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Non-local, non-convex functionals converging to Sobolev norms

Haïm Brezis [a,b,c], Hoai-Minh Nguyen [d,*]

[a] Department of Mathematics, Rutgers University, Hill Center, Busch Campus, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA
[b] Departments of Mathematics and Computer Science, Technion, Israel Institute of Technology, 32.000 Haifa, Israel
[c] Laboratoire Jacques-Louis Lions, Sorbonne Universités, UPMC Université Paris-6, 4 place Jussieu, 75005 Paris, France
[d] Department of Mathematics, EPFL SB CAMA, Station 8 CH-1015 Lausanne, Switzerland

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A B S T R A C T

We study the pointwise convergence and the Γ-convergence of a family of non-local, non-convex functionals Λ_δ in $L^p(\Omega)$ for $p > 1$. We show that the limits are multiples of $\int_\Omega |\nabla u|^p$. This is a continuation of our previous work where the case $p = 1$ was considered.

1. Introduction and statement of the main results

Assume that $\varphi : [0, +\infty) \to [0, +\infty)$ is defined at *every* point of $[0, +\infty)$, φ is continuous on $[0, +\infty)$ except at a finite number of points in $(0, +\infty)$ where it admits a limit from the left and from the right, and $\varphi(0) = 0$. Let $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ denote a domain which is either bounded and smooth, or $\Omega = \mathbb{R}^d$. Given a measurable function u on Ω, and a parameter $\delta > 0$, we define the following non-local functionals, for $p > 1$,

$$\Lambda(u, \Omega) := \int_\Omega \int_\Omega \frac{\varphi(|u(x) - u(y)|)}{|x - y|^{p+d}} \, dx \, dy \quad \text{and} \quad \Lambda_\delta(u, \Omega) := \delta^p \Lambda(u/\delta, \Omega). \tag{1.1}$$

To simplify the notation, we will often delete Ω and write $\Lambda_\delta(u)$ instead of $\Lambda_\delta(u, \Omega)$.

As in [3], we consider the following four assumptions on φ:

$$\varphi(t) \leq a t^{p+1} \text{ in } [0, 1] \text{ for some positive constant } a, \tag{1.2}$$

$$\varphi(t) \leq b \text{ in } \mathbb{R}_+ \text{ for some positive constant } b, \tag{1.3}$$

$$\varphi \text{ is non-decreasing}, \tag{1.4}$$

* Corresponding author at: Department of Mathematics, EPFL SB CAMA, Station 8 CH-1015 Lausanne, Switzerland.
E-mail addresses: brezis@math.rutgers.edu (H. Brezis), hoai-minh.nguyen@epfl.ch (H.-M. Nguyen).

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and

$$\gamma_{d,p} \int_0^\infty \varphi(t) t^{-(p+1)}\, dt = 1, \text{ where } \gamma_{d,p} := \int_{\mathbb{S}^{d-1}} |\sigma \cdot e|^p \, d\sigma \text{ for some } e \in \mathbb{S}^{d-1}. \tag{1.5}$$

In this paper, we study the pointwise and the Γ-convergence of Λ_δ as $\delta \to 0$ for $p > 1$. This is a continuation of our previous work [3] where the case $p = 1$ was investigated in great details. Concerning the pointwise convergence of Λ_δ, our main result is

Theorem 1. *Let $d \geq 1$ and $p > 1$. Assume (1.2), (1.3), and (1.5) (the monotonicity assumption (1.4) is not required here). We have*

(i) There exists a positive constant $C_{p,\Omega}$ such that

$$\Lambda_\delta(u, \Omega) \leq C_{p,\Omega} \int_\Omega |\nabla u|^p \, dx \quad \forall\, u \in W^{1,p}(\Omega), \forall\, \delta > 0; \tag{1.6}$$

moreover,

$$\lim_{\delta \to 0} \Lambda_\delta(u, \Omega) = \int_\Omega |\nabla u|^p \, dx \quad \forall\, u \in W^{1,p}(\Omega). \tag{1.7}$$

(ii) Assume in addition that φ satisfies (1.4). Let $u \in L^p(\Omega)$ be such that

$$\liminf_{\delta \to 0} \Lambda_\delta(u, \Omega) < +\infty, \tag{1.8}$$

then $u \in W^{1,p}(\Omega)$.

