbb{S}^{N-1}} \int_{\omega^\perp} \int_{\mathbb{R}} \int_{E(F, x' + s\omega, \omega)} r^{N-1} dr ds dx' d\omega.$$

We now estimate the inner most double integral. For each fixed  $\omega \in \mathbb{S}^{N-1}$  and each  $x' \in \omega^\perp$ , let  $f_{\omega, x'} \in C_c^\infty(\mathbb{R})$  be a function of one variable defined by

$$f_{\omega, x'}(t) = F(x' + t\omega), \quad t \in \mathbb{R}.$$

Then

$$E(F, x' + s\omega, \omega) = \left\{ r \in (0, \infty) : \int_0^r f_{\omega, x'}(s+t) dt \geq r^{N+1} \right\},$$

so change of variables again gives

$$(2.8) \quad \int_{\mathbb{R}} \int_{E(F, x'+s\omega, \omega)} r^{N-1} dr ds = \frac{1}{2} \iint_{E(f_{\omega, x'}, N)} |r-s|^{N-1} dr ds$$

where

$$E(f_{\omega, x'}, N) := \left\{ (s, r) \in \mathbb{R} \times \mathbb{R} : s \neq r, \left| \int_s^r f_{\omega, x'} \right| \geq |r-s|^{N+1} \right\}$$

as in Proposition 2.1 (the factor 1/2 accounts for the fact that in the integral on the left hand side of (2.8) we are only working with those  $(s, r) \in E(f_{\omega, x'}, N)$  with  $s < r$ ). Appealing to Proposition 2.1 with  $\gamma = N$ , we may now estimate the double integral in the  $(r, s)$  variables on the right hand side of (2.7). We obtain

$$\mathcal{L}^{2N}(E(F)) \leq \frac{C}{2} \cdot \frac{3^N}{N} \int_{\mathbb{S}^{N-1}} \int_{\omega^\perp} \int_{\mathbb{R}} f_{\omega, x'}(t) dt dx' d\omega = \frac{C}{2} \cdot \frac{3^N |\mathbb{S}^{N-1}|}{N} \|F\|_{L^1(\mathbb{R}^N)},$$

the last equality holding because for every  $\omega \in \mathbb{S}^{N-1}$ ,

$$\int_{\omega^\perp} \int_{\mathbb{R}} f_{\omega, x'}(t) dt dx' = \int_{\omega^\perp} \int_{\mathbb{R}} F(x' + t\omega) dt dx' = \|F\|_{L^1(\mathbb{R}^N)}. \quad \square$$

The upper bound in Theorem 1.1 follows easily from Proposition 2.2.

*Proof of Theorem 1.1, the upper bound.* If  $u \in C_c^\infty(\mathbb{R}^N)$ , then by Hölder's inequality, for every  $1 \leq p < \infty$ ,

$$|u(x) - u(y)| \leq \left| \int_y^x |\nabla u| \right| \leq |x-y|^{1-\frac{1}{p}} \left| \int_y^x |\nabla u|^p \right|^{\frac{1}{p}},$$

so for  $\lambda > 0$ ,

$$(2.9) \quad \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y, \frac{|u(x) - u(y)|}{|x-y|^{\frac{N}{p}+1}} \geq \lambda \right\} \\ \subseteq \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y, \left| \int_y^x \frac{|\nabla u|^p}{\lambda^p} \right| \geq |x-y|^{N+1} \right\}.$$

Applying Proposition 2.2 to  $F := |\nabla u|^p / \lambda^p$ , we see that

$$\mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y, \frac{|u(x) - u(y)|}{|x-y|^{\frac{N}{p}+1}} \geq \lambda \right\} \right) \leq \frac{C}{\lambda^p} \|\nabla u\|_{L^p(\mathbb{R}^N)}^p$$

with  $C = C(N)$ , as desired. □

### 3. PROOF OF THEOREM 1.2

We now prove Theorem 1.2 and hence the lower bound in Theorem 1.1.

We will use the inequalities

$$(3.1) \quad |u(x) - u(y)| \leq L|x - y| \quad \forall x, y \in \mathbb{R}^N$$

with  $L := \|\nabla u\|_{L^\infty(\mathbb{R}^N)}$  and

$$(3.2) \quad |u(x) - u(y) - \nabla u(x) \cdot (x - y)| \leq A|x - y|^2 \quad \forall x, y \in \mathbb{R}^N$$

with  $A := \|\nabla^2 u\|_{L^\infty(\mathbb{R}^N)}$ .

Fix  $x \in \mathbb{R}^N$  and a direction  $\omega \in \mathbb{S}^{N-1}$ . For a large positive number  $\lambda$ , consider the set  $E_\lambda(x, \omega)$  consisting of all  $y \in \mathbb{R}^N$  such that  $y - x$  is a positive multiple of  $\omega$  and  $(x, y) \in E_\lambda$ . We will determine two numbers  $\underline{R} = \underline{R}(x, \omega, \lambda)$  and  $\overline{R} = \overline{R}(x, \omega, \lambda)$  such that

$$\{x + r\omega : r \in (0, \underline{R}]\} \subseteq E_\lambda(x, \omega) \subseteq \{x + r\omega : r \in (0, \overline{R}]\}.$$

Using polar coordinates, we then deduce that

$$(3.3) \quad \begin{aligned} \frac{1}{N} \int_{\mathbb{S}^{N-1}} \underline{R}(x, \omega, \lambda)^N d\omega &\leq \mathcal{L}^N(\{y \in \mathbb{R}^N : (x, y) \in E_\lambda\}) \\ &\leq \frac{1}{N} \int_{\mathbb{S}^{N-1}} \overline{R}(x, \omega, \lambda)^N d\omega. \end{aligned}$$

From (3.2) we have

$$|u(x) - u(y)| \geq |\nabla u(x) \cdot (x - y)| - A|x - y|^2 \geq \lambda|x - y|^{1+\frac{N}{p}}$$

provided

$$(3.4) \quad Ar + \lambda r^{N/p} \leq |\nabla u(x) \cdot \omega|$$

where  $r := |y - x|$  and  $\omega = \frac{y-x}{|y-x|} \in \mathbb{S}^{N-1}$ .

