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Partial differential equations

The Plateau problem from the perspective of optimal transport

Le problème de Plateau vu dans la perspective du transport optimal

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ABSTRACT

Both optimal transport and minimal surfaces have received much attention in recent years. We show that the methodology introduced by Kantorovich on the Monge problem can, surprisingly, be adapted to questions involving least area, e.g., Plateau-type problems as investigated by Federer.

RÉSUMÉ

Le transport optimal, ainsi que les surfaces minimales, ont été abondamment étudiés au cours de ces dernières décennies. Nous mettons en évidence une analogie surprenante, au niveau méthodologique, entre l'approche de Kantorovich pour le problème de Monge et la minimisation de l'aire dans des problèmes géométriques de type Plateau étudiés par Federer.

1. Introduction

This note originates in [10], where one of the topics is the minimization of the $W^{1,1}$-energy of $\mathbb{S}^1$-valued maps with prescribed singularities. For example, if we consider a given closed curve Γ in $\mathbb{R}^3$, then

$$\inf\left\{\int_{\mathbb{R}^3} |\nabla u|;\ u \in C^\infty(\mathbb{R}^3 \setminus \Gamma; \mathbb{S}^1),\ \deg(u,\Gamma) = 1\right\} = 2\pi M_0(\Gamma), \tag{1}$$

where $M_0(\Gamma)$ is the least area spanned by Γ and $\deg(u,\Gamma)$ is the degree of u restricted to any small circle linking Γ. This formula was conjectured by Brezis, Coron and Lieb [9, formula (8.22)]. It was established in [9] for planar curves Γ; in full generality it is due to Almgren, Browder and Lieb [2].

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A similar formula holds in any dimension $N \geq 3$. The proof of "$\geq$" in [2] relies on geometric measure theory (= GMT) techniques, and in particular it uses the coarea formula. As we will see below (see Remark 1), (1) can be derived easily from the tools presented in Section 3.

In the special case $N = 2$, the role of Γ is played by finite collections of points (P_i), (N_i), $i = 1, \ldots, m$, and the condition $\deg(u, \Gamma) = 1$ is replaced by $\deg(u, P_i) = 1$, $\forall i$ (resp. $\deg(u, N_i) = -1$, $\forall i$), where $\deg(u, a)$ is the degree of u on a small circle around a. Then (1) holds with

$$M_0(\Gamma) = \min_{\sigma \in S_m} \sum_{i=1}^{m} |P_i - N_{\sigma(i)}|, \tag{2}$$

a quantity originally introduced by Monge in the context of optimal transport; the minimum in (2) is taken over the set S_m of all permutations of $\{1, \ldots, m\}$. Moreover, when $N = 2$, it is possible to establish "$\geq$" in (1) using the celebrated Kantorovich formula $M = D$ (see Theorem 1 below). This approach was originally used in [9]. It turns out that Federer, unaware of the Kantorovich formula, rediscovered it (thirty years later) using tools of GMT [15, Section 5.10].

This suggests a possible connection between three topics: optimal transport, the Plateau problem, and $\mathbb{S}^1$-valued maps. The main purpose of this note is to present a common methodology that fits both the Monge–Kantorovich problem (in its discrete version) and the Plateau problem in codimension 1 (i.e. minimizing the area of a hypersurface with given boundary). Concerning the connections with $\mathbb{S}^1$-valued maps, we refer the reader to [10]; note however that such maps occur in this paper as a *tool*, e.g., in Lemma 1.

There is a huge literature dealing with the Monge–Kantorovich optimal transport problem; see, e.g., Evans [12], Villani [31,32], Santambrogio [29], Brezis [8], and the references therein. In this note, we concentrate on the simplest possible setting, namely a finite number of points with a uniform distribution of masses. We first recall a basic result in this theory; see Theorem 1 below (as stated in [6] and [8]; see also [9] and [10]). Let $d = d(x, y)$ be a pseudometric (i.e. the distance between two distinct points can be zero) on a set Z. Let P_i, N_i, $i = 1, \ldots m$, be points in Z such that $P_i \neq N_j$, $\forall i, j$ (but we allow that $P_i = P_j$ or $N_i = N_j$ for some $i \neq j$). We introduce three quantities. The first one, denoted M (for Monge) is defined by

$$M = \min_{\sigma \in S_m} \sum_{i=1}^{m} d(P_i, N_{\sigma(i)}), \tag{3}$$

where the minimum in (3) is taken over the set S_m of all permutations of $\{1, 2, \ldots, m\}$.

The second one, denoted K (for Kantorovich), is defined by

$$K = \min \left\{ \sum_{i,j=1}^{m} a_{ij}\, d(P_i, N_j);\ A = (a_{ij}) \text{ is doubly stochastic} \right\}. \tag{4}$$

Recall that a matrix $A = (a_{ij})_{1 \leq i, j \leq m}$ is doubly stochastic (DS) if

$$a_{ij} \geq 0,\ \forall i, j,\ \sum_{i=1}^{m} a_{ij} = 1,\ \forall j,\ \sum_{j=1}^{m} a_{ij} = 1,\ \forall i.$$

Finally, define D (for duality) by

$$D = \sup_{\zeta : Z \to \mathbb{R}} \left\{ \sum_{i=1}^{m} \zeta(P_i) - \sum_{j=1}^{m} \zeta(N_j);\ |\zeta(x) - \zeta(y)| \leq d(x, y),\ \forall x, y \in Z \right\}. \tag{5}$$

Theorem 1. *We have*

$$M = K = D. \tag{6}$$

There are, by now, several proofs of Theorem 1; see, e.g., [8] and the references therein. The first goal of this note is to discuss a proof whose structure can be easily adapted to the Plateau problem, as explained in Section 3 below.

The main features of this proof, presented in Section 2, are the following ones.

1. As in [9], we use the Birkhoff theorem [5] on the extreme points of DS matrices combined with the Krein–Milman theorem [25] to prove that $M = K$.
2. We prove that $K = D$ via the analytic form of the Hahn–Banach theorem. This approach provides a natural alternative to the standard proofs; see, e.g., [31, pp. 23–25, pp. 34–36], which relies on convex analysis (resp. [12], based on linear programming). As we will see later, it fits well with the proof of $K(\Gamma) = D(\Gamma)$ in Theorem 2.

In Section 3, we turn to the Plateau problem in 3D.

Let $\Gamma \subset \mathbb{R}^3$ be a smooth compact connected oriented curve (without boundary). By the Frankl–Pontryagin theorem (see [17]; see also Seifert [30]), there exists a smooth compact oriented surface $S \subset \mathbb{R}^3$ with boundary Γ. We may thus consider the finite quantity

$$M_0(\Gamma) = \inf\{|S|;\ S \subset \mathbb{R}^3 \text{ is a smooth oriented surface such that } \partial S = \Gamma\}. \tag{7}$$

Here, $|S| = \mathscr{H}^2(S)$ is the area of S.

By Stokes' theorem, for each smooth compactly supported vector field, that we identify with a 1-form $\zeta \in C_c^\infty(\mathbb{R}^3; \Lambda^1)$, we have

$$\int_S d\zeta = \int_\Gamma \zeta. \tag{8}$$

If ν (resp. τ) denotes the orienting unit normal to S (resp. the orienting unit tangent vector to Γ), then (8) is equivalent to

$$\operatorname{curl}(\nu \mathscr{H}^2 \llcorner S) = \tau \mathscr{H}^1 \llcorner \Gamma \text{ in } \mathscr{D}'(\mathbb{R}^3; \mathbb{R}^3). \tag{9}$$

With an abuse of notation, we identify S with $\nu \mathscr{H}^2 \llcorner S$ (resp. Γ with $\tau \mathscr{H}^1 \llcorner \Gamma$), and then (9) reads

$$\operatorname{curl} S = \Gamma \text{ in } \mathscr{D}'(\mathbb{R}^3; \mathbb{R}^3). \tag{10}$$

One can also investigate a minimization problem involving "generalized surfaces" (that we will identify later with 2-rectifiable currents in the sense of GMT) satisfying an appropriate version of (10). More specifically, we consider a countable family of Borel subsets S_i of C^1 oriented surfaces $\Sigma_i \subset \mathbb{R}^3$ such that $\sum_i |S_i| < \infty$ and, with ν_i the unit orienting normal to Σ_i, we have

$$\operatorname{curl}\Big(\sum_i S_i\Big) = \Gamma \text{ in } \mathscr{D}'(\mathbb{R}^3; \mathbb{R}^3). \tag{11}$$

By analogy with the Monge–Kantorovich problem, we introduce three quantities, $M(\Gamma)$, $K(\Gamma)$, and $D(\Gamma)$. The first one is a "GMT version" of $M_0(\Gamma)$:

$$M(\Gamma) = \inf\{|S|;\ S = \sum_i S_i \text{ such that (11) holds}\}. \tag{12}$$

Here, $|S|$ is the mass of the vector-valued measure $S = \sum_i S_i$ (identified with $\sum_i \nu_i \mathscr{H}^2 \llcorner S_i$). When S is a classical surface, $|S|$ equals the area of S.