Remark 1. Theorem 1 provides a characterization of the Sobolev space $W^{1,p}(\Omega)$ for $p > 1$:

$$W^{1,p}(\Omega) = \Big\{ u \in L^p(\Omega); \liminf_{\delta \to 0} \Lambda_\delta(u) < +\infty \Big\}.$$

This fact is originally due to Bourgain and Nguyen [1,4] when $\varphi = \hat{\varphi}_1 := c\mathbb{1}_{(1,+\infty)}$ for an appropriate constant c.

There are some similarities but also striking differences between the cases $p > 1$ and $p = 1$.

(a) First note a similarity. Let $p = 1$ and φ satisfy (1.2)–(1.4), and assume that $u \in L^1(\Omega)$ verifies

$$\liminf_{\delta \to 0} \Lambda_\delta(u, \Omega) < +\infty,$$

then $u \in BV(\Omega)$ (see [1,3]).

(b) Next is a major difference. Let $p = 1$. There exists $u \in W^{1,1}(\Omega)$ such that, for all φ satisfying (1.2)–(1.4), one has

$$\lim_{\delta \to 0} \Lambda_\delta(u, \Omega) = +\infty$$

[3, Pathology 1]. In particular, (1.6) and (1.7) do not hold for $p = 1$. An example in the same spirit was originally constructed by Ponce and is presented in [4]. Other pathologies occurring in the case $p = 1$ can be found in [3, Section 2.2].

As we will see later, the proof of (1.6) involves the theory of maximal functions. The use of this theory was suggested independently by Nguyen [4] and Ponce and van Schaftingen (unpublished communication to the authors). The proof of (1.6) uses the same strategy as in [4].

We point out that assertion (ii) fails without the monotonicity condition (1.4) on φ. Here is an example e.g. with $\Omega = \mathbb{R}$. Let $\varphi = c\mathbb{1}_{(1,2)}$ for an appropriate, positive constant c. Let $u = \mathbb{1}_{(0,1)}$. One can easily check that $\Lambda_\delta(u) = 0$ for $\delta \in (0, 1/2)$ and it is clear that $u \notin W^{1,p}(\mathbb{R})$ for $p > 1$.

Concerning the Γ-convergence of Λ_δ, our main result is

Theorem 2. *Let $d \geq 1$ and $p > 1$. Assume (1.2)–(1.5). Then*

$$\Lambda_\delta(\cdot, \Omega) \ \Gamma\text{-converges in } L^p(\Omega) \text{ to } \Lambda_0(\cdot, \Omega) := \kappa \int_\Omega |\nabla \cdot|^p \, dx,$$

as $\delta \to 0$, for some constant κ which depends only on p and φ, and verifies

$$0 < \kappa \leq 1. \tag{1.9}$$

Theorem 2 was known earlier when $\varphi = \hat{\varphi}_1$ [5,6].

The paper is organized as follows. Theorem 1 is proved in Section 2 and the proof of Theorem 2 is given in Section 3. Throughout the paper, we denote

$$\varphi_\delta(t) := \delta^p \varphi(t/\delta) \text{ for } p > 1, \delta > 0, t \geq 0.$$

2. Proof of Theorem 1

In view of the fact that $\liminf_{t \to +\infty} \varphi(t) > 0$, assertion (1.8) is a direct consequence of [1, Theorem 1]; note that [1, Theorem 1] is stated for $\Omega = \mathbb{R}^d$ but the proof can be easily adapted to the case where Ω is bounded. It could also be deduced from Theorem 2.

We now establish assertions (1.6) and (1.7). The proof consists of two steps.

Step 1: Proof of (1.6) and (1.7) when $\Omega = \mathbb{R}^d$ and $u \in W^{1,p}(\mathbb{R}^d)$. Replacing y by $x + z$ and using polar coordinates in the z variable, we find

$$\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} \frac{\varphi_\delta(|u(x) - u(y)|)}{|x - y|^{p+d}} \, dy = \int_{\mathbb{R}^d} dx \int_0^{+\infty} dh \int_{\mathbb{S}^{d-1}} \frac{\varphi_\delta(|u(x + h\sigma) - u(x)|)}{h^{p+1}} \, d\sigma. \tag{2.1}$$