Fix  $\delta > 0$  arbitrarily small. Then by (3.4), the conditions

$$Ar \leq \delta |\nabla u(x) \cdot \omega| \quad \text{and} \quad \lambda r^{N/p} \leq (1 - \delta) |\nabla u(x) \cdot \omega|$$

imply that  $(x, y) \in E_\lambda$ . Thus we can take  $\underline{R}$  to be defined by

$$\underline{R}(x, \omega, \lambda)^N := \min \left\{ \frac{\delta^N}{A^N} |\nabla u(x) \cdot \omega|^N, \frac{(1 - \delta)^p}{\lambda^p} |\nabla u(x) \cdot \omega|^p \right\}.$$

From (3.3) we have

$$\begin{aligned} &\lambda^p \mathcal{L}^{2N}(E_\lambda) \\ &\geq \frac{1}{N} \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} \mathbf{1}_{\nabla u(x) \cdot \omega \neq 0} \min \left\{ \frac{\lambda^p \delta^N}{A^N} |\nabla u(x) \cdot \omega|^N, (1 - \delta)^p |\nabla u(x) \cdot \omega|^p \right\} d\omega dx, \end{aligned}$$

and by monotone convergence,

$$\liminf_{\lambda \rightarrow \infty} \lambda^p \mathcal{L}^{2N}(E_\lambda) \geq \frac{(1-\delta)^p}{N} \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} |\nabla u(x) \cdot \omega|^p d\omega dx.$$

Since  $\delta > 0$  is arbitrary, we conclude that

$$\liminf_{\lambda \rightarrow \infty} \lambda^p \mathcal{L}^{2N}(E_\lambda) \geq \frac{1}{N} k(p, N) \int_{\mathbb{R}^N} |\nabla u(x)|^p dx$$

where  $k(p, N)$  is defined by (1.7).

It remains to establish that

$$(3.5) \quad \limsup_{\lambda \rightarrow \infty} \lambda^p \mathcal{L}^{2N}(E_\lambda) \leq \frac{1}{N} k(p, N) \int_{\mathbb{R}^N} |\nabla u(x)|^p dx.$$

From (3.2) we have

$$|u(x) - u(y)| \leq |\nabla u(x) \cdot (x - y)| + A|x - y|^2$$

and thus if  $(x, y) \in E_\lambda$  we obtain

$$(3.6) \quad \lambda r^{N/p} \leq |\nabla u(x) \cdot \omega| + Ar$$

where again  $r = |y - x|$  and  $\omega = \frac{y-x}{|y-x|} \in \mathbb{S}^{N-1}$ . On the other hand, if  $(x, y) \in E_\lambda$ , we have from (3.1) that

$$(3.7) \quad \lambda r^{N/p} \leq L.$$

Inserting (3.7) into (3.6) yields

$$(3.8) \quad \lambda r^{N/p} \leq |\nabla u(x) \cdot \omega| + A \left( \frac{L}{\lambda} \right)^{p/N}.$$

In what follows we will consider only

$$(3.9) \quad \lambda > L.$$

Observe that if  $\text{dist}(x, \text{supp } u) > 1$  then

$$(3.10) \quad \{y \in \mathbb{R}^N : (x, y) \in E_\lambda\} = \emptyset.$$

Indeed by (3.7) and (3.9) we have, for any  $(x, y) \in E_\lambda$ , that  $|x - y| \leq 1$ . So if  $\text{dist}(x, \text{supp } u) > 1$  and  $y \in \mathbb{R}^N$  is such that  $(x, y) \in E_\lambda$ , then  $y \notin \text{supp } u$ , from which it follows that  $\lambda|x - y|^{\frac{N}{p}+1} \leq |u(x) - u(y)| = 0$ , i.e.  $x = y$ , which is a contradiction since  $(x, x) \notin E_\lambda$ .

Using (3.8) and (3.10) we may take  $\bar{R}$  to be defined by

$$\bar{R}(x, \omega, \lambda)^N = \begin{cases} \frac{1}{\lambda^p} \left( |\nabla u(x) \cdot \omega| + A \left( \frac{L}{\lambda} \right)^{p/N} \right)^p & \text{if } \text{dist}(x, \text{supp } u) \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Consequently from (3.3)

$$\lambda^p \mathcal{L}^{2N}(E_\lambda) \leq \frac{1}{N} \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} \mathbf{1}_{\text{dist}(x, \text{supp } u) \leq 1} \left( |\nabla u(x) \cdot \omega| + A \left( \frac{L}{\lambda} \right)^{p/N} \right)^p d\omega dx$$

which yields (3.5) by dominated convergence.  $\square$

*Remark 3.1.* It is instructive to compare Theorem 1.1 with a variant of the BBM formula. Consider a family  $(\rho_\varepsilon)$  of radial mollifiers in the sense that

$$(3.11) \quad \rho_\varepsilon(r) \geq 0, \quad \int_0^\infty \rho_\varepsilon(r) r^{N-1} dr = 1, \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \int_\delta^\infty \rho_\varepsilon(r) r^{N-1} dr = 0$$

for all  $\delta > 0$ . Then, for every  $1 \leq p < \infty$  and  $u \in C_c^\infty(\mathbb{R}^N)$ ,

$$(3.12) \quad \sup_{\varepsilon > 0} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^p} \rho_\varepsilon(|x - y|) dx dy = k(p, N) \|\nabla u\|_{L^p(\mathbb{R}^N)}^p,$$

where  $k(p, N)$  is defined in (1.7).