Set

$$\mathscr{R} = \{S;\ \ S \text{ is a minimizer in (12)}\}. \tag{13}$$

(A priori, $\mathscr{R}$ could be empty.)

Even more generally, we may consider finite measures $\mu \in \mathscr{M}(\mathbb{R}^3; \mathbb{R}^3)$ satisfying

$$\operatorname{curl} \mu = \Gamma \text{ in } \mathscr{D}'(\mathbb{R}^3; \mathbb{R}^3) \tag{14}$$

and the convex minimization problem

$$K(\Gamma) = \min\{\|\mu\|_{\mathscr{M}};\ \mu \in \mathscr{M}(\mathbb{R}^3; \mathbb{R}^3) \text{ is a measure such that (14) holds}\}. \tag{15}$$

(In GMT's terminology, up the action of the Hodge $*$ operator, the competitors in (12) are called integral currents with boundary Γ, while the competitors in (15) are called real currents with boundary Γ.)

From the above definitions, we have

$$M_0(\Gamma) \geq M(\Gamma) \geq K(\Gamma).$$

Consider the set

$$\mathscr{Q} = \{\mu \in \mathscr{M}(\mathbb{R}^3; \mathbb{R}^3);\ \mu \text{ is a minimizer in (15)}\}. \tag{16}$$

Clearly, $\mathscr{Q}$ is non empty, convex, and weak* compact in $\mathscr{M}(\mathbb{R}^3; \mathbb{R}^3)$. By the Krein–Milman theorem, $\mathscr{Q}$ has at least one extreme point.

We associate with (15) a "dual" problem

$$\begin{aligned} D(\Gamma) &= \sup\left\{\int_\Gamma \tau\cdot\xi\,\mathrm{d}\mathscr{H}^1;\ \xi\in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3) \text{ and } \|\operatorname{curl}\xi\|_{L^\infty}\le 1\right\} \\ &= \sup\left\{\langle\Gamma,\xi\rangle;\ \xi\in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3) \text{ and } \|\operatorname{curl}\xi\|_{L^\infty}\le 1\right\}. \end{aligned} \tag{17}$$

If μ (resp. ξ) is a competitor in (15) (resp. (17)), then

$$\int_\Gamma \tau\cdot\xi\,\mathrm{d}\mathscr{H}^1 = \langle\Gamma,\xi\rangle = \langle\operatorname{curl}\mu,\xi\rangle = \langle\mu,\operatorname{curl}\xi\rangle \le \|\mu\|_{\mathscr{M}}\,\|\operatorname{curl}\xi\|_{L^\infty}, \tag{18}$$

so that $K(\Gamma)\ge D(\Gamma)$, and therefore

$$M_0(\Gamma)\ge M(\Gamma)\ge K(\Gamma)\ge D(\Gamma). \tag{19}$$

The central result in this direction is

Theorem 2. *Let $\Gamma\subset\mathbb{R}^3$ be a smooth compact connected oriented curve. Then,*

$$M_0(\Gamma)=M(\Gamma)=K(\Gamma)=D(\Gamma). \tag{20}$$

Moreover,

$$\text{every extreme point of } \mathscr{Q} \text{ belongs to } \mathscr{R}, \tag{21}$$

and consequently

$$\overline{\operatorname{conv}\mathscr{R}}^{\text{weak}^*} = \mathscr{Q}. \tag{22}$$

Assertion (22) is an immediate consequence of (21) and of the Krein–Milman theorem.

The fact that $\mathscr{R}\ne\emptyset$ is a fundamental result in GMT, and is usually established using the Federer-Fleming compactness theorem [16]. In our presentation, this assertion is a consequence of (21) and of the existence of extreme points. Previously, Hardt and Pitts [21] established that $\mathscr{R}\ne\emptyset$ without relying on the compactness theorem.

Equality $M(\Gamma)=K(\Gamma)$ is originally due to Federer [15, Section 5.10]. Hardt and Pitts [21] devised a different proof of this result; see also Almgren, Browder and Lieb [2, Remark (3), pp. 9–10] for another approach. Assertion (21) seems to be new; its proof relies heavily on a beautiful argument due to Poliakovsky [28], who answered a question raised in [11], concerning extreme points in the framework of $\mathbb{S}^1$-valued maps.

Equality $K(\Gamma)=D(\Gamma)$ is obtained via Hahn–Banach, very much in the spirit of the proof of Theorem 1. This idea, in a slightly different context, goes back to Federer [14, Section 4.1.12]; see also Brezis, Coron and Lieb [9, Theorem 5.1] and Giaquinta, Modica and Souček [19, Section 4.2.5, Proposition 2, p. 414].

In order to obtain the equality $M_0(\Gamma)=M(\Gamma)$, we rely on the coarea formula. The effectiveness of this tool in related problems was originally pointed out in Almgren, Browder and Lieb [2]; see also Alberti, Baldo and Orlandi [1].

In Section 4, we present various generalizations of Theorem 2.

2. A proof of Theorem 1

We divide the proof into two independent parts:

$$M=K \tag{23}$$

and

$$K=D. \tag{24}$$

Proof of (23). Choosing for A in (4) a permutation matrix yields $K\le M$. The reverse inequality, $K\ge M$, relies on Birkhoff's theorem on DS matrices (also called Birkhoff–von Neumann's theorem because von Neumann [33] rediscovered it independently a few years later). It asserts that the extreme points of the convex set of DS matrices are precisely the permutation matrices. Applying the Krein–Milman theorem, one deduces that any DS matrix is a convex combination of permutation matrices, and consequently $K\ge M$. □

In fact, this argument yields an additional information. Denote by $\sigma_1, \dots, \sigma_k$ the optimal permutations in (3) and by $Q_1, \dots, Q_k$ the associated permutation matrices. Set

$$\mathscr{Q} = \{A;\ A \text{ is a DS matrix which achieves the minimum in (4)}\}. \tag{25}$$

Proposition 1. We have

$$\mathscr{Q} = \operatorname{conv}\{Q_1, \dots, Q_k\}, \tag{26}$$

and in particular the extreme points of $\mathscr{Q}$ correspond precisely to the optimal permutations in (3).

Proof. Let $A = (a_{ij})$ be any minimizer in (4). We may write $A = \sum_{n=1}^{\ell} \alpha_n \widehat{Q}_n$, with $\alpha_n > 0$, $\forall n$, $\sum_{n=1}^{\ell} \alpha_n = 1$ and each $\widehat{Q}_n$ a permutation matrix associated with a permutation $\widehat{\sigma}_n$. Then,

$$M = \sum_{i,j} a_{ij}\, d(P_i, N_j) = \sum_{i,n} \alpha_n\, d(P_i, N_{\widehat{\sigma}_n(i)}) \geq M.$$

Thus $\sum_i d(P_i, N_{\widehat{\sigma}_n(i)}) = M$, $\forall n = 1, \dots \ell$, i.e. each $\widehat{Q}_n$, $n = 1, \dots, \ell$, is an optimal permutation matrix. □

Proof of (24). Clearly,

$$D \leq K. \tag{27}$$

Indeed, if $A = (a_{ij})$ is DS and ζ satisfies $|\zeta(x) - \zeta(y)| \leq d(x, y)$, $\forall x, y \in Z$, then

$$\sum_i \zeta(P_i) = \sum_{i,j} a_{ij}\, \zeta(P_i), \tag{28}$$

$$\sum_j \zeta(N_j) = \sum_{i,j} a_{ij}\, \zeta(N_j) \tag{29}$$

and thus

$$\sum_i \zeta(P_i) - \sum_j \zeta(N_j) = \sum_{i,j} a_{ij}\, (\zeta(P_i) - \zeta(N_j)) \leq \sum_{i,j} a_{ij}\, d(P_i, N_j). \tag{30}$$

Taking $\sup_\zeta$ and $\inf_A$ in (30) yields (27).

Both quantities D and K involve the maximization (resp. minimization) of linear functionals over convex sets. We present here a very natural and elementary approach leading to the equality $D = K$, which relies on the analytic form of Hahn–Banach (i.e. extension of linear functionals). A totally similar device will be used in Section 3 in the framework of the Plateau problem.

The proof of $D \geq K$ consists of three simple steps.

Step 1. Set

$$L = \sup\left\{ \sum_{i=1}^{m} \lambda_i - \sum_{j=1}^{m} \mu_j;\ \lambda_i - \mu_j \leq d(P_i, N_j),\ \forall i, j = 1, \dots, m \right\}. \tag{31}$$

We claim that

$$L = K. \tag{32}$$

In the proof of (32), we do not use the assumption that d is a pseudometric; it could be any nonnegative cost function.

Clearly (as in (28)–(30)), $L \leq K$, and thus it remains to prove that $K \leq L$.

Adding an $\varepsilon > 0$ (and then passing to the limit as $\varepsilon \to 0$), we may assume that $d(P_i, N_j) > 0$, $\forall i, j$.