We have

$$\begin{aligned} &\int_{\mathbb{R}^d} dx \int_0^{+\infty} dh \int_{\mathbb{S}^{d-1}} \frac{\varphi_\delta(|u(x + h\sigma) - u(x)|)}{h^{p+1}} \, d\sigma \\ &= \int_{\mathbb{R}^d} dx \int_0^{+\infty} dh \int_{\mathbb{S}^{d-1}} \frac{\delta^p \varphi\Big(|u(x + h\sigma) - u(x)|/\delta\Big)}{h^{p+1}} \, d\sigma. \end{aligned} \tag{2.2}$$

Rescaling the variable h gives

$$\begin{aligned} &\int_{\mathbb{R}^d} dx \int_0^{+\infty} dh \int_{\mathbb{S}^{d-1}} \frac{\delta^p \varphi\Big(|u(x + h\sigma) - u(x)|/\delta\Big)}{h^{p+1}} \, d\sigma \\ &= \int_{\mathbb{R}^d} dx \int_0^{+\infty} dh \int_{\mathbb{S}^{d-1}} \frac{\varphi\Big(|u(x + \delta h\sigma) - u(x)|/\delta\Big)}{h^{p+1}} \, d\sigma. \end{aligned} \tag{2.3}$$

Combining (2.1), (2.2), and (2.3) yields

$$\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} \frac{\varphi_\delta(|u(x) - u(y)|)}{|x - y|^{d+p}} \, dy = \int_{\mathbb{R}^d} dx \int_0^{+\infty} dh \int_{\mathbb{S}^{d-1}} \frac{\varphi\Big(|u(x + \delta h\sigma) - u(x)|/\delta\Big)}{h^{p+1}} \, d\sigma. \tag{2.4}$$

Note that

$$\lim_{\delta \to 0} \frac{|u(x + \delta h\sigma) - u(x)|}{\delta} = |\langle \nabla u(x), \sigma \rangle| h \text{ for a.e. } (x, h, \sigma) \in \mathbb{R}^d \times [0, +\infty) \times \mathbb{S}^{d-1}. \tag{2.5}$$

Here and in what follows, $\langle .,. \rangle$ denotes the usual scalar product in $\mathbb{R}^d$. Since φ is continuous at 0 and on $(0, +\infty)$ except at a finite number of points, it follows that

$$\begin{aligned} &\lim_{\delta \to 0} \frac{1}{h^{p+1}} \varphi\Big(|u(x + \delta h\sigma) - u(x)|/\delta\Big) = \frac{1}{h^{p+1}} \varphi\Big(|\langle \nabla u(x), \sigma \rangle| h\Big) \\ &\text{for a.e. } (x, h, \sigma) \in \mathbb{R}^d \times (0, +\infty) \times \mathbb{S}^{d-1}. \end{aligned} \tag{2.6}$$

Rescaling once more the variable h gives

$$\int_0^\infty dh \int_{\mathbb{S}^{d-1}} \frac{1}{h^{p+1}} \varphi\Big(|\langle \nabla u(x), \sigma \rangle| h \Big) \, d\sigma = |\nabla u(x)|^p \int_0^\infty \varphi(t) t^{-(p+1)} \, dt \int_{\mathbb{S}^{d-1}} |\langle \sigma, e \rangle|^p \, d\sigma; \tag{2.7}$$

here we have also used the obvious fact that, for every $V \in \mathbb{R}^d$, and for any fixed $e \in \mathbb{S}^{d-1}$,

$$\int_{\mathbb{S}^{d-1}} |\langle V, \sigma \rangle|^p \, d\sigma = |V|^p \int_{\mathbb{S}^{d-1}} |\langle e, \sigma \rangle|^p \, d\sigma.$$

Thus, by the normalization condition (1.5), we obtain

$$\int_{\mathbb{R}^d} dx \int_0^\infty dh \int_{\mathbb{S}^{d-1}} \frac{1}{h^{p+1}} \varphi\Big(|\langle \nabla u(x), \sigma \rangle| h \Big) \, d\sigma = \int_{\mathbb{R}^d} |\nabla u|^p \, dx. \tag{2.8}$$

Set

$$\widetilde{\varphi}(t) = \begin{cases} a t^{p+1} & \text{for } t \in [0, 1), \\ b & \text{for } t \in [1, +\infty). \end{cases}$$

Then

$$\widetilde{\varphi} \text{ is non-decreasing and } \varphi \le \widetilde{\varphi}. \tag{2.9}$$

