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REMARKS ON SOME MINIMIZATION PROBLEMS ASSOCIATED WITH BV NORMS

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To Luis Caffarelli, a master of regularity, with esteem and affection

ABSTRACT. The purpose of this paper is twofold. Firstly I present an optimal regularity result for minimizers of a $1D$ convex functional involving the BV-norm, under Neumann boundary condition. This functional is a simplified version of models occuring in Image Processing. Secondly I investigate the existence of minimizers for the same functional under Dirichlet boundary condition. Surprisingly, this turns out to be a delicate issue, which is still widely open.

1. Introduction. Our original motivation comes from the study of minimizers of the ROF (= Rudin-Osher-Fatemi) functional

$$\int_\Omega |\nabla u| + \lambda \int_\Omega |u-f|^2,$$

where $\Omega \subset \mathbb{R}^N$ is smooth and bounded, $u \in BV(\Omega) \cap L^2(\Omega)$, $f \in L^2(\Omega)$ is given and $\lambda > 0$ is a parameter. This functional was introduced (in a slightly different form) in [17] and it has been extensively used in Image Processing (see e.g. [5] and the references therein). After scaling we may assume that $\lambda = 1/2$ and we set

$$\Phi(u) = \int_\Omega |\nabla u| + \frac{1}{2}\int_\Omega |u-f|^2. \tag{1}$$

It is standard that there exists a unique minimizer denoted $U \in BV(\Omega) \cap L^2(\Omega)$ for the problem

$$\operatorname*{Inf}_{u \in BV \cap L^2} \Phi(u). \tag{2}$$

Our first goal is to investigate the regularity of U. In variational problems one often expects a gain in regularity. For example, if we replace $\int_\Omega |\nabla u|$ in (1) by $\frac{1}{2}\int_\Omega |\nabla u|^2$, then the minimizer satisfies

$$\begin{aligned} -\Delta u + u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

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and therefore $u \in H^2(\Omega)$. In our situation there is a "modest" gain in regularity since $U \in BV(\Omega)\cap L^2(\Omega)$ while $f \in L^2(\Omega)$ only. Surprisingly this regularizing effect stops here as can be seen from the following simple example (see M. Bonforte and A. Figalli [2] and also T. Sznigir [20, 21]). Take $\Omega = (0,1)$ and let $0 = x_0 < x_1 < \ldots < x_n < x_{n+1} = 1$. Assume that f is a step function,

$$f(x) = \sum_{i=0}^{n} a_i \mathbf{1}_{(x_i,x_{i+1})}(x),$$

where $\mathbf{1}_A$ denotes the characteristic function of A, and $(a_i)_{0\le i\le n}$ are arbitrary constants, then

$$U(x) = \sum_{i=0}^{n} b_i \mathbf{1}_{(x_i,x_{i+1})}(x)$$

for some appropriate constants $(b_i)_{0\le i\le n}$.

A regularity result due V. Caselles, A. Chambolle and M. Novaga [13] asserts that if $N \le 7$ and $\nabla f \in L^\infty_{\text{loc}}(\Omega)$, then $\nabla U \in L^\infty_{\text{loc}}(\Omega)$; moreover if Ω is convex then $\nabla f \in L^\infty(\Omega)$ implies $\nabla U \in L^\infty(\Omega)$. At first sight this result seems optimal. Indeed one can easily construct examples where $\Omega = (0,1), f \in C^\infty([0,1])$ and $U \notin C^1([0,1])$, i.e., U' is discontinuous. (This comes from the fact that solutions of variational inequalities are not C^2 in general. And, as explained in Section 2, when $N = 1$, our U corresponds roughly speaking to the derivative of the solution of a variational inequality). We suspect that the result of [13] might possibly be "upgraded". Here is an improvement when $N = 1$.

Theorem 1. *Assume that $\Omega = (0,1)$ and that $f' \in BV(0,1)$, then $U' \in BV(0,1)$, and*

$$\int_0^1 |U''| \le |f'(0)| + |f'(1)| + \int_0^1 |f''|.$$

Our proof (see Section 2) involves a duality device going back to H. Brezis [3], which reduces the minimization problem (2) to a variational inequality. Much is known about the regularity of solutions of variational inequalities, see e.g. [3], [7], [10], [12], [14], [15]. The main tool we use here is a regularity result due to H. Brezis and D. Kinderlehrer [7] (valid in all dimensions). Unfortunately the duality trick holds only in $1D$. The analogue of Theorem 1 for higher dimensions remains open:

Open Problem 1. *Assume that $\Omega \subset \mathbb{R}^N$, with $N \ge 2$. Let $f \in C^\infty(\overline{\Omega})$. Is it true that the minimizer U of (2) satisfies $\nabla U \in BV(\Omega)$? Can one prove at least that ΔU is a measure?*

If one insists on using the duality device in higher dimensions we are led to the functional

$$\Psi(\vec{u}) = \int_\Omega |\text{div }\vec{u}| + \frac{1}{2}\int_\Omega |\vec{u} - \vec{f}|^2 \tag{3}$$

where $\vec{u} \in X = \{\vec{u} \in L^2(\Omega;\mathbb{R}^N);\ \text{div }\vec{u} \text{ is a finite measure}\}$, and $\vec{f} \in L^2(\Omega;\mathbb{R}^N)$ is given. It is easy to see that

$$\operatorname*{Inf}_{\vec{u}\in X} \Psi(\vec{u}) \tag{4}$$

is achieved by a unique minimizer $\vec{U} \in X$, for which we can establish the following regularity.

Theorem 2. *Assume (for simplicity) that* $\vec{f} \in C^\infty(\overline{\Omega}; \mathbb{R}^N)$. *Then* $\vec{U} \in W^{1,\infty}(\Omega; \mathbb{R}^N)$ *and* $div\, \vec{U} \in BV(\Omega)$.

Problems of the type (4) were introduced in [11] but the authors did not address there the question of regularity for the minimizers.