Let X be the linear subspace of $\mathbb{R}^{m^2}$ defined by

$$X = \left\{ \xi = (\xi_{ij}) \in \mathbb{R}^{m^2};\ \exists (\lambda_i), (\mu_j) \in \mathbb{R}^m \text{ such that } \xi_{ij} = \lambda_i - \mu_j,\ \forall i, j \right\}$$

and set, for every $\xi = (\xi_{ij}) \in X$ as above,

$$\Phi(\xi) = \sum_{i=1}^{m} \lambda_i - \sum_{j=1}^{m} \mu_j. \tag{33}$$

It is easy to see that $\Phi : X \to \mathbb{R}$ is well-defined and linear.

From (31) and scaling, we obtain

$$\Phi(\xi) \le L \left\| \left(\frac{\xi_{ij}}{d(P_i, N_j)} \right)^+ \right\|_{\ell^\infty}, \ \forall \xi = (\xi_{ij}) \in X,$$

where $\|(\eta_{ij})\|_{\ell^\infty} = \sup_{i,j} |\eta_{ij}|$. Equivalently, we have

$$\Phi(\xi) \le p(\xi), \ \forall \xi \in X, \tag{34}$$

where p is defined on $\mathbb{R}^{m^2}$ by

$$p(\xi) = L \left\| \left(\frac{\xi_{ij}}{d(P_i, N_j)} \right)^+ \right\|_{\ell^\infty}, \ \forall \xi \in \mathbb{R}^{m^2}.$$

Since p satisfies

$$p(\alpha\xi) = \alpha p(\xi), \ \forall \alpha > 0, \ \forall \xi \in \mathbb{R}^{m^2}, \ p(\xi + \eta) \le p(\xi) + p(\eta), \ \forall \xi, \eta \in \mathbb{R}^{m^2},$$

(34) and the Hahn–Banach theorem in analytic form (see, e.g., [7, Theorem 1.1]) yield the existence of a linear functional Ψ on $\mathbb{R}^{m^2}$ such that

$$\Psi(\xi) = \Phi(\xi), \ \forall \xi \in X, \tag{35}$$

$$\Psi(\xi) \le p(\xi), \ \forall \xi \in \mathbb{R}^{m^2}. \tag{36}$$

We may thus write, for some matrix $A = (a_{ij})$,

$$\Psi(\xi) = \sum_{i,j=1}^{m} a_{ij}\, \xi_{ij}, \ \forall \xi \in \mathbb{R}^{m^2}. \tag{37}$$

From (33), (35) and (37), we see that

$$\sum_{i=1}^{m} \lambda_i - \sum_{j=1}^{m} \mu_j = \sum_{i,j=1}^{m} a_{ij}(\lambda_i - \mu_j), \ \forall \lambda_i, \ \forall \mu_j, \tag{38}$$

and thus, by identification of coefficients in (38), we have

$$\sum_{j=1}^{m} a_{ij} = 1, \ \forall i = 1, \dots, m \tag{39}$$

and

$$\sum_{i=1}^{m} a_{ij} = 1, \ \forall j = 1, \dots, m. \tag{40}$$

On the other hand, choosing

$$\xi_{ij} = \begin{cases} -1, & \text{if } i = i_0, \ j = j_0 \\ 0, & \text{otherwise,} \end{cases}$$

and applying (36) yields

$$a_{i_0 j_0} \ge 0, \ \forall i_0, j_0. \tag{41}$$

By (39)–(41), the matrix $A = (a_{ij})$ is DS.

Returning to (36) and choosing $\xi_{ij} = d(P_i, N_j)$, $\forall i, j$, we find that

$$\sum_{i,j} a_{ij}\, d(P_i, N_j) \le L, \tag{42}$$

and thus

$$K \le \sum_{i,j=1}^{m} a_{ij}\, d(P_i, N_j) \le L.$$

Step 2. Let λ_i, μ_j achieve the maximum in (31). Set $Y = \{P_i;\ i = 1, \dots, m\} \cup \{N_j;\ j = 1, \dots, m\} \subset Z$ and let $\zeta : Y \to \mathbb{R}$, $\zeta(P_i) = \lambda_i$, $\forall i$, $\zeta(N_j) = \mu_j$, $\forall j$. We claim that

$$|\zeta(x) - \zeta(y)| \le d(x, y),\ \forall x, y \in Y. \tag{43}$$

Unlike in Step 1, here we use the assumption that d is a pseudometric.

The key observation is the following. Let (b_{ij}) be a DS matrix achieving the minimum in (4). By (32) and the constraints

$$\zeta(P_i) - \zeta(N_j) = \lambda_i - \mu_j \le d(P_i, N_j),\ \forall i, j, \tag{44}$$

we have

$$\sum_{i,j=1}^{m} b_{ij}\, d(P_i, N_j) = K = L = \sum_{i=1}^{m} \lambda_i - \sum_{j=1}^{m} \mu_j = \sum_{i,j=1}^{m} b_{ij}\,(\lambda_i - \mu_j) \le \sum_{i,j=1}^{m} b_{ij}\, d(P_i, N_j),$$

and thus

$$b_{ij}\,(\lambda_i - \mu_j) = b_{ij}\, d(P_i, N_j),\ \forall i, j. \tag{45}$$

Since for each i (resp. each j) there exists some k (resp. some ℓ) such that $b_{ik} > 0$ (resp. $b_{\ell j} > 0$), we find from (45) that

$$\text{for each } i \text{ there exists some } k = k(i) \text{ such that } \zeta(P_i) - \zeta(N_k) = \lambda_i - \mu_k = d(P_i, N_k) \tag{46}$$

and

$$\text{for each } j \text{ there exists some } \ell = \ell(j) \text{ such that } \zeta(P_\ell) - \zeta(N_j) = \lambda_\ell - \mu_j = d(P_\ell, N_j). \tag{47}$$

Step 2 is then a consequence of the following claim. If $\zeta : Y \to \mathbb{R}$ satisfies (44), (46) and (47) for some pseudometric d, then (43) holds.

Indeed, with $k = k(i)$, we have (using (46))

$$\zeta(P_i) - \zeta(P_j) = d(P_i, N_k) + \zeta(N_k) - \zeta(P_j) \ge d(P_i, N_k) - d(P_j, N_k) \ge -d(P_i, P_j).$$

Exchanging i and j, we find that (43) holds when $x = P_i$ and $y = P_j$. Similarly, using (47) we obtain (43) for $x = N_i$ and $y = N_j$.

Finally, using (46) and (43) for N_k and N_j, we find

$$\zeta(P_i) - \zeta(N_j) = d(P_i, N_k) + \zeta(N_k) - \zeta(N_j) \ge d(P_i, N_k) - d(N_j, N_k) \ge -d(P_i, N_j).$$

Combining this with (44) yields (43) with $x = P_i$ and $y = N_j$.

Step 3. By (43), ζ has an extension to Z such that $|\zeta(x) - \zeta(y)| \le d(x, y)$, $\forall x, y \in Z$. By (27), Steps 1 and 2, we find that

$$K = L \le D \le K,$$

and thus (24) holds. □

3. Proof of Theorem 2

In view of (19), the first assertion in Theorem 2 amounts to proving the inequalities

$$K(\Gamma) \le D(\Gamma), \tag{48}$$

$$M(\Gamma) \le K(\Gamma) \tag{49}$$

and

$$M_0(\Gamma) \le M(\Gamma). \tag{50}$$

The proof of Theorem 2 consists of five steps.

Step 1. Proof of $K(\Gamma) \le D(\Gamma)$**.** Consider the mapping

$$T : \{\operatorname{curl}\xi;\ \xi \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)\} \to \mathbb{R},\ T(\operatorname{curl}\xi) = \int_\Gamma \tau\cdot\xi\,\mathrm{d}\mathscr{H}^1 = \langle\Gamma,\xi\rangle. \tag{51}$$

Let μ be any competitor in (15). By (18), we have $T(\operatorname{curl}\xi) = \langle\mu,\operatorname{curl}\xi\rangle$, and thus T is well defined. By homogeneity and the definition of $D(\Gamma)$, we have

$$|T(\operatorname{curl}\xi)| \le D(\Gamma)\,\|\operatorname{curl}\xi\|_{L^\infty},\ \forall\,\xi \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3). \tag{52}$$

By Hahn–Banach, T extends to a linear continuous functional, still denoted T, on $C_c(\mathbb{R}^3;\mathbb{R}^3)$, of norm $D(\Gamma)$. Let $\mu \in \mathscr{M}(\mathbb{R}^3;\mathbb{R}^3)$ be the measure such that $T(\eta) = \langle\mu,\eta\rangle$, $\forall\,\eta \in C_c(\mathbb{R}^3;\mathbb{R}^3)$, and $\|\mu\|_{\mathscr{M}} = D(\Gamma)$. By the definition of T, we have

$$\langle\operatorname{curl}\mu,\xi\rangle = \langle\mu,\operatorname{curl}\xi\rangle = \int_\Gamma \tau\cdot\xi\,\mathrm{d}\mathscr{H}^1 = \langle\Gamma,\xi\rangle,\ \forall\,\xi \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3),$$

and thus μ is a competitor in (15). We find that $K(\Gamma) \le \|\mu\|_{\mathscr{M}} = D(\Gamma)$. □

Step 2. Proof of $M(\Gamma) \le K(\Gamma)$**.** This is a clear consequence of (21). □

Step 3. Proof of (21)**.** This step relies heavily on the fine structure of BV functions.