Note that, for a.e. $(x, h, \sigma) \in \mathbb{R}^d \times (0, +\infty) \times \mathbb{S}^{d-1}$,

$$\frac{|u(x + \delta h \sigma) - u(x)|}{\delta} \le \frac{1}{\delta} \int_0^{h\delta} |\langle \nabla u(x + s\sigma), \sigma \rangle| \, ds \le h M(\nabla u, \sigma)(x), \tag{2.10}$$

where

$$M(\nabla u, \sigma)(x) := \sup_{t > 0} \frac{1}{t} \int_0^t |\langle \nabla u(x + s\sigma), \sigma \rangle| \, ds.$$

Combining (2.4) and (2.10), we derive from (2.9) that

$$\begin{aligned} \Lambda_\delta(u) &\le \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^d} \int_0^\infty \frac{\widetilde{\varphi}(h |M(\nabla u, \sigma)(x)|)}{h^{p+1}} \, dh \, dx \, d\sigma \\ &= \int_0^{+\infty} \widetilde{\varphi}(t) t^{-(p+1)} \, dt \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^d} |M(\nabla u, \sigma)(x)|^p \, dx \, d\sigma. \end{aligned} \tag{2.11}$$

We claim that, for $\sigma \in \mathbb{S}^{d-1}$,

$$\int_{\mathbb{R}^d} |M(\nabla u, \sigma)(x)|^p \, dx \le C_p \int_{\mathbb{R}^d} |\nabla u(x)|^p \, dx. \tag{2.12}$$

For notational ease, we will only consider the case $\sigma = e_1$. By the theory of maximal functions (see e.g. [7]), one has, for $g \in L^p(\mathbb{R})$,

$$\int_{\mathbb{R}} \left| \sup_{t > 0} ⨍_{\xi - t}^{\xi + t} |g(s)| \, ds \right|^p d\xi \le C_p \int_{\mathbb{R}} |g(\xi)|^p \, d\xi.$$

Using this inequality with $g(x_1) = \partial_{x_1} u(x_1, x')$ for $x' \in \mathbb{R}^{d-1}$, we obtain

$$\int_{\mathbb{R}} |M(\nabla u, e_1)(x_1, x')|^p \, dx_1 \le C_p \int_{\mathbb{R}} |\partial_{x_1} u(x_1, x')|^p \, dx_1.$$

Integrating with respect to x' yields

$$\int_{\mathbb{R}^d} |M(\nabla u, e_1)(x)|^p \, dx \le C_p \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} |\partial_{x_1} u(x_1, x')|^p \, dx_1 \, dx' \le C_p \int_{\mathbb{R}^d} |\nabla u(x)|^p \, dx,$$

and (2.12) follows.

Using (2.12), we deduce from (2.11) that

$$\Lambda_\delta(u) \leq C_{p,d} \int_{\mathbb{R}^d} |\nabla u|^p \, dx,$$

which is (1.6). From (2.6), (2.7), (2.8), and (2.10) we derive, using the dominated convergence theorem, that

$$\lim_{\delta \to 0} \Lambda_\delta(u) = \int_{\mathbb{R}^d} |\nabla u|^p \, dx.$$

This completes Step 1.

Step 2: Proof of (1.6) and (1.7) when Ω is bounded and $u \in W^{1,p}(\Omega)$. We first claim that

$$\lim_{\delta \to 0} \Lambda_\delta(u) = \int_\Omega |\nabla u|^p \text{ for } u \in W^{1,p}(\Omega). \tag{2.13}$$

Indeed, consider an extension of u in $\mathbb{R}^d$ which belongs to $W^{1,p}(\mathbb{R}^d)$, and is still denoted by u. By the same method as in the case $\Omega = \mathbb{R}^d$, we have

$$\lim_{\delta \to 0} \int_\Omega dx \int_{\mathbb{R}^d} \frac{\varphi_\delta(|u(x) - u(y)|)}{|x-y|^{p+d}} \, dy = \int_\Omega |\nabla u|^p dx \tag{2.14}$$

and, for $D \Subset \Omega$ and $\varepsilon > 0$,

$$\lim_{\delta \to 0} \int_D dx \int_{B(x,\varepsilon)} \frac{\varphi_\delta(|u(x) - u(y)|)}{|x-y|^{p+d}} \, dy = \int_D |\nabla u|^p \, dx. \tag{2.15}$$

Combining (2.14) and (2.15) yields (2.13).