Since we do not impose any boundary condition in (2), we expect that the minimizers U will satisfy the *Neumann boundary condition*

$$\frac{\partial U}{\partial n} = 0 \quad \text{ on } \partial\Omega. \tag{5}$$

Unfortunately, if $N \geq 2$, we do not have sufficient information on U in order to give a meaning to (5) (the fact that $\nabla U \in L^\infty$ does not suffice). However when $N = 1$ we know that $U' \in BV$ (by Theorem 1), and thus $U'(0), U'(1)$ are well-defined provided $f' \in BV$. We will indeed prove that $U'(0) = 0$ and $U'(1) = 0$. In fact we establish a stronger conclusion:

Corollary 1. *Assume that* $\Omega = (0, 1)$ *and that* $f \in L^2(0, 1)$. *Then the minimizer* U *in (2) is constant near* 0 *and* 1.

Open problem 2. *Assume that* $N \geq 2$ *and* $f \in C^\infty(\overline{\Omega})$. *Can one show that* $\partial U/\partial n = 0$ *on* $\partial\Omega$*, at least in some weak sense (to be determined)?*

Next, we investigate questions similar to (2) associated with a *Dirichlet boundary condition.* Here is a typical example. Fix a (smooth) boundary condition g and consider the minimization problem

$$\text{Inf } \{\Phi(u); u \in BV(\Omega) \cap L^2(\Omega), u = g \text{ on } \partial\Omega\}. \tag{6}$$

Recall that BV functions admit a trace in L^1 of the boundary (see e.g. [1, Theorem 3.87]) and thus the condition $u = g$ on $\partial\Omega$ makes sense. If we take a minimizing sequence (u_n) in (6), we may assume that $u_{n_k} \to u$ in $L^1(\Omega)$ with $u \in BV(\Omega)$, and it may happen that the boundary condition is "lost" in the limit, i.e., u does *not* satisfy $u = g$ on $\partial\Omega$. (Similar questions for functionals which do *not* involve the term $\int_\Omega |u - f|^2$ have been considered e.g. in [18] and [19], but the situation seems to be quite different from ours). It turns out that the existence of a minimizer in (6) is a very delicate issue:

Open Problem 3. *Find (necessary and sufficient) conditions on* f *and* g *such that a minimizer in (6) exists. This problem is open even for* $N = 1$. *In particular it would nice to find a statement which covers both Theorems 3 and 4 below.*

We do not know any result concerning this problem when $N \geq 2$. If $N = 1$ we have an answer in two special cases:

Theorem 3. *Fix* $a_0, a_1 \in \mathbb{R}$, *and consider the minimization problem*

$$\text{Inf} \left\{ \int_0^1 |u'| + \frac{1}{2}\int_0^1 |u|^2; u \in BV(0,1), u(0) = a_0 \textit{ and } u(1) = a_1 \right\}. \tag{7}$$

A minimizer exists if and only if

$$a_0 = a_1 = a \textit{ with } |a| \leq 2, \tag{8}$$

and in this case the unique minimizer is the constant function a.

The above result shows that a minimizer in (6) exists only under *very* restrictive assumptions. A partial result in the same spirit was established by T. Sznigir [20, Theorem 3.16]. An interesting application of Theorem 3 to the study of some "regularized interpolation" problems is presented in [6]. Here is another case where the Inf in (6) is attained.

Theorem 4. *Assume that $f \in C([0,1])$ and set $a_0 = f(0), a_1 = f(1)$. Then the minimization problem*

$$\operatorname{Inf}\left\{\int_0^1 |u'| + \frac{1}{2}\int_0^1 |u-f|^2; u \in BV(0,1),\ u(0)=a_0 \text{ and } u(1)=a_1\right\} \tag{9}$$

admits a (unique) minimizer $U \in BV(0,1)\cap C([0,1])$. If in addition $f \in W^{1,p}(0,1)$ for some $1 \leq p \leq \infty$, (resp. $f' \in BV(0,1)$), then $U \in W^{1,p}(0,1)$, (resp. $U' \in BV(0,1)$).

2. Proofs of Theorem 1, 2 and Corollary 1 via duality. We will first transform problem (2) with $N = 1$, and problem (4) with general $N \geq 1$, into a variational inequality following a duality technique introduced in [3, Section I.1.3] (see also [9]); a similar idea was rediscovered in [11], but our approach is simpler. Related change of unknown in $1D$ are used in [22] and [2]. For the convenience of the reader we present this device in Lemma 1 below (see also the proof of Theorem 2).

Lemma 1. *Let $\Omega = (0,1)$ and $f \in L^2(0,1)$. Consider the following variational inequality*

$$\operatorname{Min}\left\{\frac{1}{2}\int_0^1 |v'|^2 + \int_0^1 fv'; v \in H_0^1(0,1) \text{ and } |v| \leq 1 \text{ in } (0,1).\right\} \tag{10}$$

Let $V \in H_0^1(0,1)$ be the unique minimizer of (10) and set

$$U = V' + f. \tag{11}$$

Then $U \in BV(0,1)$, and U is the unique minimizer of (2). Moreover

$$\frac{1}{2}\int_0^1 |V'|^2 + \int_0^1 fV' = -\left(\int_0^1 |U'| + \frac{1}{2}\int_0^1 |U-f|^2\right). \tag{12}$$

Proof. Let (f_n) be a sequence of smooth functions such that $f_n \to f$ in L^2. Denote by V_n the corresponding minimizer in (10). From standard regularity theory for variational inequalities (see e.g. [3], [7], [10], [12], [14], [15]) we know that

$$V_n \in W^{2,\infty}(0,1) \tag{13}$$

In addition

$$-V_n'' + Z_n = f_n' \text{ on } (0,1), \tag{14}$$

where

$$Z_n \in \gamma(V_n) \text{ on } (0,1), \tag{15}$$

and γ is the multivalued maximal monotone graph

$$\gamma(t) = \begin{cases} [0,\infty) & \text{if } t = +1 \\ 0 & \text{if } -1 < t < +1 \\ (-\infty, 0] & \text{if } t = -1, \end{cases} \tag{16}$$

i.e., $\gamma = \partial j$ where $j(t)$ is the convex function defined by

$$j(t) = \begin{cases} 0 & \text{if } |t| \le 1 \\ +\infty & \text{otherwise.} \end{cases} \tag{17}$$