A proof of the upper bound in (3.12) may be found e.g. in [39], while the lower bound is a consequence of the BBM formula, which asserts that (3.12) holds with  $\sup_{\varepsilon > 0}$  replaced by  $\lim_{\varepsilon \rightarrow 0}$ . Choosing in particular  $\rho_\varepsilon(r) = \frac{N}{\varepsilon^N} \mathbf{1}_{(0, \varepsilon)}(r)$  yields

$$\sup_{\varepsilon > 0} \frac{1}{\varepsilon^N} \iint_{|x-y| < \varepsilon} \frac{|u(x) - u(y)|^p}{|x - y|^p} dx dy = \frac{k(p, N)}{N} \|\nabla u\|_{L^p(\mathbb{R}^N)}^p.$$

A final comment concerning the upper bound in (3.12) is in order. It is standard and straightforward (see e.g. [9, Proposition 9.3]) that

$$(3.13) \quad \int_{\mathbb{R}^N} |u(x+h) - u(x)|^p dx \leq |h|^p \int_{\mathbb{R}^N} |\nabla u(x)|^p dx \quad \forall p \geq 1, \forall h \in \mathbb{R}^N,$$

which implies that, for every  $m \in L^1(\mathbb{R}^N)$

$$(3.14) \quad \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^p} m(x - y) dx dy \leq \|m\|_{L^1(\mathbb{R}^N)} \|\nabla u\|_{L^p(\mathbb{R}^N)}^p.$$

Choosing  $m(z) = \rho_\varepsilon(|z|)$  where  $\rho_\varepsilon$  satisfies (3.11) yields

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^p} \rho_\varepsilon(|x - y|) dx dy \leq \mathcal{L}^{N-1}(\mathbb{S}^{N-1}) \|\nabla u\|_{L^p(\mathbb{R}^N)}^p$$

which misses, when  $N > 1$ , the best constant  $k(p, N)$  proper to radial  $\rho$ 's.

#### 4. GENERALIZATIONS OF THEOREMS 1.1 AND 1.2 TO WEIGHTED INTEGRALS

Theorem 1.1 and Theorem 1.2 can be viewed as special cases of the following results concerning weighted integrals.

**Theorem 4.1.** *For every  $N \geq 1$ , there exists a constant  $C = C(N)$  such that for all  $\gamma > 0$ ,  $1 \leq p < \infty$ ,  $u \in C_c^\infty(\mathbb{R}^N)$  and  $\lambda > 0$ ,*

$$(4.1) \quad \lambda^p \iint_{E_{\lambda, \gamma/p}} |x - y|^{\gamma - N} dx dy \leq C \frac{3^\gamma}{\gamma} \|\nabla u\|_{L^p(\mathbb{R}^N)}^p.$$

where

$$(4.2) \quad E_{\lambda, \gamma/p} := \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y, |u(x) - u(y)| \geq \lambda |x - y|^{\frac{\gamma}{p} + 1}\}.$$

As a result, for any  $N \geq 1$  and any  $\gamma > 0$ , if  $\mu$  is the measure  $|x - y|^{\gamma - N} dx dy$  on  $\mathbb{R}^N \times \mathbb{R}^N$ , then for any  $u \in C_c^\infty(\mathbb{R}^N)$  and any  $1 \leq p < \infty$ , the Marcinkiewicz  $M^p(\mathbb{R}^N \times \mathbb{R}^N, \mu)$  quasi-norm of the function  $|u(x) - u(y)|/|x - y|^{\frac{\gamma}{p} + 1}$  is bounded by a multiple of  $\|\nabla u\|_{L^p(\mathbb{R}^N)}$ . The case  $\gamma = N$  of Theorem 4.1 is precisely the upper bound in Theorem 1.1. The proof of Theorem 4.1 is similar to the proof of the upper bound in Theorem 1.1, using instead of Proposition 2.2 the following generalization of Propositions 2.1 and 2.2:

**Proposition 4.2.** *For every  $N \geq 1$ , there exists a constant  $C = C(N)$  such that for all  $\gamma > 0$  and all  $F \in C_c(\mathbb{R}^N)$ ,*

$$\iint_{E(F, \gamma)} |x - y|^{\gamma - N} dx dy \leq C \frac{3^\gamma}{\gamma} \|F\|_{L^1(\mathbb{R}^N)},$$

where

$$E(F, \gamma) := \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y, \left| \int_y^x F \right| \geq |x - y|^{\gamma + 1} \right\}.$$

The case  $N = 1$  (respectively  $\gamma = N$ ) of Proposition 4.2 is precisely Proposition 2.1 (respectively 2.2). When  $N > 1$  Proposition 4.2 can be deduced from Proposition 2.1 via an argument similar to the proof of Proposition 2.2.

Next we have the following generalization of Theorem 1.2 with essentially the same proof (the case  $\gamma = N$  of Theorem 4.3 is precisely Theorem 1.2).

**Theorem 4.3.** *Let  $N \geq 1$ ,  $\gamma > 0$ ,  $1 \leq p < \infty$  and  $u \in C_c^\infty(\mathbb{R}^N)$ . Then*

$$(4.3) \quad \lim_{\lambda \rightarrow +\infty} \lambda^p \iint_{E_{\lambda, \gamma/p}} |x - y|^{\gamma - N} dx dy = \frac{1}{\gamma} k(p, N) \|\nabla u\|_{L^p(\mathbb{R}^N)}^p$$

where  $E_{\lambda, \gamma/p}$  is defined in (4.2) and  $k(p, N)$  comes from (1.7).

*Remark 4.4.* There is a ‘‘family resemblance’’ between the above results and earlier results of H.-M. Nguyen [31–33]; see also [12, 13]. They assert in particular that for all  $u \in C_c^\infty(\mathbb{R}^N)$  and  $p \geq 1$ ,

$$(4.4) \quad \lim_{\delta \searrow 0} \delta^p \iint_{|u(x) - u(y)| > \delta} \frac{dx dy}{|x - y|^{N+p}} = \frac{1}{p} k(p, N) \|\nabla u\|_{L^p(\mathbb{R}^N)}^p.$$

Note that (4.4) is analogous to (4.3) with  $\gamma = -p$ , replacing  $\gamma$  by  $|\gamma|$  on the right hand side and taking the limit as  $\lambda \searrow 0$  (instead of  $\lambda \rightarrow +\infty$ ) on the left hand side. We plan to return in the future to similar questions involving negative  $\gamma$ .