We next show that

$$\Lambda_\delta(u) \leq C_{p,\Omega} \int_\Omega |\nabla u|^p \, dx \text{ for } u \in W^{1,p}(\Omega). \tag{2.16}$$

Without loss of generality, we may assume that $\int_\Omega u = 0$. Consider an extension U of u in $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} |\nabla U|^p \, dx \leq C_{p,\Omega} \int_\Omega |\nabla u|^p \, dx.$$

Such an extension exists since Ω is smooth and $\int_\Omega u = 0$, see, e.g., [2, Chapter 9]. Using the fact

$$\Lambda_\delta(u, \Omega) \leq \Lambda_\delta(U, \mathbb{R}^d) \leq C_{p,d} \int_{\mathbb{R}^d} |\nabla U|^p \, dx,$$

we get (2.16). The proof is complete. $\square$

3. Proof of Theorem 2

We first recall the meaning of Γ-convergence. One says that $\Lambda_\delta(\cdot, \Omega) \xrightarrow{\Gamma} \Lambda_0(\cdot, \Omega)$ in $L^p(\Omega)$ as $\delta \to 0$ if

(G1) For each $g \in L^p(\Omega)$ and for *every* family $(g_\delta) \subset L^p(\Omega)$ such that (g_δ) converges to g in $L^p(\Omega)$ as $\delta \to 0$, one has

$$\liminf_{\delta \to 0} \Lambda_\delta(g_\delta, \Omega) \geq \Lambda_0(g, \Omega).$$

(G2) For each $g \in L^p(\Omega)$, there *exists* a family $(g_\delta) \subset L^p(\Omega)$ such that (g_δ) converges to g in $L^p(\Omega)$ as $\delta \to 0$, and

$$\limsup_{\delta \to 0} \Lambda_\delta(g_\delta, \Omega) \leq \Lambda_0(g, \Omega).$$

Denote Q the unit open cube, i.e., $Q = (0,1)^d$ and set

$$U(x) = d^{-1/2} \sum_{j=1}^{d} x_j \text{ in } Q,$$

so that $|\nabla U| = 1$ in Q.

In the following two subsections, we establish properties (G1) and (G2) where κ is the constant defined by

$$\kappa = \inf \liminf_{\delta \to 0} \Lambda_\delta(v_\delta, Q). \tag{3.1}$$

Here the infimum is taken over all families of functions $(v_\delta) \subset L^p(Q)$ such that $v_\delta \to U$ in $L^p(Q)$ as $\delta \to 0$.

3.1. Proof of Property (G1)

We begin with

Lemma 1. *Let $d \geq 1$, $p > 1$, S be an open bounded subset of $\mathbb{R}^d$ with Lipschitz boundary, and let g be an affine function. Then*

$$\inf \liminf_{\delta \to 0} \Lambda_\delta(g_\delta, S) = \kappa |\nabla g|^p |S|, \tag{3.2}$$

where the infimum is taken over all families $(g_\delta) \subset L^p(S)$ such that $g_\delta \to g$ in $L^p(S)$ as $\delta \to 0$.

Proof. The proof of Lemma 1 is based on the definition of κ in (3.1) and a covering argument. It is identical to the one of the first part of [3, Lemma 6]. The details are omitted. $\square$

The proof of Property (G1) for $p > 1$ relies on the following lemma with roots in [6].

Lemma 2. *Let $d \geq 1$, $p > 1$, and $\varepsilon > 0$. There exist two positive constants $\hat{\delta}_1, \hat{\delta}_2$ such that for every open cube $\widetilde{Q}$ which is an image of Q by a dilation, for every $a \in \mathbb{R}^d$, every $b \in \mathbb{R}$, and every $h \in L^p(\widetilde{Q})$ satisfying*

$$\fint_{\widetilde{Q}} |h(x) - (\langle a, x\rangle + b)|^p \, dx \leq \hat{\delta}_1 |a|^p |\widetilde{Q}|^{p/d}, \tag{3.3}$$

one has

$$\Lambda_\delta(h, \widetilde{Q}) \geq (\kappa - \varepsilon)|a|^p |\widetilde{Q}| \text{ for } \delta \in (0, \hat{\delta}_2 |a| |\widetilde{Q}|^{1/d}). \tag{3.4}$$

Hereafter, as usual, we denote $\fint_A f = \frac{1}{|A|}\int_A f$.