Moreover it is easy to see that

$$V_n \to V \text{ in } H_0^1(0,1) \text{ as } n \to \infty. \tag{18}$$

We claim that

$$U_n = V_n' + f_n \tag{19}$$

satisfies

$$\int_0^1 |U_n'| + \frac{1}{2}\int_0^1 |U_n - f_n|^2 \le \int_0^1 |u'| + \frac{1}{2}\int_0^1 |u - f_n|^2 \quad \forall u \in W^{1,1}(0,1). \tag{20}$$

Assuming (20) holds we see that (U_n) is bounded in $BV(0,1)$ and thus (U_n) converges in L^2 to a limit denoted $\hat{U} \in BV(0,1)$. From (18) and (19) we deduce that

$$\hat{U} = V' + f \tag{21}$$

and from (20) we obtain

$$\int_0^1 |\hat{U}'| + \frac{1}{2}\int_0^1 |\hat{U} - f|^2 \le \int_0^1 |u'| + \frac{1}{2}|u - f|^2 \quad \forall u \in W^{1,1}(0,1). \tag{22}$$

On the other hand (see e.g. [1, Theorem 3.9] or [8, Appendix 18.7]), given any $u \in BV(0,1)$ there exists a sequence (u_n) in $W^{1,1}(0,1)$ such that

$$u_n \to u \text{ in } L^2(0,1) \text{ and } \int_0^1 |u_n'| \to \int_0^1 |u'|. \tag{23}$$

Therefore (22) holds for every $u \in BV(0,1)$ and hence $\hat{U}$ is a minimizer in (2). By uniqueness we infer that $\hat{U} = U$, which is the desired conclusion of the first part of Lemma 1.

We now turn to the proof of (20). As usual we denote by Sign the monotone graph which is the inverse of γ defined in (16), i.e.,

$$\text{Sign}(s) = \begin{cases} +1 & \text{if } s > 0 \\ [-1,+1] & \text{if } s = 0\,. \\ -1 & \text{if } s < 0 \end{cases} \tag{24}$$

From (15) we see that
$$V_n \in \text{ Sign } Z_n. \tag{25}$$
But
$$Z_n = V_n^{''} + f_n' = U_n' \text{ by (14) and (19).} \tag{26}$$
Combining (25) and (26) yields, for every $u \in W^{1,1}(0,1)$,
$$|u'| - |U_n'| \geq V_n(u' - U_n'), \tag{27}$$
and therefore
$$\begin{array}{c} \int_0^1 |u'| + \frac{1}{2}\int_0^1 |u - f_n|^2 - \int_0^1 |U_n'| - \frac{1}{2}\int_0^1 |U_n - f_n|^2 \\ \geq \int_0^1 V_n u' - V_n U_n' + \frac{1}{2}u^2 - \frac{1}{2}U_n^2 - f_n u + f_n U_n \end{array}. \tag{28}$$
Integrating by parts and using (19) we obtain
$$\int_0^1 V_n u' - V_n U_n' = -\int_0^1 (U_n - f_n)(u - U_n). \tag{29}$$
From (29) we are led to
$$\begin{array}{c} \int_0^1 V_n u' - V_n U_n' + \frac{1}{2}u^2 - \frac{1}{2}U_n^2 - f_n u + f_n U_n \\ = \frac{1}{2}\int_0^1 (u - U_n)^2 \geq 0 \end{array}. \tag{30}$$
Combining (28) and (30) yields (20).

We now turn to the proof of (12). We have, by (25)-(26),
$$V_n U_n' = |U_n'|.$$
and thus, by (19),
$$\left\{ \begin{array}{l} \int_0^1 |U_n'| + \frac{1}{2}|U_n - f_n|^2 = \int_0^1 V_n U_n' + \frac{1}{2}|V_n'|^2 = \\ = \int_0^1 -V_n' U_n + \frac{1}{2}|V_n'|^2 = \int_0^1 -V_n'(V_n' + f_n) + \frac{1}{2}|V_n'|^2 \\ = -\frac{1}{2}\int_0^1 |V_n'|^2 - \int_0^1 f_n V_n'. \end{array} \right. \tag{31}$$
(Recall that $V_n \in H^2 \cap H_0^1$ and $U_n \in H^1$). Returning to (20) we obtain as above, for every $u \in BV$,
$$\limsup_{n\to\infty} \int_0^1 |U_n'| \leq -\frac{1}{2}\int_0^1 |U - f|^2 + \int_0^1 |u'| + \frac{1}{2}|u - f|^2. \tag{32}$$
Choosing in particular $u = U$ in (32) we conclude that $\limsup\limits_{n\to\infty} \int_0^1 |U_n'| \leq \int_0^1 |U'|$. Since
$$\liminf_{n\to\infty} \int_0^1 |U_n'| \geq \int_0^1 |U'|$$

we infer that $\lim_{n\to\infty} \int_0^1 |U_n'| = \int_0^1 |U'|$. Passing to the limit in (31)(using (18)) yields the desired conclusion (12). □

Proof of Theorem 1. We know that solutions of variational inequalities such as (10) enjoy the property that $V^{''} \in BV_{\text{loc}}$ provided f is sufficiently smooth; see Brezis-Kinderlehrer [BK, Theorem 3]. Keeping (11) in mind we will deduce that $U' \in BV_{\text{loc}}$. In our situation we can even extend this result up to the boundary so that $V^{''} \in BV$ and therefore $U' \in BV$, which is the desired conclusion. For this purpose we adapt the proof from [BK]. (A similar technique, applied to evolution equations, has been rediscovered in [16]). We first assume that f is smooth and we consider, as in [3], an approximation of the graph γ (defined in (16)) by smooth monotone functions γ_ε such that $\gamma_\varepsilon(0) = 0$. Let V_ε be the (smooth) solution of the problem

$$\begin{cases} - \quad V_\varepsilon^{''} + \gamma_\varepsilon(V_\varepsilon) = f' \text{ in } (0,1) \\ \quad V_\varepsilon(0) = V_\varepsilon(1) = 0. \end{cases} \tag{33}$$