## 5. PROOFS OF COROLLARIES 1.5 AND 1.6

### 5.1. Proof of Corollary 1.5

Corollary 1.5 can be derived as an immediate consequence of (the upper bound in) Theorem 1.1 (applied with  $p = 1$ ) and the fact that

$$\frac{|u(x) - u(y)|}{|x - y|^{\frac{N+1}{p}}} \geq \lambda \quad \text{implies} \quad \frac{|u(x) - u(y)|}{|x - y|^{N+1}} \geq \frac{\lambda^p}{(2\|u\|_{L^\infty})^{p-1}}.$$

Hence

$$\mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N+1}{p}}} \geq \lambda \right\} \right) \leq \frac{2^{p-1}C}{\lambda^p} \|u\|_{L^\infty(\mathbb{R}^N)}^{p-1} \|\nabla u\|_{L^1(\mathbb{R}^N)},$$

where  $C = C(N)$  is as in Theorem 1.1; note that  $(2^{p-1}C)^{1/p}$  can be dominated by a constant depending only on  $N$ . This proves Corollary 1.5.  $\square$

For the enjoyment of the reader we also present an elementary qualitative argument for Corollary 1.5 which does not make use of Theorem 1.1. It relies on the following estimate occurring in [6]; unfortunately it yields a constant  $C$  in (1.16) which depends on  $p$  and  $N$ , and which deteriorates as  $p \searrow 1$ . Note that (5.1) is a straightforward consequence of the inequality

$$\int_{\mathbb{R}^N} |u(x+h) - u(x)| dx \leq |h| \int_{\mathbb{R}^N} |\nabla u|, \quad \forall h \in \mathbb{R}^N, \quad \forall u \in C_c^\infty(\mathbb{R}^N).$$

**Lemma 5.1.** *For every  $u \in C_c^\infty(\mathbb{R}^N)$  and  $\rho \in L^1(\mathbb{R}^N)$ ,*

$$(5.1) \quad \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|}{|x - y|} \rho(x - y) dx dy \leq \|\rho\|_{L^1(\mathbb{R}^N)} \int_{\mathbb{R}^N} |\nabla u|,$$

and in particular choosing  $\rho(z) = \mathbf{1}_{B_r(0)}(z)/|z|^{N-\delta}$ ,  $\delta > 0$ , we obtain

$$(5.2) \quad \iint_{\substack{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N \\ |x-y| \leq r}} \frac{|u(x) - u(y)|}{|x - y|^{N+1-\delta}} dx dy \leq C(N) \frac{r^\delta}{\delta} \int_{\mathbb{R}^N} |\nabla u|.$$

We now define the set

$$(5.3) \quad E_\lambda := \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N+1}{p}}} \geq \lambda \right\}.$$

Observe that

$$E_\lambda \subseteq K_\lambda := \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : |x - y| \leq \left( 2\|u\|_{L^\infty(\mathbb{R}^N)} / \lambda \right)^{\frac{p}{N+1}} \right\}.$$

Thus

$$\mathbf{1}_{E_\lambda} \leq \mathbf{1}_{K_\lambda} \frac{1}{\lambda} \frac{|u(x) - u(y)|}{|x - y|^{\frac{N+1}{p}}}.$$

which implies

$$\mathcal{L}^{2N}(E_\lambda) \leq \frac{1}{\lambda} \iint_{K_\lambda} \frac{|u(x) - u(y)|}{|x - y|^{\frac{N+1}{p}}} dx dy.$$

It then follows by (5.2), with  $\delta := (N+1)(1 - \frac{1}{p}) > 0$  and  $r := (2\|u\|_{L^\infty(\mathbb{R}^N)} / \lambda)^{\frac{p}{N+1}}$ , that

$$(5.4) \quad \mathcal{L}^{2N}(E_\lambda) \leq \frac{C(N) (2\|u\|_{L^\infty(\mathbb{R}^N)})^{p-1}}{(N+1)(1 - \frac{1}{p})\lambda^p} \int_{\mathbb{R}^N} |\nabla u|.$$

□

To conclude this subsection we mention another estimate in the spirit of Gagliardo–Nirenberg interpolation between  $L^\infty$  and  $W^{1,1}$ . It is originally due to Figalli–Serra [23, Lemma 3.1] when  $p = 2$  and  $q = \infty$ , with roots in Figalli–Jerison [22, Lemma 2.1] (see also [24, Lemma 2.2 and Corollary 2.3] for a simpler proof and more general version).

**Corollary 5.2.** *Let  $N \geq 1$ ,  $1 < p < \infty$  and  $N < q \leq \infty$ . There exists a constant  $C = C(N, p, q)$  such that*

$$(5.5) \quad \iint_{B_1 \times B_1} \frac{|u(x) - u(y)|^p}{|x - y|^{N+1}} dx dy \leq C \|u\|_{L^\infty(B_1)}^{p-1} \|\nabla u\|_{L^1(B_1)} \left( 1 + \log \max \left\{ \frac{\|\nabla u\|_{L^q(B_1)}}{\|u\|_{L^\infty(B_1)}}, 1 \right\} \right)$$

for every  $u \in C^1(\overline{B_1})$ .

Here  $B_1$  denotes the unit ball in  $\mathbb{R}^N$ .