Set $\dot{\mathrm{BV}}(\mathbb{R}^N) = \{\varphi \in L^1_{\mathrm{loc}}(\mathbb{R}^N);\ D\varphi \in \mathscr{M}(\mathbb{R}^N;\mathbb{R}^N)\}$; we define similarly $\dot{W}^{1,1}(\mathbb{R}^N)$.

Let μ be an extreme point of $\mathscr{Q}$. Let $S \subset \mathbb{R}^3$ be any smooth compact oriented surface in $\mathbb{R}^3$ such that (10) holds, and set $\mu_0 = \mu - S$.

Since μ_0 is a measure satisfying $\operatorname{curl}\mu_0 = 0$ in $\mathscr{D}'(\mathbb{R}^3;\mathbb{R}^3)$, there exists some $\psi \in \dot{\mathrm{BV}}(\mathbb{R}^3)$ such that $\mu_0 = D\psi$, and thus

$$\mu = S + D\psi. \tag{53}$$

We claim that

$$\text{there exists some constant } C \in \mathbb{R} \text{ such that } \psi - C \text{ is } \mathbb{Z}\text{-valued.} \tag{54}$$

This remarkable assertion is essentially due to Poliakovsky [28]; we postpone its proof, which follows closely [28], to Step 4.

Assuming the claim proved, we continue as follows. By the Fleming–Rishel coarea formula for BV functions (see, e.g., [3, Theorem 3.40]), for a.e. $t \in \mathbb{R}$ the set $[\psi > t] = \{x \in \mathbb{R}^3;\ \psi(x) > t\}$ has finite perimeter, denoted $\operatorname{Per}[\psi > t]$, and we have

$$\|D\psi\|_{\mathscr{M}} = \int_{-\infty}^{\infty} \operatorname{Per}[\psi > t]\,\mathrm{d}t. \tag{55}$$

Equivalently, for a.e. $t \in \mathbb{R}$ we have $\chi_{[\psi>t]} \in \dot{\mathrm{BV}}(\mathbb{R}^3)$, and

$$\|D\psi\|_{\mathscr{M}} = \int_{-\infty}^{\infty} \|D\chi_{[\psi>t]}\|_{\mathscr{M}}\,\mathrm{d}t.$$

Let $j \in \mathbb{Z}$ and set $F_j = [\psi \ge j]$. Since, for each $t \in (j-1, j)$, we have $[\psi > t] = F_j$, we find that F_j has finite perimeter for every j, and that

$$\|D\psi\|_{\mathscr{M}} = \sum_{j=-\infty}^{\infty} \operatorname{Per} F_j. \tag{56}$$

Let now, for $j \in \mathbb{Z}$,

$$E_j = \begin{cases} F_j, & \text{if } j \ge 0 \\ (F_{j+1})^c = \mathbb{R}^3 \setminus F_{j+1}, & \text{if } j < 0. \end{cases} \tag{57}$$

By (56), we have

$$\|D\psi\|_{\mathscr{M}} = \sum_{j=-\infty}^{\infty} \operatorname{Per} E_j. \tag{58}$$

We claim that

$$\psi = \sum_{j=-\infty}^{\infty} (\operatorname{sgn} j)\, \chi_{E_j} \text{ in } \dot{\mathrm{BV}}(\mathbb{R}^3). \tag{59}$$

This is proved as follows. Let $m \geq 1$ be an integer and set

$$\psi_m = \begin{cases} \psi, & \text{if } |\psi| \leq m \\ m, & \text{if } \psi \geq m \\ -m, & \text{if } \psi \leq -m. \end{cases}$$

Clearly, we have

$$\psi_m \to \psi \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^3) \text{ as } m \to \infty. \tag{60}$$

The coarea formula yields

$$\|D(\psi - \psi_m)\|_{\mathscr{M}} = \sum_{|j|>m} \operatorname{Per} F_j \to 0 \text{ as } m \to \infty. \tag{61}$$

On the other hand, we clearly have

$$\psi_m = \sum_{j=-m}^{m} (\operatorname{sgn} j)\, \chi_{E_j} \text{ in } \dot{\mathrm{BV}}(\mathbb{R}^3). \tag{62}$$

We obtain (59) from (60)–(62).

Granted (54), we complete the proof of (21) as follows. For each j, let R_j be the reduced boundary of E_j, which is a 2-rectifiable set, and let ν_j be the (measure theoretic) inner unit normal to R_j. Then $\operatorname{Per} E_j = \mathscr{H}^2(R_j)$ and $D\chi_{E_j} = R_j$; see, e.g., [3, Section 3.5]. Using this and (59), we find that

$$\mu = S + \mu_0 = S + \sum_{j\in\mathbb{Z}} (\operatorname{sgn} j)\, R_j;$$

this leads (via (58)) to (21). □

Step 4. Proof of (54)**.** Argue by contradiction and assume that (54) does not hold. This is equivalent to the fact that (at least) one of the functions $\sin(2\pi\psi)$, $\cos(2\pi\psi)$ is not constant. Assume, e.g., that $\sin(2\pi\psi)$ is not constant, and set

$$\psi_\pm = \psi \pm \frac{1}{2\pi}\sin(2\pi\psi), \;\; \mu_\pm = S + D\psi_\pm.$$

By (53) and the assumption on ψ, we have $\mu_\pm \neq \mu$ and $\mu = (\mu_+ + \mu_-)/2$. On the other hand, $\mu_\pm$ is a competitor in (15). We will prove that

$$\|\mu\|_{\mathscr{M}} = \frac{\|\mu_+\|_{\mathscr{M}} + \|\mu_-\|_{\mathscr{M}}}{2}. \tag{63}$$

Clearly, this contradicts the fact that μ is an extreme point of $\mathscr{Q}$.

In order to prove (63), we rely on the structure of $D\psi$ with $\psi \in \dot{\mathrm{BV}}$ and on Volpert's chain rule; see, e.g., [3, Chapter 3]. Recall that, if $\psi \in \dot{\mathrm{BV}}$, then the measure $D\psi$ can be (uniquely) written as a sum of an absolutely continuous part with respect to the Lebesgue measure, $D^a\psi$, whose density is denoted $\nabla\psi$, a Cantor part $D^c\psi$ and a jump part $D^j\psi$. With an abuse of notation, we write this decomposition as:

$$D\psi = \nabla\psi + D^c\psi + D^j\psi = \nabla\psi + D^c\psi + (\psi^+ - \psi^-)\, J_\psi. \tag{64}$$

Here, J_ψ is the jump set of ψ, which is a 2-rectifiable set, ν is an orienting unit normal to J_ψ, and $\psi^\pm$ are the approximate one-sided limits of ψ on J_ψ.

On the other hand, Volpert's chain rule asserts that, when f is C^1 and Lipschitz, we have (for the precise representative of ψ)

$$D(f \circ \psi) = f'(\psi)\nabla\psi + f'(\psi)D^c\psi + (f(\psi^+) - f(\psi^-))\, J_\psi. \tag{65}$$

Using (65), we find that

$$\begin{aligned} D\psi_\pm =& [1 \pm \cos(2\pi\psi)]\nabla\psi + [1 \pm \cos(2\pi\psi)]D^c\psi \\ &+ [(\psi^+ - \psi^-) \pm (\sin(2\pi\psi^+) - \sin(2\pi\psi^-))/(2\pi)]\, J_\psi \\ =& \nabla\psi_\pm + D^c\psi_\pm + D^j\psi_\pm. \end{aligned} \tag{66}$$

We next note the following. If S, S' are 2-rectifiable sets, with orienting unit normals ν and ν', then $\nu = \pm\nu'$ $\mathscr{H}^2$-a.e. on $S \cap S'$; see, e.g., [18, Section 2.1.4]. With no loss of generality, we may assume that, $\mathscr{H}^2$-a.e. on $S \cap J_\psi$, S and J_ψ have the same orienting unit normal ν, and then (66) yields

$$\begin{aligned} D\mu_\pm =& \nabla\psi_\pm + D^c\psi_\pm + (S \setminus J_\psi) \\ &+ [(\psi^+ - \psi^-) \pm (\sin(2\pi\psi^+) - \sin(2\pi\psi^-))/(2\pi)]\,(J_\psi \setminus S) \\ &+ [1 + (\psi^+ - \psi^-) \pm (\sin(2\pi\psi^+) - \sin(2\pi\psi^-))/(2\pi)]\,(J_\psi \cap S). \end{aligned} \tag{67}$$

We now make the following observations:

$$1 \pm \cos(2\pi\psi) \geq 0, \tag{68}$$

$$(\psi^+ - \psi^-) \pm (\sin(2\pi\psi^+) - \sin(2\pi\psi^-))/(2\pi) \text{ and } \psi^+ - \psi^- \text{ have the same sign,} \tag{69}$$

$$1 + (\psi^+ - \psi^-) \pm (\sin(2\pi\psi^+) - \sin(2\pi\psi^-))/(2\pi) \text{ and } 1 + \psi^+ - \psi^- \text{ have the same sign;} \tag{70}$$

(69) and (70) are immediate consequences of the fact that $t \mapsto t \pm \sin t$ is non decreasing.