Let (θ_δ) be a smooth approximation of the graph Sign defined in (24) (e.g. $\theta_\delta(t) = \frac{t}{|t^2+\delta^2|^{1/2}}$). Differentiating (33) and multiplying by $\theta_\delta(V_\varepsilon')$ yields

$$-V_\varepsilon^{'''}\theta_\delta(V_\varepsilon') + \gamma_\varepsilon'(V_\varepsilon)V_\varepsilon'\theta_\delta(V_\varepsilon') = f^{''}\theta_\delta(V_\varepsilon'). \tag{34}$$

Integrating (34) on $(0,1)$, using the monotonicity of θ_δ, and (33), we find

$$\int_0^1 \gamma_\varepsilon'(V_\varepsilon)V_\varepsilon'\theta_\delta(V_\varepsilon') \le |f'(0)| + |f'(1)| + \int_0^1 |f^{''}|. \tag{35}$$

As $\delta \to 0$ (with fixed ε) we see that

$$\int_0^1 |\gamma_\varepsilon(V_\varepsilon)'| \le |f'(0)| + |f'(1)| + \int_0^1 |f^{''}|. \tag{36}$$

Returning to (33) we deduce from (36) that

$$\int_0^1 |(V_\varepsilon' + f)^{''}| \le |f'(0)| + |f'(1)| + \int_0^1 |f^{''}|.$$

Clearly $V_\varepsilon \to V$ in $H_0^1(0,1)$ as $\varepsilon \to 0$, where V is defined in Lemma 1. We conclude that $(V' + f)' \in BV$. Keeping in mind (11) we find that the minimizer U of (2) satisfies $U' \in BV$ and

$$\int_0^1 |U^{''}| \le |f'(0)| + |f'(1)| + \int_0^1 |f^{''}|. \tag{37}$$

Thus estimate (37) has been established for smooth f. By a density argument (as in (23)–see also Lemma 2 below) we reach the same conclusion assuming only $f' \in BV$. □

Proof of Corollary 1. By Lemma 1 we know that U is given by $U = V' + f$ and $V \in H_0^1(0,1)$ is the solution of the variational inequality (10). Since $V \in C([0,1])$

and $V(0) = V(1) = 0$ we infer that $|V(x)| \le \frac{1}{2}$ for x in some neighborhood N_0 of 0 (resp. N_1 of 1). In these neighborhoods the constraint $|V| \le 1$ is not saturated and thus $-V'' = f'$ in $H^{-1}(N_0)$ (resp. $H^{-1}(N_1)$). Hence U is constant on N_0 and N_1. □

Proof of Theorem 2. We follow the same strategy as in the proof of Theorem 1 with some substantial modifications. We start as above with the solution V of the variational inequality

$$\text{Min } \left\{ \frac{1}{2} \int_\Omega |\nabla v|^2 + \int_\Omega \vec{f} \cdot \nabla v;\ v \in H_0^1(\Omega) \text{ and } |v| \le 1 \text{ on } \Omega \right\}. \tag{38}$$

From classical results on variational inequalities (see e.g. [3],[7]) we know that V satisfies the following properties:

$$-\Delta V + Z = \text{ div } \vec{f} \text{ on } \Omega, \tag{39}$$

where

$$Z \in \gamma(V) \text{ on } \Omega, \tag{40}$$

and γ is defined in (16),

$$\Delta V \in L^\infty(\Omega) \text{ and } \|\Delta V\|_{L^\infty} \le 2\|\text{div } \vec{f}\|_{L^\infty}, \tag{41}$$

$$-\Delta V = \text{ div } \vec{f} \text{ in a neighborhood } N \text{ of } \partial\Omega \tag{42}$$

and in particular

$$V \text{ is smooth in } N, \tag{43}$$

$$V \in W^{2,\infty}(\Omega), \tag{44}$$

$$\Delta V \in BV(\Omega). \tag{45}$$

This requires some explanations. In [7] we have only established that $\Delta V \in BV_{\text{loc}}(\Omega)$. Combining this with (43) we conclude that (45) holds. Also we have only established (44) for the one-obstacle problem. However we may recover (44) by applying this result in neighborhoods of the sets $[v = +1]$ and $[v = -1]$. Let

$$\vec{U} = \nabla V + \vec{f}, \tag{46}$$

so that $\vec{U} \in W^{1,\infty}(\Omega; \mathbb{R}^N)$, div $\vec{U} \in BV(\Omega)$ and

$$\text{div } \vec{U} = \Delta V + \text{ div } \vec{f} = Z \text{ by (39)}. \tag{47}$$

We claim that $\vec{U}$ is the unique minimizer of (4) and this will complete the proof of Theorem 2. Fix

$$\vec{u} \in L^2(\Omega; \mathbb{R}^N) \text{ such that div } \vec{u} \text{ is a finite measure }. \tag{48}$$

We need to check that

$$\int_\Omega |\text{div } \vec{U}| + \frac{1}{2} \int_\Omega |\vec{U} - \vec{f}|^2 \le \int_\Omega |\text{div } \vec{u}| + \frac{1}{2} \int_\Omega |\vec{u} - \vec{f}|^2. \tag{49}$$

From (40) we have $V \in \text{Sign } Z = \text{Sign }(\text{div } \vec{U})$ by (47). Hence

$$|\text{div } \vec{U}| = V \text{div } \vec{U} \text{ on } \Omega,$$

and

$$\int_\Omega |\text{div } \vec{U}| = \int_\Omega V \text{div } \vec{U} = -\int_\Omega (\nabla V) \cdot \vec{U} = -\int_\Omega (\vec{U} - \vec{f}) \cdot \vec{U} \text{ by (46).}$$

Therefore

$$\int_\Omega |\text{div } \vec{U}| + \frac{1}{2}\int_\Omega |\vec{U} - \vec{f}|^2 = -\frac{1}{2}\int_\Omega |\vec{U}|^2 + \frac{1}{2}\int_\Omega |\vec{f}|^2. \tag{50}$$

On the other hand

$$\int_\Omega (\vec{U} - \vec{f}) \cdot \vec{u} = \int_\Omega \nabla V \cdot \vec{u} \text{ by (46).} \tag{51}$$

Next we claim that

$$\int_\Omega \nabla V \cdot \vec{u} = -\int_\Omega V \text{div } \vec{u}. \tag{52}$$

This needs to be justified carefully since we only know that div $\vec{u}$ is a finite measure and $\int_\Omega V \text{div } \vec{u}$ is to be understood in the sense of duality of measures and continuous functions. It suffices to invoke the following:

Fact: Given $V \in H_0^1(\Omega) \cap C(\overline{\Omega})$, there exists a sequence $W_n \in C_c^\infty(\Omega)$ such that $W_n \to V$ in $H^1(\Omega)$ and in $C(\overline{\Omega})$.