*Proof.* We may always extend  $u$  to  $B_3$  with control of norms and assume that  $\|u\|_{L^\infty(B_1)} = 1$ . By the Sobolev–Morrey embedding we have (since  $q > N$ )

$$(5.6) \quad |u(x) - u(y)| \leq C \min \left\{ 1, |x - y|^\alpha \|\nabla u\|_{L^q(B_1)} \right\}, \quad \text{for all } x, y \in B_1,$$

where  $\alpha = 1 - \frac{N}{q}$ . Thus

$$|u(x) - u(y)|^p \leq C \min \left\{ 1, |x - y|^{\alpha(p-1)} \|\nabla u\|_{L^q(B_1)}^{p-1} \right\} |u(x) - u(y)|$$

and therefore

$$(5.7) \quad \iint_{B_1 \times B_1} \frac{|u(x) - u(y)|^p}{|x - y|^{N+1}} dx dy \leq C \int_{B_2} dh \int_{B_1} \min\{1, |h|^{\alpha(p-1)} \|\nabla u\|_{L^q(B_1)}^{p-1}\} \frac{|u(x+h) - u(x)|}{|h|^{N+1}} dx.$$

Since

$$\int_{B_1} |u(x+h) - u(x)| dx \leq |h| \|\nabla u\|_{L^1(B_3)} \leq C|h| \|\nabla u\|_{L^1(B_1)}, \quad \text{for all } h \in B_2,$$

we conclude that

$$(5.8) \quad \iint_{B_1 \times B_1} \frac{|u(x) - u(y)|^p}{|x - y|^{N+1}} dx dy \leq C \|\nabla u\|_{L^1(B_1)} \int_{B_2} \min\{1, |h|^{\alpha(p-1)} \|\nabla u\|_{L^q(B_1)}^{p-1}\} \frac{dh}{|h|^N} \\ = C' \|\nabla u\|_{L^1(B_1)} \int_0^2 \min\{1, r^{\alpha(p-1)} \|\nabla u\|_{L^q(B_1)}^{p-1}\} \frac{dr}{r}$$

and the conclusion follows from a straightforward computation.  $\square$

## 5.2. Proof of Corollary 1.6

Note that by (1.17),

$$\frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p} + s}} = \left( \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p_1} + s_1}} \right)^\theta \left( \frac{|u(x) - u(y)|}{|x - y|^{N+1}} \right)^{1-\theta}.$$

Since

$$(5.9) \quad \frac{1}{p} = \frac{\theta}{p_1} + (1 - \theta),$$

by Hölder's inequality for Lorentz spaces,

$$(5.10) \quad \left[ \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p} + s}} \right]_{M^p(\mathbb{R}^N \times \mathbb{R}^N)} \\ \leq 2^{1/p} \left[ \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p_1} + s_1}} \right]_{M^{p_1}(\mathbb{R}^N \times \mathbb{R}^N)}^\theta \left[ \frac{|u(x) - u(y)|}{|x - y|^{N+1}} \right]_{M^1(\mathbb{R}^N \times \mathbb{R}^N)}^{1-\theta}.$$

Indeed, for any  $\lambda > 0$ , we have

$$\left\{ \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p} + s}} \geq \lambda \right\} \subseteq \left\{ \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p_1} + s_1}} \geq A^{1-\theta} \lambda \right\} \cup \left\{ \frac{|u(x) - u(y)|}{|x - y|^{N+1}} \geq A^{-\theta} \lambda \right\}$$

for any  $A > 0$ , and the  $\mathcal{L}^{2N}$ -measure of the right hand side is bounded by

$$\frac{G^{p_1}}{A^{(1-\theta)p_1} \lambda^{p_1}} + \frac{A^\theta H}{\lambda},$$

where  $G$  and  $H$  are shorthands for

$$G := \left[ \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p_1} + s_1}} \right]_{M^{p_1}(\mathbb{R}^N \times \mathbb{R}^N)}, \quad H := \left[ \frac{|u(x) - u(y)|}{|x - y|^{N+1}} \right]_{M^1(\mathbb{R}^N \times \mathbb{R}^N)}.$$

We take  $A$  so that  $A^{\theta+(1-\theta)p_1} = \left(\frac{G}{\lambda}\right)^{p_1} \left(\frac{H}{\lambda}\right)^{-1}$ , which in light of (5.9) says  $A^{\frac{p_1}{p}} = \left(\frac{G}{\lambda}\right)^{p_1} \left(\frac{H}{\lambda}\right)^{-1}$ . Then

$$\frac{G^{p_1}}{A^{(1-\theta)p_1} \lambda^{p_1}} + \frac{A^\theta H}{\lambda} = 2 \frac{A^\theta H}{\lambda} = 2 \left(\frac{G}{\lambda}\right)^{p\theta} \left(\frac{H}{\lambda}\right)^{1-\frac{p\theta}{p_1}} = \frac{2}{\lambda^p} G^{p\theta} H^{p(1-\theta)},$$

where we have used (5.9) again in the last equality to show that  $1 - \frac{p\theta}{p_1} = p\left(\frac{1}{p} - \frac{\theta}{p_1}\right) = p(1 - \theta)$ . This yields

$$\begin{aligned} \mathcal{L}^{2N} \left( \left\{ \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p} + s}} \geq \lambda \right\} \right) \\ \leq \frac{2}{\lambda^p} \left[ \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p_1} + s_1}} \right]_{M^{p_1}(\mathbb{R}^N \times \mathbb{R}^N)}^{p\theta} \left[ \frac{|u(x) - u(y)|}{|x - y|^{N+1}} \right]_{M^1(\mathbb{R}^N \times \mathbb{R}^N)}^{p(1-\theta)} \end{aligned}$$

and (5.10) follows.

Going back to (5.10), we may make the first factor larger by replacing the  $M^{p_1}$  quasi-norm with the  $L^{p_1}$  norm, and bound the second factor by Theorem 1.1 applied with  $p = 1$ . Thus the right hand side of (5.10) is

$$\leq C(N) \|u\|_{W^{s_1, p_1}(\mathbb{R}^N)}^\theta \|\nabla u\|_{L^1(\mathbb{R}^N)}^{1-\theta},$$

as desired. □

*Remark 5.3.* The proof of Corollary 1.6 works as well when  $s_1 p_1 < 1$ , but in this case we have the better strong-type estimate (1.18) (which requires some work [17]).