Using (68)–(70), we find that

$$\begin{aligned} \frac{\|\mu_+\|_{\mathscr{M}} + \|\mu_-\|_{\mathscr{M}}}{2} =& \int_\Omega |\nabla\psi| + \|D^c\psi\|_{\mathscr{M}} + \mathscr{H}^2(S \setminus J_\psi) + \int_{J_\psi \setminus S} |\psi^+ - \psi^-|\,\mathrm{d}\mathscr{H}^2 \\ &+ \int_{J_\psi \cap S} |1 + \psi^+ - \psi^-|\,\mathrm{d}\mathscr{H}^2 = \|\mu\|_{\mathscr{M}}, \end{aligned}$$

whence (63). □

Step 5. Proof of $M_0(\Gamma) \leq M(\Gamma)$. We rely on the following auxiliary results.

Lemma 1. *Let $\Gamma \subset \mathbb{R}^3$ be as in Theorem 2. Then,*

1. *There exist some $\varepsilon > 0$ and an orientation preserving diffeomorphism*

$$\Phi : \Gamma \times \overline{\mathbb{D}}(0, \varepsilon) \to \{x \in \mathbb{R}^3;\ \mathrm{dist}(x, \Gamma) \leq \varepsilon\}$$

such that:
(a) $\Phi(x, z) \in N_x$ *(the normal plane at x to Γ), $\forall x \in \Gamma$, $\forall z \in \overline{\mathbb{D}}(0, \varepsilon)$.*
(b) $|\Phi(x, z) - x| = |z|$, $\forall x \in \Gamma$, $\forall z \in \overline{\mathbb{D}}(0, \varepsilon)$.

2. *There exist some $u \in C^\infty(\mathbb{R}^3 \setminus \Gamma; \mathbb{S}^1)$ and $f \in C^\infty(\Gamma; \mathbb{S}^1)$ such that*

$$u(\Phi(x, r\mathrm{e}^{\mathrm{i}\theta})) = f(x)\,\mathrm{e}^{\mathrm{i}\theta},\ \forall x \in \Gamma,\ \forall 0 < r \leq \varepsilon/3,\ \forall \theta \in \mathbb{R} \tag{71}$$

and

$$\nabla u \in L^1(\mathbb{R}^3). \tag{72}$$

3. *Any $u = u_1 + \mathrm{i}u_2$ as in item 2 satisfies*

$$\frac{1}{2\pi}\,\mathrm{curl}\,(u \wedge \nabla u) = \frac{1}{2\pi}\,\mathrm{curl}\,(u_1\,\nabla u_2 - u_2\,\nabla u_1) = \Gamma \text{ in } \mathscr{D}'(\mathbb{R}^3; \mathbb{R}^3). \tag{73}$$

Here, "$\wedge$" stands for the vector product of complex numbers: if $u = u_1 + \mathrm{i}u_2$, then

$$u \wedge \nabla u = u_1\nabla u_2 - u_2\nabla u_1. \tag{74}$$

Lemma 2. *Let $F \in L^1(\mathbb{R}^N; \mathbb{R}^N)$ and $\psi \in \dot{\mathrm{BV}}(\mathbb{R}^N)$. Then, there exists a sequence $(\psi_n) \subset C^\infty(\mathbb{R}^N)$ such that:*

(1) $\psi_n \in \dot{W}^{1,1}(\mathbb{R}^N)$, $\forall n$;

(2) $\liminf_{n\to\infty} \int_\Omega |F + \nabla\psi_n| \le \|F + D\psi\|_{\mathscr{M}}$;

(3) *let $K \subset \mathbb{R}^N$ be a compact set such that $\mathscr{H}^{N-1}(K) = 0$. Then we may choose ψ_n such that, in addition, $\psi_n = 0$ in a neighborhood of K.*

Granted Lemmas 1 and 2, we prove the inequality $M_0(\Gamma) \le M(\Gamma)$ as follows. Let $S = \sum_i S_i$ be a competitor in (12). By Lemma 1 item 3, we have

$$\operatorname{curl}\left(S - \frac{1}{2\pi} u \wedge \nabla u\right) = 0 \text{ in } \mathscr{D}'(\mathbb{R}^3; \mathbb{R}^3),$$

and thus (using Lemma 1, item 2)

$$S = \frac{1}{2\pi} u \wedge \nabla u + D\psi \tag{75}$$

for some $\psi \in \dot{\mathrm{BV}}(\mathbb{R}^3)$.

Set $F = u \wedge \nabla u \in L^1(\mathbb{R}^3; \mathbb{R}^3)$ and $K = \Gamma$. Let (ψ_n) be as in Lemma 2 and set $u_n = u\, \mathrm{e}^{\mathrm{i}\psi_n}$.

Clearly, u_n is smooth in $\mathbb{R}^3 \setminus \Gamma$. Let $\alpha = \mathrm{e}^{\mathrm{i}\xi} \in \mathbb{S}^1$ be a regular value of u_n in $\mathbb{R}^3 \setminus \Gamma$ and set $S^\alpha = [u_n = \alpha]$, which is a smooth 2-submanifold of $\mathbb{R}^3 \setminus \Gamma$, oriented by $u_n \wedge \nabla u_n$. Since $\psi_n = 0$ near Γ, for each $x \in \Gamma$ there exists some $\varepsilon_0 > 0$ such that near x, we have

$$S^\alpha = \Phi\left(\{(y, r\mathrm{e}^{\mathrm{i}\alpha(y)});\ y \in \Gamma,\ 0 < r < \varepsilon_0\}\right), \text{ with } \alpha(y) = \xi + \mathrm{i} \ln f(y). \tag{76}$$

Here, $-\mathrm{i} \ln f$ is a smooth local phase of f.

By (76), $S^\alpha \cup \Gamma$ has boundary Γ, and thus $S^\alpha \cup \Gamma$ is a competitor in (7). Combining this with the coarea formula, we obtain

$$\int_{\mathbb{R}^3} |\nabla u_n| = \int_{\mathbb{R}^3 \setminus \Gamma} |\nabla u_n| = \int_{\mathbb{S}^1} |S^\alpha|\, \mathrm{d}\alpha = \int_{\mathbb{S}^1} |S^\alpha \cup \Gamma|\, \mathrm{d}\alpha \ge 2\pi M_0(\Gamma). \tag{77}$$

On the other hand, we have

$$|\nabla u_n| = |u_n \wedge \nabla u_n| = |u \wedge \nabla u + \nabla\psi_n|. \tag{78}$$

Combining Lemma 2 item 2, (75), (77) and (78), we find that for every competitor $S = \sum_i S_i$ in (15) we have $|S| \ge M_0(\Gamma)$. Granted Lemmas 1 and 2, the proof of Theorem 2 is complete. □

Remark 1. We return here to (1), which we derive from Theorem 2 and its proof.

Step 1. Proof of "$\ge$" in (1). If u is as in (1), then

$$\operatorname{curl}(u \wedge \nabla u) = 2\pi\Gamma \text{ in } \mathscr{D}'(\mathbb{R}^3; \mathbb{R}^3) \tag{79}$$

(see [9, equation (8.30)]). Therefore, for every $\xi \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^3)$ satisfying $\|\operatorname{curl}\xi\|_{L^\infty} \le 1$, we have

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla u| &= \int_{\mathbb{R}^3} |u \wedge \nabla u| \ge \int_{\mathbb{R}^3} (u \wedge \nabla u) \cdot \operatorname{curl}\xi = \langle u \wedge \nabla u, \operatorname{curl}\xi\rangle \\ &= \langle \operatorname{curl}(u \wedge \nabla u), \xi\rangle = 2\pi\langle\Gamma, \xi\rangle = 2\pi \int_\Gamma \tau \cdot \xi\, \mathrm{d}\mathscr{H}^1. \end{aligned} \tag{80}$$

Taking in (80) sup over ξ and using the equality $M_0(\Gamma) = D(\Gamma)$, we obtain "$\ge$" in (1).

Step 2. Proof of "$\le$" in (1). Let $\mu \in \mathscr{Q}$, so that $\|\mu\|_{\mathscr{M}} = K(\Gamma) = M_0(\Gamma)$ (by Theorem 2). Let $u \in C^\infty(\mathbb{R}^3 \setminus \Gamma) \cap \dot{W}^{1,1}(\mathbb{R}^3)$ be a competitor in (1). By (79), we have $\operatorname{curl}(2\pi\mu - u \wedge \nabla u) = 0$ in $\mathscr{D}'(\mathbb{R}^3; \mathbb{R}^3)$, and thus there exists some $\psi \in \dot{\mathrm{BV}}(\mathbb{R}^3; \mathbb{R})$ such that $2\pi\mu = u \wedge \nabla u + D\psi$. By Lemma 2, there exists a sequence $(\psi_n) \subset C^\infty(\mathbb{R}^3; \mathbb{R}) \cap \dot{W}^{1,1}$ such that

$$2\pi M_0(\Gamma) = 2\pi\|\mu\|_{\mathscr{M}} = \lim_{n\to\infty} \int_{\mathbb{R}^3} |u \wedge \nabla u + \nabla\psi_n|. \tag{81}$$

Set $u_n = u\,\mathrm{e}^{\mathrm{i}\psi_n}$, which is clearly a competitor in (1). We thus have (using (81))

$$\text{lhs of (1)} \le \lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_n| = \lim_{n\to\infty} \int_{\mathbb{R}^3} |u_n \wedge \nabla u_n| = \lim_{n\to\infty} \int_{\mathbb{R}^3} |u \wedge \nabla u + \nabla \psi_n| = 2\pi M_0(\Gamma).$$

Remark 2. There is an alternative proof of Theorem 1 (presented in Brezis [6,8]) which avoids completely Birkhoff, Krein–Milman and Hahn–Banach; it is totally self-contained (and reminds of the original proof of Kantorovich [23]). The heart of the matter is the construction of an *explicit* function ζ such that $|\zeta(x) - \zeta(y)| \le d(x, y)$, $\forall x, y \in Z$, and $\sum_i \zeta(P_i) - \sum_j \zeta(N_j) \ge M$.