This fact can be established as in [4, proof of Theorem 9.17].

Combining (51), (52) and $|V| \leq 1$ on Ω, we are led to

$$\left| \int_\Omega (\vec{U} - \vec{f}) \cdot \vec{u} \right| \leq \int_\Omega |\text{div } \vec{u}|. \tag{53}$$

From (50) and (53) we deduce that

$$\begin{aligned}
&\int_\Omega |\text{div } \vec{u}| + \frac{1}{2}|\vec{u} - \vec{f}|^2 - |\text{div } \vec{U}| - \frac{1}{2}|\vec{U} - f|^2 \\
&\geq \int_\Omega -(\vec{U} - \vec{f}) \cdot \vec{u} + \frac{1}{2}|\vec{u} - \vec{f}|^2 + \frac{1}{2}|\vec{U}|^2 - \frac{1}{2}|\vec{f}|^2 \\
&= \frac{1}{2}\int_\Omega |\vec{u} - \vec{U}|^2 \geq 0.
\end{aligned}$$

□

3. Minimizing under Dirichlet condition. Proofs of Theorems 3 and 4. We now turn to the minimization problem (6). When $N = 1$ it takes the form

$$A = \text{ Inf } \left\{ \int_0^1 |u'| + \frac{1}{2}\int_0^1 |u - f|^2 ; u \in BV(0,1),\, u(0) = a_0 \text{ and } u(1) = a_1 \right\}, \tag{54}$$

where $f \in L^2(0,1), a_0$ and a_1 are given.

Since problem (54) is delicate and need not have a solution we replace it by a relaxed problem which always admits a solution.

We start with a basic inequality familiar to the experts (see e.g. [8, Appendix 18.8])

Lemma 2. *Let (u_n) be a bounded sequence in BV such that $u_n \to u$ in $L^1, u_n(0) \to \alpha_0$ and $u_n(1) \to \alpha_1$. Then $u \in BV$ and*

$$\liminf_{n\to\infty} \int_0^1 |u_n'| \geq \int_0^1 |u'| + |u(0) - \alpha_0| + |u(1) - \alpha_1|. \tag{55}$$

Proof. Fix any function $h \in C_c^\infty(\mathbb{R})$ such that $h(0) = \alpha_0$ and $h(1) = \alpha_1$. Consider the functions

$$v_n(t) = \begin{cases} h(t) & \text{if } t < 0 \\ u_n(t) & \text{if } 0 \leq t \leq 1 \\ h(t) & \text{if } t \geq 1 \end{cases}, \quad v(t) = \begin{cases} h(t) & \text{if } t < 0 \\ u(t) & \text{if } 0 \leq t \leq 1 \\ h(t) & \text{if } t > 1 \end{cases}.$$

Clearly $v_n, v \in BV(\mathbb{R})$ and

$$\begin{aligned} \int_{\mathbb{R}} |v_n'| &= \int_{-\infty}^0 |h'| + \int_0^1 |u_n'| + \int_1^\infty |h'| + |u_n(0) - \alpha_0| + |u_n(1) - \alpha_1|, \\ \int_{\mathbb{R}} |v'| &= \int_{-\infty}^0 |h'| + \int_0^1 |u'| + \int_1^\infty |h'| + |u(0) - \alpha_0| + |u(1) - \alpha_1|. \end{aligned} \tag{56}$$

Since $v_n \to v$ in $L^1(\mathbb{R})$ we know that

$$\liminf_{n\to\infty} \int_{\mathbb{R}} |v_n'| \geq \int_{\mathbb{R}} |v'|.$$

Combining this with (56) yields (55). □

Given $u \in BV(0,1)$ set

$$F(u) = \int_0^1 |u'| + \frac{1}{2}\int_0^1 |u - f|^2 + |u(0) - a_0| + |u(1) - a_1|. \tag{57}$$

Our next lemma asserts that F is lower semi-continuous on BV; more precisely:

Lemma 3. *Let (u_n) be a bounded sequence in $BV(0,1)$ such that $u_n \to u$ in $L^2(0,1)$, then $u \in BV((0,1))$ and*

$$\liminf_{n\to\infty} F(u_n) \geq F(u). \tag{58}$$

Proof. Passing to a subsequence we may assume that $u_n(0) \to \alpha_0$ and $u_n(1) \to \alpha_1$. From Lemma 2 we have

$$\begin{aligned} \liminf_{n\to\infty} F(u_n) &\geq \int_0^1 |u'| + |u(0) - \alpha_0| + |u(1) - \alpha_1| \\ &\quad + \frac{1}{2}\int_0^1 |u - f|^2 + |\alpha_0 - a_0| + |\alpha_1 - a_1| \\ &\geq F(u) \text{ by the triangle inequality.} \end{aligned}$$

□

From Lemma 3 we deduce that

$$B = \operatorname*{Inf}_{u \in BV} F(u) \text{ is achieved.} \tag{59}$$

Denote by U the unique minimizer in (59).