*Remark 5.4.* A weaker version of Corollary 1.6 can also be proved relying on Lemma 5.1 instead of Theorem 1.1 by starting from the estimate

$$\begin{aligned} \mathcal{L}^{2N} \left( \left\{ \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p} + s}} \geq \lambda \right\} \right) &\leq \left( \frac{1}{\lambda} \iint_{\substack{x, y \in \mathbb{R}^N \times \mathbb{R}^N \\ |x - y| \leq r}} \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p} + s}} dx dy \right. \\ &\quad \left. + \frac{1}{\lambda^{p_1}} \iint_{\substack{x, y \in \mathbb{R}^N \times \mathbb{R}^N \\ |x - y| > r}} \frac{|u(x) - u(y)|^{p_1}}{r^\beta |x - y|^{N + s_1 p_1}} dy dx \right), \end{aligned}$$

where  $\beta := p_1(1 - \theta)(N(1 - \frac{1}{p_1}) + 1 - s_1)$ , applying Lemma 5.1 and optimizing the right-hand side with respect to  $r$ . With this method we obtain (1.19) with a constant which (unfortunately!) deteriorates as  $\theta \searrow 0$ .

## 6. OPTIMALITY OF THEOREM 1.1 AND COROLLARIES 1.3 AND 1.5 IN THE LORENTZ SCALE

We now turn to refinements of the previous results in the scale of Lorentz spaces. Recall that the Lorentz space  $L^{p,q}(X, \mu)$ , with  $1 \leq p < \infty$  and  $1 \leq q \leq \infty$ , is characterized by (see for example [16, Chapter 6; 25, Section 1.4; 28; 40, Section 1.8]), when  $q < \infty$

$$(6.1) \quad [f]_{L^{p,q}(X, \mu)}^q = p \int_0^\infty \lambda^q \mu(\{x \in X : |f(x)| \geq \lambda\})^{\frac{q}{p}} \frac{d\lambda}{\lambda} < +\infty,$$

and when  $q = \infty$  by  $[f]_{L^{p,\infty}(X, \mu)} = [f]_{M^p(X, \mu)} < +\infty$ .

Since  $L^{p,\infty} = M^p$ , Theorem 1.1 and Corollaries 1.3, 1.5 and 1.6 can be restated equivalently as Lorentz spaces estimates. One may wonder whether they can be improved in the Lorentz scale. (Recall that for any fixed  $p$  the Lorentz spaces  $L^{p,q}$  increase as  $q$  increases.) We will see below that Theorem 1.1 and Corollaries 1.3 and 1.5 cannot be improved in the Lorentz scale. On the other hand, improvement for Corollary 1.6 is possible, and the details are presented in Section 7.

### 6.1. Optimality of Theorem 1.1

Obviously the lower bound in Theorem 1.1 cannot be improved since  $M^p = L^{p,\infty}$  is larger than any  $L^{p,q}$  space (for any fixed  $p$ ). The upper bound in Theorem 1.1 also cannot be improved. This is a consequence of the following lemma.

**Lemma 6.1.** *Assume that  $1 \leq p < \infty$  and  $1 \leq q < \infty$ . Then*

$$(6.2) \quad \left[ \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p} + 1}} \right]_{L^{p,q}(\mathbb{R}^N \times \mathbb{R}^N)} = \infty, \quad \forall u \in C_c^\infty(\mathbb{R}^N), u \not\equiv 0.$$

*Proof.* Suppose that for some  $u \in C_c^\infty(\mathbb{R}^N)$ ,

$$\left[ \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p} + 1}} \right]_{L^{p,q}(\mathbb{R}^N \times \mathbb{R}^N)} < \infty.$$

It follows immediately from definition (6.1) that

$$(6.3) \quad \liminf_{\lambda \rightarrow \infty} \lambda^p \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p} + 1}} \geq \lambda \right\} \right) = 0,$$

and we deduce from Theorem 1.2 that  $\|\nabla u\|_{L^p(\mathbb{R}^N)} = 0$ , so that  $u \equiv 0$ .  $\square$

## 6.2. Optimality of Corollaries 1.3 and 1.5

Corollaries 1.3 and 1.5 cannot be improved. This is a consequence of the following lemma and its proof.

**Lemma 6.2.** *Assume that  $1 \leq p < \infty$ . If*

$$(6.4) \quad \left[ \frac{u(x) - u(y)}{|x - y|^{\frac{N+1}{p}}} \right]_{L^{p,q}(\mathbb{R}^N \times \mathbb{R}^N)} \leq C \|u\|_{L^\infty(\mathbb{R}^N)}^{1-\frac{1}{p}} \|\nabla u\|_{L^1(\mathbb{R}^N)}^{\frac{1}{p}}, \quad \forall u \in C_c^\infty(\mathbb{R}^N)$$

*holds for some  $1 \leq q \leq \infty$ , then  $q = \infty$ .*

*Proof.* We consider the case  $N = 1$ , the case  $N > 1$  being similar. By an approximation argument, it follows that (6.4) holds for every  $u \in BV(\mathbb{R})$  with compact support. However, if  $u := \mathbf{1}_{[0,1]}$ , we have

$$\begin{aligned} & \left\{ (x, y) \in (-1, 0) \times (0, 1) : |x - y| \leq \lambda^{-p/2} \right\} \\ & \subseteq E_\lambda := \left\{ (x, y) \in \mathbb{R} \times \mathbb{R} : \frac{|u(x) - u(y)|}{|x - y|^{\frac{2}{p}}} \geq \lambda \right\}, \end{aligned}$$

and thus, if  $\lambda \geq 1$ ,

$$\mathcal{L}^2(E_\lambda) \geq \frac{c}{\lambda^p}.$$

Hence, if  $1 \leq q < \infty$ ,

$$\left[ \frac{u(x) - u(y)}{|x - y|^{2/p}} \right]_{L^{p,q}(\mathbb{R} \times \mathbb{R})}^q = p \int_0^\infty \lambda^q \mathcal{L}^2(E_\lambda)^{\frac{q}{p}} \frac{d\lambda}{\lambda} \geq p c^{\frac{q}{p}} \int_1^\infty \frac{d\lambda}{\lambda} = \infty,$$

which contradicts (6.4).  $\square$

### 6.3. Further thoughts

The above proof of Lemma 6.1 suggests interesting directions of research. In particular, one may wonder whether any measurable function  $u$  satisfying (6.3) for some  $p \geq 1$  must be a constant. Here is a partial answer related to property (1.2) mentioned in the Introduction.