It would be very interesting to perform a similar construction in the framework of Theorem 2. More precisely, given Γ, can one find an *explicit* $\zeta \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^3)$ such that $\|\operatorname{curl} \zeta\|_{L^\infty} \le 1$ and $\int_\Gamma \tau \cdot \zeta \, \mathrm{d}\mathscr{H}^1 \ge M(\Gamma) - \varepsilon$ (with $\varepsilon > 0$ arbitrarily small)?

Finally, we turn to the auxiliary results used in the proof of Theorem 2.

Proof of Lemma 1, item 1. Let $S \subset \mathbb{R}^3$ be a smooth compact oriented surface with boundary Γ. Let, for $x \in \Gamma$, $X(x)$ denote the outward unit normal to S at x. Let $\tau(x)$ be the orienting unit tangent vector at $x \in \Gamma$. Let $Y(x) \in \mathbb{R}^3$ be the unique unit vector such that $(X(x), Y(x), \tau(x))$ is a direct orthonormal basis of $\mathbb{R}^3$. Clearly, X and Y are smooth and, for each $x \in \Gamma$, we have $X(x), Y(x) \in N_x$. By the inverse function theorem and the properties of the nearest point projection on Γ, for sufficiently small $\varepsilon > 0$, the map

$$\Phi(x, (y_1, y_2)) := x + y_1 X(x) + y_2 Y(x), \ x \in \Gamma, \ 0 \le |(y_1, y_2)| \le \varepsilon$$

has all the required properties. □

Proof of Lemma 1, item 2. When $v \in C^\infty(\mathbb{R}^3 \setminus \Gamma; \mathbb{S}^1)$, we may define the "degree of v around Γ", $\deg(v, \Gamma)$, as follows. Let $x \in \Gamma$ and let N_x be the normal plane to Γ at x. On N_x, we have a natural orientation induced by the orientation of Γ (such that a direct basis of N_x, completed with $\tau(x)$, forms a direct basis of $\mathbb{R}^3$). Let, for small δ, $C(x, \delta) = \{y \in N_x;\ |y - x| = \delta\}$. This circle inherits an orientation from N_x and does not intersect Γ. We let $\deg(v, \Gamma) = \deg(v, C(x, \delta))$. By a homotopy argument, this definition does not depend on x or on small δ. One can define similarly $\deg(v, \Gamma)$ when v is merely defined on $\Phi(\Gamma \times \overline{\mathbb{D}}(0, \varepsilon))$.

We now invoke the existence of some $v \in C^\infty(\mathbb{R}^3 \setminus \Gamma; \mathbb{S}^1)$ such that $\deg(v, \Gamma) = 1$. Moreover, we may choose such v satisfying $\nabla v \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$; see [1, Section 4]. We next modify v at infinity as follows. Let $R > 0$ be such that $\Phi(\Gamma \times \overline{\mathbb{D}}(0, \varepsilon)) \subset B(0, R)$. On $\mathbb{R}^3 \setminus B(0, R)$, we may write $v = \mathrm{e}^{\mathrm{i}\varphi}$ for some smooth φ. By replacing φ with an appropriate smooth function $\widetilde{\varphi}$ that agrees with φ near $S(0, R)$, we may assume that $v = 1$ on $\mathbb{R}^3 \setminus B(0, R + 1)$, and thus $\nabla v \in L^1(\mathbb{R}^3)$.

Define, for $x \in \Gamma$, $0 < r \le \varepsilon$ and $\theta \in \mathbb{R}$, $w(\Phi(x, r\mathrm{e}^{\mathrm{i}\theta})) = \mathrm{e}^{\mathrm{i}\theta}$.

We now note the following straightforward result, whose proof is left to the reader.

Lemma 3. *Let $g \in C^\infty(\Gamma \times (\overline{\mathbb{D}}(0, \varepsilon) \setminus \{0\}); \mathbb{S}^1)$ be such that $\deg(g(x, \cdot), C(0, \varepsilon)) = 0$, $\forall x \in \Gamma$.*

Set $f(x) = g(x, \varepsilon)$, $\forall x \in \Gamma$. Then there exists some smooth function $\psi : \Gamma \times (\overline{\mathbb{D}}(0, \varepsilon) \setminus \{0\}) \to \mathbb{R}$ such that $g(x, z) = f(x)\,\mathrm{e}^{\mathrm{i}\psi(x,z)}$, $\forall (x, z) \in \Gamma \times (\overline{\mathbb{D}}(0, \varepsilon) \setminus \{0\})$.

Clearly, the above lemma applies to $g = (v\overline{w}) \circ \Phi : \Gamma \times \left(\overline{\mathbb{D}}(0, \varepsilon) \setminus \{0\}\right) \to \mathbb{S}^1$.

Set $U = \Gamma \times (\overline{\mathbb{D}}(0, \varepsilon_1) \setminus \{0\})$. Consider some $\eta \in C^\infty(U; \mathbb{R})$ such that $\eta(x, z) = \psi(x, z)$ if $|z| > \varepsilon/2$ and $\eta(x, z) = 0$ if $0 < |z| < \varepsilon/3$.

Define $h(x, z) = f(x)\,\mathrm{e}^{\mathrm{i}\psi(x,z)}$ and set, for $y \in \mathbb{R}^3 \setminus \Gamma$,

$$u(y) = \begin{cases} h(\Phi^{-1} y)\, w(y), & \text{if } 0 < \operatorname{dist}(y, \Gamma) < \varepsilon \\ v(y), & \text{if } \operatorname{dist}(y, \Gamma) \ge \varepsilon. \end{cases}$$

It is easy to see that u has all the required properties. □

Proof of Lemma 1, item 3. See [9, equation (8.30)]. □

Proof of Lemma 2. When $F = 0$, the existence of a sequence (ψ_n) satisfying items 1 and 2 and such that $\psi_n \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ when $n \to \infty$ is classical; see, e.g., [20, Theorem 1.17]. The case of an arbitrary $F \in L^1(\mathbb{R}^N; \mathbb{R}^N)$ is established in an appendix of [10].

Item 3 follows from the fact that the $W^{1,1}$-capacity of K is zero (see, e.g., [13, Section 4.7.1, Theorem 2]). □

4. Generalizations of Theorem 2

We first discuss the generalization of Theorem 2 to $\mathbb{R}^N$ with $N \geq 4$. It will be more convenient to adopt the terminology of GMT. In this language, (8) asserts that $\partial S = \Gamma$, where this time ∂ stands for the boundary operator (not the geometric boundary). Let us recall the definition of ∂ (which coincides with d^*, the formal adjoint of d acting on forms), first in 3D. By definition, ∂S (i.e. ∂ acting on the 2-current S) is the 1-current satisfying

$$\langle \partial S, \zeta \rangle = \langle S, \mathrm{d}\zeta \rangle = \int_S \nu \cdot \operatorname{curl} \zeta = \langle \operatorname{curl} S, \zeta \rangle, \ \forall \zeta \in C_c^\infty(\mathbb{R}^3; \Lambda^1) \simeq C_c^\infty(\mathbb{R}^3; \mathbb{R}^3). \tag{82}$$

More generally, if $T \in \mathscr{D}'(\mathbb{R}^N; \Lambda^k)$ for some $1 \leq k \leq N$, then $\partial T \in \mathscr{D}'(\mathbb{R}^N; \Lambda^{k-1})$ is defined by

$$\langle \partial T, \zeta \rangle = \langle T, \mathrm{d}\zeta \rangle, \ \forall \zeta \in C_c^\infty(\mathbb{R}^N; \Lambda^{k-1}). \tag{83}$$

This applies in particular to the case where $S \subset \mathbb{R}^N$ is an oriented k-dimensional manifold with (geometric) boundary Γ. Then S defines a k-current (still denoted S) through the formula

$$\langle S, \xi \rangle = \int_S \xi, \ \forall \xi \in C_c^\infty(\mathbb{R}^N; \Lambda^k),$$

and

$$\langle \partial S, \zeta \rangle = \langle S, \mathrm{d}\zeta \rangle = \int_S \mathrm{d}\zeta = \int_\Gamma \zeta \text{ (by Stokes)} = \langle \Gamma, \zeta \rangle, \ \forall \zeta \in C_c^\infty(\mathbb{R}^N; \Lambda^{k-1}),$$

where Γ is viewed as a $(k-1)$-current.