Lemma 4. *We have*

$$F(U) = B = A. \tag{60}$$

Proof. Clearly

$$\begin{cases} B & = \operatorname*{Inf}_{u\in BV} F(u) \leq \ \text{Inf}\ \{F(u); u\in BV, u(0)=a_0 \text{ and } u(1)=a_1\} \\ & = \ \text{Inf}\ \left\{\int_0^1 |u'| + \frac{1}{2}\int_0^1 |u-f|^2; u\in BV, u(0)=a_0 \text{ and } u(1)=a_1\right\} \\ & = A. \end{cases} \tag{61}$$

On the other hand, set for $\delta > 0$ small

$$U_\delta(x) = \begin{cases} a_0 & \text{if } 0 < x < \delta \\ U & \text{if } \delta \leq x \leq 1-\delta, \\ a_1 & \text{if } 1-\delta < x < 1 \end{cases}.$$

Clearly $U_\delta \in BV$ and, for a.e. δ,

$$\begin{aligned} F(U_\delta) &= \int_0^1 |U_\delta'| + \frac{1}{2}\int_0^1 |U_\delta - f|^2 \\ &= |U(\delta) - a_0| + |U(1-\delta) - a_1| + \int_\delta^{1-\delta} |U'| + \frac{1}{2}\int_0^1 |U_\delta - f|^2 \end{aligned}.$$

As $\delta \to 0$ we obtain

$$F(U_\delta) \to F(U) = B.$$

But $A \leq F(U_\delta)$ (by definition of A) and thus $A \leq B$. Combining this with (61) yields $A = B$. ☐

Problem (54) usually admits *no* minimizer, while Problem (59) *always* admits a minimizer denoted U. If (by chance!) Problem (54) admits a minimizer $\widetilde{U}$, then $\widetilde{U} = U$. Indeed we have (by assumption) $\widetilde{U} \in BV, \widetilde{U}(0) = a_0, \widetilde{U}(1) = a_1$ and

$$\int_0^1 |\widetilde{U}'| + \frac{1}{2}\int_0^1 |\tilde{U} - f|^2 = A.$$

Thus $F(\widetilde{U}) = A = B$ (by Lemma 4) and $\widetilde{U}$ is a minimizer for (59); from uniqueness we deduce that $\widetilde{U} = U$.

On the other hand if we happen to know that the minimizer U of (59) satisfies $U(0) = a_0$ and $U(1) = a_1$, then U is a minimizer of (54). Indeed we have

$$A = B = F(U) = \int_0^1 |U'| + \frac{1}{2}\int_0^1 |U - f|^2 \geq A.$$

Conclusion. The existence of a minimizer for Problem (54) boils down to the question whether U satisfies

$$U(0) = a_0 \text{ and } U(1) = a_1. \tag{62}$$

Proof of Theorem 3. In view of the above discussion we introduce the relaxed problem

$$\operatorname*{Inf}_{u\in BV} G(u), \tag{63}$$

where

$$G(u) = \int_0^1 |u'| + \frac{1}{2}\int_0^1 |u|^2 + |u(0) - a_0| + |u(1) - a_1|, u \in BV(0,1),$$

(recall that $f = 0$ in Theorem 3). We denote again by U the minimizer in (63). We distinguish two cases.

Case 1. $a_0 a_1 \le 0$.

Case 2. $a_0 a_1 > 0$.

In Case 1 we have $U = 0$. Indeed for every $u \in BV$ we have

$$\begin{aligned} G(u) &\ge \int_0^1 |u'| + |u(0) - a_0| + |u(1) - a_1| \\ &\ge |u(1) - u(0)| + |u(0) - a_0| + |u(1) - a_1| \\ &\ge |a_0 - a_1| = |a_0| + |a_1|. \end{aligned}$$

On the other hand $G(0) = |a_0| + |a_1|$ and therefore 0 is the unique minimizer in (63). We conclude in Case 1 that our original problem (7) admits a solution only if $a_0 = a_1 = 0$ and in this case $U = 0$ is the minimizer of (7).

We now turn to Case 2. Without loss of generality we may assume that

$$0 < a_0 \le a_1. \tag{64}$$

Our next result gives a complete description of U in this case.

Lemma 5. *Assume (64). Then*

$$U \equiv \begin{cases} a_0 & \text{if } a_0 \le 2. \\ 2 & \text{if } a_0 > 2 \end{cases}.$$

Proof. Clearly $G(u^+) \le G(u) \quad \forall u \in BV$, and thus U^+ is also a minimizer of (63). Hence $U^+ = U$ and therefore

$$U \ge 0. \tag{65}$$

Let $\xi \in (0,1)$ be any point of continuity of U. Recall that $U \in BV$ and thus U is continuous except at a countable numbers of points.

Using the constant function $u \equiv U(\xi)$ as a testing function in (63) yields

$$\begin{aligned} &\int_0^1 |U'| + \frac{1}{2}\int_0^1 |U|^2 + |U(0) - a_0| + |U(1) - a_1| \\ &\le \frac{1}{2}|U(\xi)|^2 + |U(\xi) - a_0| + |U(\xi) - a_1| \\ &\le \frac{1}{2}|U(\xi)|^2 + |U(\xi) - U(0)| + |U(0)| - a_0| + |U(1) - U(\xi)| + |U(1) - a_1| \\ &\le \frac{1}{2}|U(\xi)|^2 + \int_0^\xi |U'| + \int_\xi^1 |U'| + |U(0) - a_0| + |U(1) - a_1| \\ &\le \frac{1}{2}|U(\xi)|^2 + \int_0^1 |U'| + |U(0) - a_0| + |U(1) - a_1|, \end{aligned}$$

and thus

$$|U(\xi)|^2 - \int_0^1 |U|^2 \geq 0 \text{ for a.e. } \xi. \tag{66}$$

Since

$$\int_0^1 \left(|U(\xi)|^2 - \int_0^1 |U|^2 \right) d\xi = 0,$$

we conclude that $|U(\xi)|^2 = \int_0^1 |U|^2$ for a.e. ξ. In view of (65) we deduce that U is a constant. Therefore we are led to the minimization of the function

$$f(t) = \frac{1}{2}t^2 + |t - a_0| + |t - a_1|, \; t \geq 0.$$

An easy inspection show that $\operatorname*{Min}_{t \geq 0} f(t)$ is achieved at $t = 2$ if $a_0 > 2$ and at $t = a_0$ if $a_0 \leq 2$. □

Proof of Theorem 3 completed. From Lemma 5 we see that in Case 2 the conditions $U(0) = a_0$ and $U(1) = a_1$ are satisfied if and only if $a_0 = a_1 = a$ and $|a| \leq 2$. In Case 1 the conditions $U(0) = a_0$ and $U(1) = a_1$ are satisfied if and only if $a_0 = a_1 = 0$. □

Proof of Theorem 4. We use a totally different strategy. We rely instead on regularity estimates as in the proof of Theorem 1. Set

$$j_\varepsilon(t) = (\varepsilon^2 + t^2)^{1/2} - \varepsilon + \varepsilon t^2, \, t \in \mathbb{R}$$

so that $j_\varepsilon(t) \to |t|$ as $\varepsilon \to 0$, $j_\varepsilon(t) \leq |t| + \varepsilon t^2 \quad \forall t$, and $j_\varepsilon''(t) \geq 2\varepsilon \quad \forall t$.