**Proposition 6.3.** *Let  $1 < p < \infty$ . If  $u : \mathbb{R}^N \rightarrow \mathbb{R}$  is measurable and if*

$$(6.5) \quad \lim_{\lambda \rightarrow \infty} \lambda^p \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p} + 1}} \geq \lambda \right\} \right) = 0,$$

*then  $u$  is constant.*

*Proof.* Let  $E_\lambda \subset \mathbb{R}^N \times \mathbb{R}^N$  denote the set in the left-hand side of (6.5). First observe that for each  $\lambda > 0$ ,

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \left( \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p} + 1}} - \lambda \right)_+ dx dy \leq \int_\lambda^\infty \mathcal{L}^{2N}(E_t) dt \leq \frac{1}{(p-1)\lambda^{p-1}} \sup_{t \geq \lambda} t^p \mathcal{L}^{2N}(E_t).$$

Hence, we have

$$(6.6) \quad \lim_{\lambda \rightarrow \infty} \lambda^{p-1} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \left( \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p} + 1}} - \lambda \right)_+ dx dy = 0.$$

We next use an argument similar to the one in [20, 35] and [38, Proof of Proposition 5.1]. From the triangle inequality and change of variable, we obtain

$$(6.7) \quad \begin{aligned} & \iint_{\mathbb{R}^N \times \mathbb{R}^N} \left( \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p} + 1}} - \lambda \right)_+ dx dy \\ & \leq \iint_{\mathbb{R}^N \times \mathbb{R}^N} \left( \frac{|u(x) - u(\frac{x+y}{2})|}{|x - y|^{\frac{N}{p} + 1}} - \frac{\lambda}{2} \right)_+ dx dy + \iint_{\mathbb{R}^N \times \mathbb{R}^N} \left( \frac{|u(\frac{x+y}{2}) - u(y)|}{|x - y|^{\frac{N}{p} + 1}} - \frac{\lambda}{2} \right)_+ dx dy \\ & = 2^{\frac{N}{p}(p-1)} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \left( \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p} + 1}} - 2^{\frac{N}{p}} \lambda \right)_+ dx dy. \end{aligned}$$

Iterating (6.7), we have in view of (6.6),

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \left( \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p} + 1}} - \lambda \right)_+ dx dy = 0, \quad \forall \lambda > 0,$$

from which it follows that  $u$  is constant. □

Here are natural questions related to Proposition 6.3:

**Open Problem 1.** Does the conclusion of Proposition 6.3 still hold when  $p = 1$ ?

**Open Problem 2.** Does the conclusion of Proposition 6.3 still hold if “lim” is replaced by “lim inf” in (6.5)?

## 7. IMPROVING COROLLARY 1.6 IN THE LORENTZ SCALE

Our main improvement of Corollary 1.6 is the following

**Corollary 7.1.** *For every  $N \geq 1$ ,  $p_1 \in (1, \infty)$ ,  $\theta \in (0, 1)$  and  $q \in [p_1/\theta, \infty)$ , there exists a constant  $C = C(N, p_1, \theta, q)$  such that for all  $s_1 \in (0, 1)$  with  $s_1 p_1 \geq 1$ , we have*

$$\left[ \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p} + s}} \right]_{L^{p,q}(\mathbb{R}^N \times \mathbb{R}^N)} \leq C |u|_{W^{s_1, p_1}(\mathbb{R}^N)}^\theta \|\nabla u\|_{L^1(\mathbb{R}^N)}^{1-\theta}, \quad \forall u \in C_c^\infty(\mathbb{R}^N),$$

where  $0 < s < 1$  and  $1 < p < \infty$  are defined by (1.17).

Note that the choice  $q = p$  is *not admissible* since  $p < \frac{p_1}{\theta}$ . This is consistent with the fact that (1.18) fails when  $s_1 p_1 \geq 1$ .

*Proof of Corollary 7.1.* We may use the same proof as Corollary 1.6: just note that if  $q \geq p_1/\theta$ , then since (1.17) holds, Hölder’s inequality for Lorentz spaces [34, Theorem 3.4] give

$$\begin{aligned} \left[ \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p} + s}} \right]_{L^{p,q}(\mathbb{R}^N \times \mathbb{R}^N)} &\lesssim_{p_1, \theta, q} \left[ \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p_1} + s_1}} \right]_{L^{p_1}(\mathbb{R}^N \times \mathbb{R}^N)}^\theta \left[ \frac{|u(x) - u(y)|}{|x - y|^{N+1}} \right]_{M^1(\mathbb{R}^N \times \mathbb{R}^N)}^{1-\theta} \end{aligned}$$

in place of (5.10), and the conclusion follows from Theorem 1.1.  $\square$

The “optimality” of Corollary 7.1 follows from

**Lemma 7.2.** *Fix  $N \in \mathbb{N}$ ,  $s_1 \in (0, 1)$ ,  $p_1 \in (1, \infty)$  such that  $s_1 p_1 \geq 1$  and  $\theta \in (0, 1)$ . Let  $0 < s < 1$  and  $1 < p < \infty$  be defined by (1.17). If*

$$\left[ \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p} + s}} \right]_{L^{p,q}(\mathbb{R}^N \times \mathbb{R}^N)} \leq C |u|_{W^{s_1, p_1}}^\theta \|\nabla u\|_{L^1(\mathbb{R}^N)}^{1-\theta}, \quad \forall u \in C_c^\infty(\mathbb{R}^N)$$

*holds for some  $1 \leq q \leq \infty$ , then  $q \geq \frac{p_1}{\theta}$ .*

*Proof of Lemma 7.2 when  $s_1 p_1 = 1$ .* We concentrate on the case  $N = 1$ , the case  $N > 1$  being similar. Following [10, Proof of Lemma 4.1, Step 1], we define the function

$$u_k(x) = \varphi(k(|x| - 1/2)).$$

where  $\varphi \in C^1(\mathbb{R})$ ,  $\varphi = 1$  on  $(-\infty, -1]$  and  $\varphi = 0$  on  $[1, \infty)$ . We have as in [10]

$$(7.1) \quad \|u'_k\|_{L^1(\mathbb{R})} \leq C \quad \text{and} \quad |u_k|_{W^{s_1, p_1}(\mathbb{R})} \leq C(\log k)^{\frac{1}{p_1}}.$$