We now return to the higher dimensional version of Theorem 2. Let $N \geq 3$ and let $\Gamma \subset \mathbb{R}^N$ be a smooth compact connected oriented $(N-2)$-manifold (without boundary).

Remark 3. In 3D, Γ is a curve and its orientability is not an issue. However, when $N \geq 4$ we have to assume Γ orientable, since this "does not come with Γ". (Think of the Klein bottle.)

Given such Γ, there exists a smooth compact oriented hypersurface $S \subset \mathbb{R}^N$ with (geometric) boundary Γ; see, e.g., Kirby [24, Theorem 3, p. 50]. We may thus associate with Γ the finite quantities

$$M_0(\Gamma) = \inf\{|S|;\ S \subset \mathbb{R}^N \text{ is a smooth oriented hypersurface with boundary } \Gamma\}, \tag{84}$$

$$M(\Gamma) = \inf\{|S|;\ S \text{ is an } (N-1)\text{-rectifiable current in } \mathbb{R}^N \text{ such that } \partial S = \Gamma\}, \tag{85}$$

$$K(\Gamma) = \min\{\|\mu\|_{\mathscr{M}};\ \mu \in \mathscr{M}(\mathbb{R}^N; \Lambda^{N-1}) \text{ is a measure such that } \partial\mu = \Gamma\}, \tag{86}$$

$$D(\Gamma) = \sup\left\{\int_\Gamma \zeta;\ \zeta \in C_c^\infty(\mathbb{R}^N; \Lambda^{N-2}) \text{ and } \|\mathrm{d}\zeta\|_{L^\infty} \leq 1\right\}. \tag{87}$$

Here, $|S|$ is the mass of S; it coincides with $\mathscr{H}^{N-1}(S)$ when S is smooth.

Consider also the (possibly empty) set

$$\mathscr{R} = \{S;\ S \text{ is a minimizer in (85)}\} \tag{88}$$

and the non-empty set

$$\mathscr{Q} = \{\mu \in \mathscr{M}(\mathbb{R}^N; \Lambda^{N-2});\ \mu \text{ is a minimizer in (86)}\}. \tag{89}$$

We have the following result.

Theorem 3. *Let $\Gamma \subset \mathbb{R}^N$ be a smooth compact connected oriented $(N-2)$-manifold. Then*

$$M_0(\Gamma) = M(\Gamma) = K(\Gamma) = D(\Gamma). \tag{90}$$

Moreover,

$$\text{every extreme point of } \mathscr{Q} \text{ is a minimizer in (85)}, \tag{91}$$

and consequently

$$\overline{\operatorname{conv} \mathscr{R}}^{\text{weak}^*} = \mathscr{Q}. \tag{92}$$

The proof of Theorem 3 is very similar to the one of Theorem 2; see [10, Chapter 4] for the full proofs and more general results. We mention below the main adaptations required.

1. If S is a competitor in (85) and μ is a competitor in (86), then $\partial(\mu - S) = 0$ and thus $*\mu = *S + \mathrm{d}\psi$ for some $\psi \in \dot{\mathrm{BV}}(\mathbb{R}^N)$. Here, $*$ stands for the Hodge operator, and thus $*\mu$ is a 1-current with coefficient finite measures, which we may identify with a vector field. The same for $*S$.
2. If $u \in C(\mathbb{R}^N \setminus \Gamma; \mathbb{S}^1)$, then we may define the integer $\deg(u, \Gamma)$, thanks to the fact that the (two-dimensional) normal plane N_x at some $x \in \Gamma$ has a natural orientation (such that a direct basis of N_x completed with a direct basis of $T_x(\Gamma)$ forms a direct basis of $\mathbb{R}^N$).
3. Lemma 1, items 1 and 2, holds (with the same proof) in any dimension.
4. If u is as in the previous item, then (see, e.g., Jerrard and Soner [22, Section 3, Example 4])

$$\partial(*(u \wedge \mathrm{d}u)) = 2\pi\Gamma \text{ in } \mathscr{D}'(\mathbb{R}^N). \tag{93}$$

5. Let u be as above and let ψ_n be as in Lemma 2 (with $F = u \wedge \nabla u \;\; K = \Gamma$). Set $u_n = u\,\mathrm{e}^{\mathrm{i}\psi_n}$. Then, for a.e. $\alpha \in \mathbb{S}^1$, the set $[u_n = \alpha] \cup \Gamma$ is a smooth hypersurface with boundary Γ.

We next go beyond smooth Γ's. Set

$$\begin{aligned}\mathscr{F} = \{\Gamma;\ &\Gamma \text{ is an } (N-2)\text{-current in } \mathbb{R}^N \text{ such that } \Gamma = \partial S \text{ for some}\\ &(N-1)\text{-rectifiable current } S \text{ in } \mathbb{R}^N\}.\end{aligned} \tag{94}$$

Given $\Gamma \in \mathscr{F}$, we define

$$M(\Gamma) = \inf\{|S|;\ S \text{ is an } (N-1)\text{-rectifiable current in } \mathbb{R}^N \text{ such that } \partial S = \Gamma\}, \tag{95}$$

$$K(\Gamma) = \inf\left\{\|\mu\|_{\mathscr{M}};\ \mu \in \mathscr{M}(\mathbb{R}^N; \Lambda^{N-1}) \text{ is a measure such that } \partial\mu = \Gamma\right\}, \tag{96}$$

$$D(\Gamma) = \sup\left\{\langle \Gamma, \zeta\rangle;\ \zeta \in C_c^\infty(\mathbb{R}^N; \Lambda^{N-2}),\ \|\mathrm{d}\zeta\|_{L^\infty} \le 1\right\}, \tag{97}$$

$$\mathscr{R} = \{S;\ S \text{ is a minimizer in (95)}\} \tag{98}$$

and

$$\mathscr{Q} = \{\mu \in \mathscr{M}(\mathbb{R}^N; \Lambda^{N-2});\ \mu \text{ is a minimizer in (96)}\}. \tag{99}$$

Then we have the following straightforward extension of Theorems 2–3 (without M_0).

Theorem 4. *Let $N \ge 2$. Then*

$$M(\Gamma) = K(\Gamma) = D(\Gamma),\ \forall \Gamma \in \mathscr{F}. \tag{100}$$

Moreover,

$$\textit{every extreme point of } \mathscr{Q} \textit{ is a minimizer in } (95), \tag{101}$$

and consequently

$$\overline{\mathrm{conv}\,\mathscr{R}}^{\mathrm{weak}^*} = \mathscr{Q}. \tag{102}$$

The proof of Theorem 4 is essentially the same as the one of Theorem 2.

Remark 4.

1. The above theorem does not hold under the weaker assumption that $\Gamma = \partial\mu$ for some $\mu \in \mathscr{M}(\mathbb{R}^N; \Lambda^{N-1})$. Indeed, consider, for example, $N = 2$ and $\Gamma = (1/2)(\delta_P - \delta_N)$, where $P, N \in \mathbb{R}^2$ are distinct points. Then $\Gamma = \mathrm{div}\, F = -\partial F$ for some vector field $F \in L^1_c(\mathbb{R}^2; \mathbb{R}^2)$. However, there exists no 1-rectifiable current S such that $\Gamma = \partial S$. For otherwise, by Federer and Fleming's boundary rectifiability theorem [14, Theorem 4.2.16 (2), p. 413], Γ would be a finite sum of Dirac masses with integer multiplicities.

2. With more work (see [10, Chapter 4]), one may prove that

$$\mathscr{F} = \left\{ \sum_j \partial S_j \text{ in } \mathscr{D}'(\mathbb{R}^N);\ S_j \subset \mathbb{R}^N \text{ is a compact oriented hypersurface with boundary} \right.$$
$$\left. \text{and } \sum_j |S_j| < \infty \right\}. \tag{103}$$

Remark 5. One may consider, more generally, for $0 \le k \le N-2$, the class

$$\mathscr{F}^k = \{\Gamma;\ \Gamma \text{ is a } k\text{-current such that } \Gamma = \partial S \text{ for some } (k+1)\text{-rectifiable current } S \text{ in } \mathbb{R}^N\}, \tag{104}$$

and define, for $\Gamma \in \mathscr{F}^k$, the quantities

$$M^k(\Gamma) = \inf\{|S|;\ S \text{ is a } (k+1)\text{-rectifiable current in } \mathbb{R}^N \text{ such that } \partial S = \Gamma\}, \tag{105}$$
$$K^k(\Gamma) = \inf\{\|\mu\|_{\mathscr{M}};\ \mu \in \mathscr{M}(\mathbb{R}^N; \Lambda^{k+1}) \text{ is a measure such that } \partial\mu = \Gamma\} \tag{106}$$

and

$$D^k(\Gamma) = \sup\{\langle \Gamma, \zeta\rangle;\ \zeta \in C_c^\infty(\mathbb{R}^N; \Lambda^k) \text{ and } \|\mathrm{d}\zeta\|_{L^\infty} \le 1\}. \tag{107}$$