By standard variational theory the minimization problem

$$\operatorname{Inf} \left\{ \int_0^1 j_\varepsilon(u') + \frac{1}{2}\int_0^1 |u - f|^2; u \in H^1(0,1), u(0) = a_0 \text{ and } u(1) = a_1 \right\} \tag{67}$$

admits a unique minimizer U_ε. Moreover U_ε satisfies the equation

$$-(\beta_\varepsilon(U_\varepsilon'))' + U_\varepsilon = f \text{ on } (0,1) \tag{68}$$

where $\beta_\varepsilon = j_\varepsilon'$, and U_ε'' has the same regularity as f, e.g. $U_\varepsilon'' \in L^2$ if $f \in L^2$.

We now proceed as in the proof of Theorem 1, namely we set

$$V_\varepsilon = \beta_\varepsilon(U_\varepsilon') \tag{69}$$

so that, by (68), we have

$$-V_\varepsilon' + U_\varepsilon = f \tag{70}$$

and thus

$$-V_\varepsilon'' + U_\varepsilon' = f'. \tag{71}$$

Since β_ε is strictly monotone and bijective we may introduce its inverse function β_ε^{-1} which is denoted γ_ε.

Thus

$$U_\varepsilon' = \gamma_\varepsilon(V_\varepsilon). \tag{72}$$

Combining (72) and (71) yields

$$-V_\varepsilon^{''} + \gamma_\varepsilon(V_\varepsilon) = f'. \tag{73}$$

This is the dual problem to (67). Note that, by (70)

$$V_\varepsilon'(0) = U_\varepsilon(0) - f(0) = a_0 - f(0) = 0,\ V_\varepsilon'(1) = U_\varepsilon(1) - f(1) = a_1 - f(1) = 0, \tag{74}$$

so that V_ε satisfies a *Neumann condition* (by contrast, in Theorem 1, U satisfied a Neumann condition and V satisfied a Dirichlet condition).

We now divide the proof in 5 steps.

Step 1. We assume here that $f \in W^{1,p}(0,1)$ with $1 < p \leq \infty$ and establish the existence of a minimizer U for (9) with $U \in W^{1,p}(0,1)$.

For simplicity we consider only the case $p < \infty$; the case $p = \infty$ is similar. Multiplying (73) by $|\gamma_\varepsilon(V_\varepsilon)|^{p-2}\gamma_\varepsilon(V_\varepsilon)$, and using (74) yields

$$\|\gamma_\varepsilon(V_\varepsilon)\|_{L^p} \leq \|f'\|_{L^p}.$$

From (72) we derive that $\|U_\varepsilon\|_{W^{1,p}} \leq C$ as $\varepsilon \to 0$. Passing to a subsequence we may assume that

$$U_{\varepsilon_n} \rightharpoonup U \text{ weakly in } W^{1,p}.$$

On the other hand (67) implies that

$$\begin{cases} \int_0^1 |U_\varepsilon'| + \frac{1}{2}\int_0^1 |U_\varepsilon - f|^2 & \leq \varepsilon + \int_0^1 j_\varepsilon(u') + \frac{1}{2}\int_0^1 |u-f|^2 \\ & \leq \varepsilon + \int_0^1 |u'| + \varepsilon\int_0^1 |u'|^2 + \frac{1}{2}\int_0^1 |u-f|^2, \end{cases} \tag{75}$$

for every $u \in H^1(0,1)$ such that $u(0) = a_0$ and $u(1) = a_1$. Thus U satisfies: $U \in W^{1,p}(0,1)$, $U(0) = a_0$, $U(1) = a_1$, and

$$\int_0^1 |U'| + \frac{1}{2}\int_0^1 |U-f|^2 \leq \int_0^1 |u'| + \frac{1}{2}\int_0^1 |u-f|^2$$

for every $u \in H^1(0,1)$ such that $u(0) = a_0$ and $u(1) = a_1$.

In order to conclude it suffices to show that given any $u \in BV(0,1)$ there exists a sequence (u_n) in $H^1(0,1)$ such that, as $n \to \infty$, $u_n \to u$ in L^2, $\int_0^1 |u_n'| \to \int_0^1 |u'|$, and $u_n(0) = u(0), u_n(1) = u(1)$ for all n. This is established e.g. in [8, Appendix 18.8].

Step 2. Let $f, \tilde{f} \in W^{1,p}(0,1)$ with $1 < p \leq \infty$, be such that $f(0) = \tilde{f}(0)$ and $f(1) = \tilde{f}(1)$. Let $U, \widetilde{U} \in W^{1,p}(0,1)$ be the corresponding minimizers given by Step 1. Then

$$\|U - \widetilde{U}\|_{L^p} \leq \|f - \tilde{f}\|_{L^p} \tag{76}$$

and

$$\|U' - \widetilde{U}'\|_{L^1} \leq \|f' - \tilde{f}'\|_{L^1}. \tag{77}$$

From (68) we have

$$-(\beta_\varepsilon(U_\varepsilon') - \beta_\varepsilon(\widetilde{U}_\varepsilon'))' + U_\varepsilon - \widetilde{U}_\varepsilon = f - \tilde{f}. \tag{78}$$

Let θ be any (smooth) monotone function on $\mathbb{R}$ such that $\theta(0) = 0$. Multiplying (78) by $\theta(U_\varepsilon - \widetilde{U}_\varepsilon)$, integrating over $(0,1)$ and using the fact that $U_\varepsilon(0) = \widetilde{U}_\varepsilon(0) = a_0$, $U_\varepsilon(1) = \widetilde{U}_\varepsilon(1) = a_1$, we find

$$\int_0^1 (U_\varepsilon - \widetilde{U}_\varepsilon)\theta(U_\varepsilon - \widetilde{U}_\varepsilon) \le \int_0^1 (f - \tilde{f})\theta(U_\varepsilon - \widetilde{U}_\varepsilon). \tag{79}$$

Choosing $\theta(t) = |t|^{p-2}t$ yields

$$\|U_\varepsilon - \widetilde{U}_\varepsilon\|_{L^p} \le \|f - \tilde{f}\|_{L^p},$$

which implies (76) as $\varepsilon \to 0$.