Proof. By a change of variables, without loss of generality, it suffices to prove Lemma 2 in the case $\widetilde{Q} = Q$, $|a| = 1$, and $b = 0$. We prove this by contradiction. Suppose that this is not true. There exist $\varepsilon_0 > 0$, a sequence of measurable functions $(h_n) \subset L^p(Q)$, a sequence $(a_n) \subset \mathbb{R}^d$, and a sequence (δ_n) converging to 0 such that $|a_n| = 1$,

$$\int_Q |h_n(x) - \langle a_n, x\rangle|^p \leq \frac{1}{n}, \quad \text{ and } \quad \Lambda_{\delta_n}(h_n, Q) < \kappa - \varepsilon_0.$$

Without loss of generality, we may assume that (a_n) converges to a for some $a \in \mathbb{R}^d$ with $|a| = 1$. It follows that (h_n) converges to $\langle a, .\rangle$ in $L^p(Q)$. Applying Lemma 1 with $S = Q$ and $g = \langle a, \cdot\rangle$, we obtain a contradiction. The conclusion follows. $\square$

The second key ingredient in the proof of Property (G1) is the following useful property of functions in $W^{1,p}(\mathbb{R}^d)$.

Lemma 3. *Let $d \geq 1$, $p > 1$, and $u \in W^{1,p}(\mathbb{R}^d)$. Given $\varepsilon_1 > 0$, there exist a subset $B = B(\varepsilon_1)$ of Lebesgue points of u and ∇u, and an integer $\ell = \ell(\varepsilon_1) \geq 1$ such that*

$$\int_{\mathbb{R}^d \setminus B} |\nabla u|^p \, dx \leq \varepsilon_1 \int_{\mathbb{R}^d} |\nabla u|^p \, dx, \tag{3.5}$$

and, for every open cube Q' with $|Q'|^{1/d} \leq 1/\ell$ and $Q' \cap B \neq \emptyset$, and for every $x \in Q' \cap B$,

$$\frac{1}{|Q'|^p} \fint_{Q'} \left| u(y) - u(x) - \langle \nabla u(x), y - x \rangle \right|^p dy \leq \varepsilon_1 \tag{3.6}$$

and

$$|\nabla u(x)|^p \geq (1 - \varepsilon_1) \fint_{Q'} |\nabla u(y)|^p \, dy. \tag{3.7}$$

Proof. We first recall the following property of $W^{1,p}(\mathbb{R}^d)$ functions (see e.g., [8, Theorem 3.4.2]): for a.e. $x \in \mathbb{R}^d$,

$$\lim_{r \to 0} \frac{1}{r^p} \fint_{Q(x,r)} \left| u(y) - u(x) - \langle \nabla u(x), y - x \rangle \right|^p dy = 0, \tag{3.8}$$

where $Q(x,r) := x + (-r, r)^d$ for $x \in \mathbb{R}^d$ and $r > 0$.

Given $n \in \mathbb{N}$, define, for a.e. $x \in \mathbb{R}^d$,

$$\rho_n(x) = \sup \left\{ \frac{1}{r^p} \fint_{Q(x,r)} \left| u(y) - u(x) - \langle \nabla u(x), y - x \rangle \right|^p dy; \ r \in (0, 1/n) \right\} \tag{3.9}$$

and

$$\tau_n(x) = \sup \left\{ \fint_{Q(x,r)} |\nabla u(y) - \nabla u(x)|^p \, dy; r \in (0, 1/n) \right\}. \tag{3.10}$$

Note that, by (3.8), $\rho_n(x) \to 0$ for a.e. $x \in \mathbb{R}^d$ as $n \to +\infty$. We also have, $\tau_n(x) \to 0$ for a.e. $x \in \mathbb{R}^d$ as $n \to +\infty$ (and in fact at every Lebesgue point of ∇u). For $m \geq 1$, set

$$D_m = \Big\{ x \in (-m, m)^d; x \text{ is a Lebesgue point of } u \text{ and } \nabla u, \text{ and } |\nabla u(x)| \geq 1/m \Big\}.$$

Since

$$\lim_{m \to +\infty} \int_{\mathbb{R}^d \setminus D_m} |\nabla u|^p \, dx = 0,$$

there exists $m \geq 1$ such that

$$\int_{\mathbb{R}^d \setminus D_m} |\nabla u|^p \, dx \leq \frac{\varepsilon_1}{2} \int_{\mathbb{R}^d} |\nabla u|^p \, dx. \tag{3.11}$$

Fix such an m. By Egorov's theorem, there exists a subset $B \subset D_m$ such that (ρ_n) and (τ_n) converge to 0 uniformly on B, and

$$\int_{D_m \setminus B} |\nabla u|^p \, dx \leq \frac{\varepsilon_1}{2} \int_{\mathbb{R}^d} |\nabla u|^p \, dx. \tag{3.12}$$

Combining (3.11) and (3.12) yields (3.5).