1. Theorem 4 asserts that $M^k(\Gamma) = K^k(\Gamma) = D^k(\Gamma)$ when $k = N-2$ and $\Gamma \in \mathscr{F}^k$. An easy argument (based on Theorem 1) shows that the same holds when $k = 0$.
   In this special case ($k = 0$), with $\Gamma = \sum_{i=1}^m \delta_{P_i} - \sum_{i=1}^m \delta_{N_i}$, the equalities $M^0(\Gamma) = K^0(\Gamma) = D^0(\Gamma)$ take (in any dimension $N \ge 2$) the form

$$\min_{\sigma \in S_m} \sum_{i=1}^m |P_i - N_{\sigma(i)}| = \inf\left\{ \|\mu\|_{\mathscr{M}};\ \mu \in \mathscr{M}(\mathbb{R}^N; \mathbb{R}^N) \text{ and } \operatorname{div}\mu = \sum_{i=1}^m (\delta_{P_i} - \delta_{N_i}) \right\}$$
$$= \sup\left\{ \sum_{i=1}^m (\zeta(P_i) - \zeta(N_i));\ \zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}) \text{ and } \|\nabla\zeta\|_{L^\infty} \le 1 \right\}.$$

   Such results are discussed with further details in [9, Section V], where the motivation came from liquid crystals. They also appear in Santambrogio [29, Chapter 4] under the name "Beckmann's problem". Beckmann's motivation in 1952 came from mathematical economics [4]; he was unaware of Kantorovich's work.
2. In the remaining cases $1 \le k \le N-3$, we still have $K^k(\Gamma) = D^k(\Gamma)$. However, in general we have $M^k(\Gamma) > K^k(\Gamma)$; see, for $N = 4$ and $k = 1$, Young [35], White [34] and Morgan [27].
3. We may also introduce a smooth analogue of $M^k(\Gamma)$. More precisely, assume that $0 \le k \le N-2$ and

$$\Gamma \text{ is the (geometric) boundary of some smooth compact oriented } (k+1)\text{-manifold } S_0 \subset \mathbb{R}^N. \tag{108}$$

   Set

$$M_0^k(\Gamma) = \inf\{|S|;\ S \subset \mathbb{R}^N \text{ is a smooth compact oriented } (k+1)\text{-manifold with boundary } \Gamma\}.$$

   Theorem 3 asserts that, when $k = N-2$ and Γ is connected, we have $M_0^k(\Gamma) = M^k(\Gamma)$. The same holds without assuming Γ connected (see [10, Chapter 4]). This equality also holds when $k = 0$; see, e.g., [10, Chapter 4]. This led us to raise the question whether

$$M_0^k(\Gamma) = M^k(\Gamma),\ \forall\, 1 \le k \le N-3 \text{ (assuming (108))?} \tag{109}$$

   Very recently, F.H. Lin [26] informed us that he gave a positive answer to (109).

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References

[1] G. Alberti, S. Baldo, G. Orlandi, Functions with prescribed singularities, J. Eur. Math. Soc. 5 (3) (2003) 275–311.
[2] F. Almgren, W. Browder, E.H. Lieb, Co-area, liquid crystals, and minimal surfaces, in: Partial Differential Equations, Tianjin, 1986, in: Lecture Notes in Mathematics, vol. 1306, Springer, Berlin, 1988, pp. 1–22.
[3] L. Ambrosio, N. Fusco, D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 2000.
[4] M. Beckmann, A continuous model of transportation, Econometrica 20 (1952) 643–660.
[5] G. Birkhoff, Tres observaciones sobre el algebra lineal, Rev. Univ. Nac. Tucumán Ser. A, Mat. Fis. Teor. 5 (1946) 147–150.
[6] H. Brezis, Liquid crystals and energy estimates for $\mathbb{S}^2$-valued maps, in: J. Ericksen, D. Kinderlehrer (Eds.), Theory and Applications of Liquid Crystals, Minneapolis, MN, USA, 1985, Springer, 1987, pp. 31–52; See also article 112L1 in http://sites.math.rutgers.edu/~brezis/publications.html.
[7] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Universitext, Springer, New York, 2011.
[8] H. Brezis, Remarks on the Monge–Kantorovich problem in the discrete setting, C. R. Acad. Sci. Paris, Ser. I 356 (2) (2018) 207–213.
[9] H. Brezis, J.-M. Coron, E.H. Lieb, Harmonic maps with defects, Commun. Math. Phys. 107 (4) (1986) 649–705.
[10] H. Brezis, P. Mironescu, Sobolev Maps with Values Into the Circle, Birkhäuser, 2019, to appear.
[11] H. Brezis, P. Mironescu, A. Ponce, $W^{1,1}$-maps with values into S^1, in: Geometric Analysis of PDE and Several Complex Variables, in: Contemporary Mathematics, vol. 368, Amer. Math. Soc., Providence, RI, USA, 2005, pp. 69–100.
[12] L.C. Evans, Partial differential equations and Monge–Kantorovich mass transfer, in: Current Developments in Mathematics, Cambridge, MA, USA, 1997, Int. Press, Boston, MA, USA, 1999, pp. 65–126.
[13] L.C. Evans, R.F. Gariepy, Measure Theory and Fine Properties of Functions, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, USA, 1992.
[14] H. Federer, Geometric Measure Theory, Die Grundlehren der mathematischen Wissenschaften, vol. 153, Springer-Verlag New York Inc., New York, 1969.
[15] H. Federer, Real flat chains, cochains and variational problems, Indiana Univ. Math. J. 24 (1974–1975) 351–407.
[16] H. Federer, W.H. Fleming, Normal and integral currents, Ann. of Math. (2) 72 (1960) 458–520.
[17] F. Frankl, L. Pontrjagin, Ein Knotensatz mit Anwendung auf die Dimensionstheorie, Math. Ann. 102 (1) (1930) 785–789.
[18] M. Giaquinta, G. Modica, J. Souček, Cartesian Currents in the Calculus of Variations. I. Cartesian Currents, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics (Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics), vol. 37, Springer-Verlag, Berlin, 1998.
[19] M. Giaquinta, G. Modica, J. Souček, Cartesian Currents in the Calculus of Variations. II. Variational Integrals, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics (Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics), vol. 38, Springer-Verlag, Berlin, 1998.
[20] E. Giusti, Minimal Surfaces and Functions of Bounded Variation, Monographs in Mathematics, vol. 80, Birkhäuser Verlag, Basel, Switzerland, 1984.
[21] R.M. Hardt, J.T. Pitts, Solving Plateau's problem for hypersurfaces without the compactness theorem for integral currents, in: Geometric Measure Theory and the Calculus of Variations, Arcata, Calif., 1984, in: Proceedings of Symposia in Pure Mathematics, vol. 44, Amer. Math. Soc., Providence, RI, USA, 1986, pp. 255–259.
[22] R.L. Jerrard, H.M. Soner, Functions of bounded higher variation, Indiana Univ. Math. J. 51 (3) (2002) 645–677.
[23] L.V. Kantorovitch, On the translocation of masses, Dokl. Akad. Nauk SSSR 37 (1942) 227–229; English translation: J. Math. Sci. 133 (2006) 1381–1382.
[24] R.C. Kirby, The Topology of 4-Manifolds, Lecture Notes in Mathematics, vol. 1374, Springer-Verlag, Berlin, 1989.
[25] M. Krein, D. Milman, On extreme points of regular convex sets, Stud. Math. 9 (1940) 133–138.
[26] F.H. Lin, Personal communication, July 2019.
[27] F. Morgan, Area-minimizing currents bounded by higher multiples of curves, Rend. Circ. Mat. Palermo 33 (1) (1984) 37–46.
[28] A. Poliakovsky, On a minimization problem related to lifting of BV functions with values in S^1, C. R. Acad. Sci., Sér. 1 339 (12) (2004) 855–860.
[29] F. Santambrogio, Optimal Transport for Applied Mathematicians. Calculus of Variations, PDEs, and Modeling, Progress in Nonlinear Differential Equations and Their Applications, vol. 87, Birkhäuser/Springer, Cham, 2015.
[30] H. Seifert, Über das Geschlecht von Knoten, Math. Ann. 110 (1) (1935) 571–592.
[31] C. Villani, Topics in Optimal Transportation, Graduate Studies in Mathematics, vol. 58, American Mathematical Society, Providence, RI, USA, 2003.
[32] C. Villani, Optimal Transport. Old and New, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 338, Springer-Verlag, Berlin, 2009.
[33] J. von Neumann, A Certain Zero-Sum Two-Person Game Equivalent to the Optimal Assignment Problem, Contributions to the Theory of Games, vol. 2, Princeton University Press, 1953, pp. 5–12.
[34] B. White, The least area bounded by multiples of a curve, Proc. Amer. Math. Soc. 90 (2) (1984) 230–232.
[35] L.C. Young, Some extremal questions for simplicial complexes. V. The relative area of a Klein bottle, Rend. Circ. Mat. Palermo 12 (1963) 257–274.