Next we have by (73)

$$-(V_\varepsilon - \widetilde{V}_\varepsilon)'' + \gamma_\varepsilon(V_\varepsilon) - \gamma_\varepsilon(\widetilde{V}_\varepsilon) = f' - \tilde{f}'. \tag{80}$$

Multiplying (80) by $\theta_\delta(V_\varepsilon - \widetilde{V}_\varepsilon)$, where θ_δ is a smooth approximation of Sign (e.g. $\theta_\delta(t) = \frac{t}{|t^2+\delta^2|^{1/2}}$), and integrating over $(0,1)$ gives

$$\int_0^1 |\gamma_\varepsilon(V_\varepsilon) - \gamma_\varepsilon(\widetilde{V}_\varepsilon)| \le \int_0^1 |f' - \tilde{f}'|;$$

here we have used the fact that

$$V_\varepsilon'(0) = \widetilde{V}_\varepsilon'(0) = V_\varepsilon'(1) = \widetilde{V}_\varepsilon'(1) = 0.$$

Hence by (72)

$$\|U_\varepsilon' - \widetilde{U}_\varepsilon'\|_{L^1} \le \|f' - \tilde{f}'\|_{L^1},$$

which implies (77) as $\varepsilon \to 0$.

Step 3. We assume here that $f \in W^{1,1}(0,1)$ and establish the existence of a minimizer U for (9) with $U \in W^{1,1}(0,1)$.

Let (f_n) be a sequence of smooth functions such that $f_n \to f$ in $W^{1,1}(0,1)$ as $n \to \infty$, $f_n(0) = f(0)$, and $f_n(1) = f(1)$ for all n. (The existence of such a sequence is a consequence of the density of $C_c^\infty(0,1)$ in $W_0^{1,1}(0,1)$). Denote by $U_n \in W^{1,\infty}(0,1)$ the minimizer given by Step 1, corresponding to f_n in place of f. From (77) in Step 2 we have

$$\|U_n' - U_m'\|_{L^1} \le \|f_n' - f_m'\|_{L^1} \quad \forall m, n.$$

Therefore (U_n) is a Cauchy sequence in $W^{1,1}(0,1)$. We may thus assume that $U_n \to U$ in $W^{1,1}(0,1)$ as $n \to \infty$, for some $U \in W^{1,1}(0,1)$ satisfying $U(0) = a_0$ and $U(1) = a_1$. By construction we know that

$$\begin{gathered}\int_0^1 |U_n'| + \frac{1}{2}\int_0^1 |U_n - f_n|^2 \le \int_0^1 |u'| + \frac{1}{2}\int_0^1 |u - f_n|^2 \\ \forall n, \forall u \in BV(0,1), u(0) = a_0, u(1) = a_1.\end{gathered} \tag{81}$$

Passing to the limit as $n \to \infty$ completes the proof of Step 3.

Step 4. We establish Theorem 4 assuming only $f \in C([0,1])$.

Let (f_n) be a sequence of smooth functions such that $f_n \to f$ in $f \in C([0,1])$ as $n \to \infty, f_n(0) = f(0), f_n(1) = f(1)$ for all n. By Steps 1 and 2 we see that the corresponding sequence (U_n) in $W^{1,\infty}(0,1)$ satisfies $U_n(0) = a_0, U_n(1) = a_1$ for all n, and

$$\|U_n - U_m\|_{L^\infty} \leq \|f_n - f_m\|_{L^\infty} \quad \forall m, n.$$

Therefore $U_n \to U$ in $C([0,1])$ as $n \to \infty$, for some $U \in C([0,1])$ satisfying $U(0) = a_0$ and $U(1) = a_1$. Passing to the limit in (81) as $n \to \infty$ completes the proof of Step 4.

Step 5. We now turn to the last assertion in Theorem 4, i.e., $U' \in BV$ if $f' \in BV$.

Differentiating (73) we find

$$-V_\varepsilon^{'''} + \gamma_\varepsilon'(V_\varepsilon)V_\varepsilon' = f^{''}. \tag{82}$$

Let θ_δ be a smooth approximation of the graph Sign (e.g. $\theta_\delta(t) = \frac{t}{(t^2+\delta^2)^{1/2}}$). Multiplying (82) by $\theta_\delta(V_\varepsilon')$ and integrating yields

$$\int_0^1 |V_\varepsilon^{''}|^2 \theta_\delta'(V_\varepsilon') + \int_0^1 \gamma_\varepsilon'(V_\varepsilon)V_\varepsilon'\theta_\delta(V_\varepsilon') \leq \int_0^1 |f^{''}|;$$

here we use once more in a crucial way the fact that $V_\varepsilon'(1) = V_\varepsilon'(0) = 0$ by (74), which comes from the *specific prescribed Dirichlet condition* for u in (67). Consequently

$$\int_0^1 \gamma_\varepsilon'(V_\varepsilon)V_\varepsilon'\theta_\delta(V_\varepsilon') \leq \int_0^1 |f^{''}|.$$

As $\delta \to 0$ (with $\varepsilon > 0$ fixed) we obtain

$$\int_0^1 |\gamma_\varepsilon'(V_\varepsilon)V_\varepsilon'| \leq \int_0^1 |f^{''}|,$$

i.e.,

$$\int_0^1 |(\gamma_\varepsilon(V_\varepsilon))'| \leq \int_0^1 |f^{''}|.$$

Going back to (72) we are led to

$$\int_0^1 |U_\varepsilon^{''}| \leq \int_0^1 |f^{''}|.$$

Therefore U defined in Step 1 satisfies $U' \in BV$. □

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