By the triangle inequality, we have, for every non-empty, open cube Q' and a.e. $x \in \mathbb{R}^d$ (in particular for $x \in Q' \cap B$),

$$\left(\fint_{Q'} |\nabla u(y)|^p \, dy \right)^{1/p} \leq \left(\fint_{Q'} |\nabla u(y) - \nabla u(x)|^p \, dy \right)^{1/p} + |\nabla u(x)| \leq \frac{|\nabla u(x)|}{(1 - \varepsilon_1)^{1/p}}, \tag{3.13}$$

provided

$$\left(\fint_{Q'} |\nabla u(y) - \nabla u(x)|^p \, dy \right)^{1/p} \leq \left(\frac{1}{(1 - \varepsilon_1)^{1/p}} - 1 \right) 1/m \quad \text{and} \quad |\nabla u(x)| \geq 1/m.$$

Since (ρ_n) and (τ_n) converge to 0 uniformly on B and $|\nabla u(x)| \geq 1/m$ for $x \in B$, it follows from (3.13) that there exists an $\ell \geq 1$ such that (3.6) and (3.7) hold when $|Q'|^{1/d} \leq 1/\ell$ and $Q' \cap B \neq \emptyset$, and $x \in Q' \cap B$. The proof is complete. □

We are ready to give the

Proof of Property (G1). We only consider the case $\Omega = \mathbb{R}^d$. The other case can be handled as in [3] and is left to the reader. We follow the same strategy as in [6].

In order to establish Property (G1), it suffices to prove that

$$\liminf_{k \to +\infty} \Lambda_{\delta_k}(g_k, \mathbb{R}^d) \geq \kappa \int_{\mathbb{R}^d} |\nabla g|^p \, dx \tag{3.14}$$

for every $g \in L^p(\mathbb{R}^d)$, $(\delta_k) \subset \mathbb{R}_+$ and $(g_k) \subset L^p(\mathbb{R}^d)$ such that $\delta_k \to 0$ and $g_k \to g$ in $L^p(\mathbb{R}^d)$.

Without loss of generality, we may assume that $\liminf_{k \to +\infty} \Lambda_{\delta_k}(g_k, \mathbb{R}^d) < +\infty$. It follows from [6] that $g \in W^{1,p}(\mathbb{R}^d)$. Fix $\varepsilon > 0$ (arbitrary) and let $\hat{\delta}_1$ be the positive constant in Lemma 2. Set, for $m \geq 1$,

$$A_m = \Big\{ x \in \mathbb{R}^d;\ x \text{ is a Lebesgue point of } g \text{ and } \nabla g, \text{ and } |\nabla g(x)| \leq 1/m \Big\}.$$

Since

$$\lim_{m \to +\infty} \int_{A_m} |\nabla g|^p \, dx = 0,$$

there exists $m \geq 1$ such that

$$\int_{A_m} |\nabla g|^p \, dx \leq \frac{\varepsilon}{2} \int_{\mathbb{R}^d} |\nabla g|^p \, dx. \tag{3.15}$$

Fix such an integer m. By Lemma 3 applied to $u = g$ and $\varepsilon_1 = \min\{\varepsilon/2, \delta_1/(2m)^p\}$, there exist a subset B of Lebesgue points of g and ∇g, and a positive integer ℓ such that

$$\int_{\mathbb{R}^d \setminus B} |\nabla g|^p \, dx \leq \varepsilon_1 \int_{\mathbb{R}^d} |\nabla g|^p \, dx \leq \frac{\varepsilon}{2} \int_{\mathbb{R}^d} |\nabla g|^p \, dx, \tag{3.16}$$

and for every open cube Q' with $|Q'|^{1/d} \leq 1/\ell$ and $Q' \cap B \neq \emptyset$, and, for every $x \in Q' \cap B$,

$$\frac{1}{|Q'|^{p/d}} \fint_{Q'} \big| g(y) - g(x) - \langle \nabla g(x), y - x \rangle \big|^p \, dy \leq \varepsilon_1 \leq \hat{\delta}_1/(2m)^p \tag{3.17}$$

and

$$|\nabla g(x)|^p |Q'| \geq (1 - \varepsilon_1) \int_{Q'} |\nabla g|^p \, dy \geq (1 - \varepsilon) \int_{Q'} |\nabla g|^p \, dy. \tag{3.18}$$