Given  $\lambda > 0$ , we have since  $sp = 1$ ,

$$\begin{aligned} & \left\{ (x, y) \in [-1, 1] \times [-1, 1] : \frac{|u_k(x) - u_k(y)|}{|x - y|^{\frac{1}{p} + s}} \geq \lambda \right\} \\ & \supseteq \left\{ (x, y) \in [0, \frac{1}{2} - \frac{1}{k}] \times [\frac{1}{2} + \frac{1}{k}, 1] : |x - y| \leq \lambda^{-p/2} \right\}. \end{aligned}$$

Hence, there is  $c > 0$  such that if  $\lambda \leq (k/4)^{2/p}$ ,

$$\mathcal{L}^2(E_\lambda) \geq \frac{c}{\lambda^p}.$$

It follows from (6.1) that

$$\left[ \frac{|u_k(x) - u_k(y)|}{|x - y|^{\frac{N}{p} + s}} \right]_{L^{p, q}(\mathbb{R} \times \mathbb{R})} \geq \left( \int_1^{(k/4)^{2/p}} \frac{c \, d\lambda}{\lambda} \right)^{\frac{1}{q}} \geq c'(\log k)^{\frac{1}{q}}.$$

By assumption and by (7.1), we have

$$(\log k)^{\frac{1}{q}} \leq C(\log k)^{\frac{\theta}{p_1}},$$

and it follows thus that  $q \geq \frac{p_1}{\theta}$ . □

*Proof of Lemma 7.2 for  $s_1 p_1 > 1$ .* We concentrate on the case  $N = 1$ , the case  $N > 1$  being similar. We adapt the proof from [10, Proof of Lemma 4.1], where functions  $w_j^k$  are constructed and satisfy

$$(7.2) \quad \|w_j^k\|_{L^1([0,1])} = 1, \quad \limsup_{k \rightarrow \infty} |w_j^k|_{W^{s_1, p_1}([0,1])} \leq Cj^{1/p_1}$$

and

$$(7.3) \quad \limsup_{k \rightarrow \infty} |w_j^k|_{W^{s, p}([0,1])} \geq \frac{j^{1/p}}{C}.$$

We improve (7.3) to cover the case  $q \neq p$  in the Lorentz scale  $L^{p, q}$ .

Given  $\lambda > 0$ , we have

$$\begin{aligned} & \left\{ (x, y) \in [0, 1] \times [0, 1] : \frac{|w_j^k(x) - w_j^k(y)|}{|x - y|^{\frac{2}{p}+s}} \geq \lambda \right\} \\ & \supseteq \left\{ (x, y) \in [0, 1] \times [0, 1] : |w_j^k(x) - w_j^k(y)| \geq \lambda \right\}, \end{aligned}$$

and thus if  $\lambda \leq \frac{1}{3}$ , we have

$$\mathcal{L}^2 \left( \left\{ (x, y) \in [0, 1] \times [0, 1] : \frac{|w_j^k(x) - w_j^k(y)|}{|x - y|^{\frac{1}{p}+s}} \geq \lambda \right\} \right) \geq c$$

for some constant  $c > 0$ .

Next by the inductive definition of  $w_j^k$  and by scaling, we have

$$\begin{aligned} & \mathcal{L}^2 \left( \left\{ (x, y) \in [0, 1] \times [0, 1] : \frac{|w_j^k(x) - w_j^k(y)|}{|x - y|^{\frac{1}{p}+s}} \geq \lambda \right\} \right) \\ & \geq \sum_{\ell=1}^k \mathcal{L}^2 \left( \left\{ (x, y) \in I_k^\ell \times I_k^\ell : \frac{|w_j^k(x) - w_j^k(y)|}{|x - y|^{\frac{1}{p}+s}} \geq \lambda \right\} \right) \\ & \geq \frac{1}{k^{\frac{2}{\alpha}-1}} \mathcal{L}^2 \left( \left\{ (x, y) \in [0, 1] \times [0, 1] : \frac{|w_{j-1}^k(x) - w_{j-1}^k(y)|}{|x - y|^{\frac{1}{p}+s}} \geq \frac{\lambda}{k^{\frac{1}{p}(\frac{2}{\alpha}-1)}} \right\} \right), \end{aligned}$$

where  $\alpha := \frac{s - \frac{1}{p}}{1 - \frac{1}{p}}$ , since

$$\frac{\frac{1}{p} + s}{\alpha} - 1 = \frac{2}{p\alpha} + \frac{s - \frac{1}{p}}{\alpha} - 1 = \frac{1}{p} \left( \frac{2}{\alpha} - 1 \right).$$

By induction, for each  $i \in \{1, \dots, j\}$  and  $\lambda \leq k^{\frac{i-1}{p}(\frac{2}{\alpha}-1)}/3$ , we have

$$\mathcal{L}^2 \left( \left\{ (x, y) \in [0, 1] \times [0, 1] : \frac{|w_j^k(x) - w_j^k(y)|}{|x - y|^{\frac{1}{p}+s}} \geq \lambda \right\} \right) \geq \frac{c}{k^{(i-1)(\frac{2}{\alpha}-1)}}.$$

We finally estimate in view of (6.1)

$$(7.4) \quad \left[ \frac{w_j^k(x) - w_j^k(y)}{|x - y|^{\frac{1}{p}+s}} \right]_{L^{p,q}([-1,1] \times [-1,1])} \geq c' \sum_{i=1}^j \int_{k^{\frac{i-1}{p}(\frac{2}{\alpha}-1)}/3}^{k^{\frac{i}{p}(\frac{2}{\alpha}-1)}/3} \frac{\lambda^{q-1}}{k^{\frac{q}{p}(i-1)(\frac{2}{\alpha}-1)}} d\lambda \geq c'' j.$$

The conclusion follows from the assumptions combined with the estimates (7.2) and (7.4).  $\square$

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