Fix such a set B and such an integer ℓ. Set

$$B_m := B \setminus A_m.$$

Since $\mathbb{R}^d \setminus (B \setminus A_m) \subset (\mathbb{R}^d \setminus B) \cup A_m$, it follows that

$$\int_{\mathbb{R}^d \setminus B_m} |\nabla g|^p \, dx = \int_{\mathbb{R}^d \setminus (B \setminus A_m)} |\nabla g|^p \, dx \leq \int_{\mathbb{R}^d \setminus B} |\nabla g|^p \, dx + \int_{A_m} |\nabla g|^p \, dx.$$

We deduce from (3.15) and (3.16) that

$$\int_{\mathbb{R}^d \setminus B_m} |\nabla g|^p \, dx \leq \varepsilon \int_{\mathbb{R}^d} |\nabla g|^p \, dx. \tag{3.19}$$

Set $P_\ell = \frac{1}{\ell}\mathbb{Z}^d$. Let $\mathbf{\Omega}_\ell$ be the collection of all open cubes with side length $1/\ell$ whose vertices belong to P_ℓ and denote

$$\mathbf{J}_\ell = \Big\{ Q' \in \mathbf{\Omega}_\ell;\ Q' \cap B_m \neq \emptyset \Big\}.$$

Take $Q' \in \mathbf{J}_\ell$ and $x \in Q' \cap B_m$. Since $g_k \to g$ in $L^p(Q')$, from (3.17), we obtain, for large k,

$$\frac{1}{|Q'|^{p/d}} \fint_{Q'} \big| g_k(y) - g(x) - \langle \nabla g(x), y - x \rangle \big|^p \, dy < \hat{\delta}_1 / m^p \leq \hat{\delta}_1 |\nabla g(x)|^p,$$

since $|\nabla g(x)| \geq 1/m$ for $x \in B_m \subset \mathbb{R}^d \setminus A_m$. Next, we apply Lemma 2 with $\widetilde{Q} = Q'$, $h = g_k$, $a = \nabla g(x)$, $b = g(x)$, and large k; we have

$$\Lambda_\delta(g_k, Q') \geq (\kappa - \varepsilon) |\nabla g(x)|^p |Q'| \text{ for } \delta \in (0, \hat{\delta}_2 |\nabla g(x)|^p |Q'|^{1/d}),$$

which implies, by (3.18),

$$\liminf_{k \to +\infty} \Lambda_{\delta_k}(g_k, Q') \geq (\kappa - \varepsilon)(1 - \varepsilon) \int_{Q'} |\nabla g|^p \, dy. \tag{3.20}$$

Since

$$\liminf_{k \to +\infty} \Lambda_{\delta_k}(g_k, \mathbb{R}^d) \geq \sum_{Q' \in \mathbf{J}_\ell} \liminf_{k \to +\infty} \Lambda_\delta(g_k, Q'),$$

it follows from (3.20) that

$$\begin{aligned} \liminf_{k \to +\infty} \Lambda_{\delta_k}(g_k, \mathbb{R}^d) &\geq (\kappa - \varepsilon)(1 - \varepsilon) \sum_{Q' \in \mathbf{J}_\ell} \int_{Q'} |\nabla g|^p \, dx \\ &\geq (\kappa - \varepsilon)(1 - \varepsilon) \int_{B_m} |\nabla g|^p \, dx \overset{(3.19)}{\geq} (\kappa - \varepsilon)(1 - \varepsilon)^2 \int_{\mathbb{R}^d} |\nabla g|^p \, dx; \end{aligned}$$

in the second inequality, we have used the fact B_m is contained in $\bigcup_{Q' \in \mathbf{J}_\ell} Q'$ up to a null set. Since $\varepsilon > 0$ is arbitrary, one has

$$\liminf_{k \to +\infty} \Lambda_{\delta_k}(g_k, \mathbb{R}^d) \geq \kappa \int_{\mathbb{R}^d} |\nabla g|^p \, dx.$$

The proof is complete. □

3.2. Proof of Property (G2)

The proof of Property (G2) for $p > 1$ is the same as the one for $p = 1$ given in [3]. The details are omitted.

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