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Minimizers of the $W^{1,1}$-energy of $\mathbb{S}^1$-valued maps with prescribed singularities. Do they exist?

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A R T I C L E I N F O

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For our dear friend Carlo Sbordone on his 70th birthday, wishing him continued success and inspiration.

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A B S T R A C T

The paper is concerned with the least $W^{1,1}$-energy required to produce maps from a domain $\Omega \subset \mathbb{R}^2$ with values into $\mathbb{S}^1$ having prescribed singularities $(a_i)_{1\le i\le k}$. The value of the infimum has been known for a long time and corresponds to the length of minimal configurations connecting the points (a_i) between themselves and/or to the boundary. We tackle here the question whether the infimum of this $W^{1,1}$-energy is achieved. This natural topic turns out to be delicate and we have a complete answer only when $k = 1$. The bottom line for $k \ge 1$ is that the infimum is "rarely" achieved. As a "substitute", we give a full description of the asymptotic behavior of all minimizing sequences and show that they "concentrate" along "convex combinations" of minimal configurations.

1. Introduction

Throughout this paper, the domain $\Omega \subset \mathbb{R}^2$ is smooth, bounded and simply connected. Given $u \in W^{1,1}(\Omega; \mathbb{S}^1)$, we write $u = (u_1, u_2)$. To such u, we may associate various objects and quantities which play a fundamental role in the study of $W^{1,1}(\Omega; \mathbb{S}^1)$ (see e.g. [5,7,9]).

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One of these objects is the Jacobian of u, Ju. This distribution is defined by

$$Ju := \frac{1}{2}\left[\frac{\partial}{\partial x_1}\left(u_1\frac{\partial u_2}{\partial x_2} - u_2\frac{\partial u_1}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(u_1\frac{\partial u_2}{\partial x_1} - u_2\frac{\partial u_1}{\partial x_1}\right)\right]. \tag{1.1}$$

It is well-known (see e.g. [5,8] and [7]) that

$$J(uv) = Ju + Jv,\ \forall\, u, v \in W^{1,1}(\Omega; \mathbb{S}^1), \tag{1.2}$$

$$J\,\overline{u} = -Ju,\ \forall\, u \in W^{1,1}(\Omega; \mathbb{S}^1), \tag{1.3}$$

and

$$Ju = 0 \text{ if and only if } u = e^{\imath\varphi} \text{ for some } \varphi \in W^{1,1}(\Omega; \mathbb{R}). \tag{1.4}$$

If, in addition,

$$u \in W^{1,1}(\Omega; \mathbb{S}^1) \cap C(\Omega \setminus \{a_1, \ldots, a_k\}; \mathbb{S}^1),$$

where $a_1, \ldots, a_k \in \Omega$, then (see e.g. [6,5,8] and [7])

$$Ju = \pi \sum_{j=1}^{k} M_j \delta_{a_j} \ \text{ in } \mathscr{D}'(\Omega). \tag{1.5}$$

Here, the integer $M_j := \deg(u, a_j)$ is the degree of u restricted to any small circle centered at a_j. (This integer does not depend on the small circle.) Roughly speaking, when u has a finite number of singularities, the Jacobian Ju detects the location and the topological strength of the singularities of u.

Given k points $a_1, \ldots, a_k$ in Ω and k integers $M_1, \ldots, M_k \in \mathbb{Z}$, we will use the notation $\mathbf{a} := (a_1, \ldots, a_k)$ and $\mathbf{M} := (M_1, \ldots, M_k)$. When $k = 1$, we let $a := a_1$, $M := M_1$ and write a, M instead of $\mathbf{a}$ and $\mathbf{M}$.

The quantity $L(\mathbf{a}, \mathbf{M})$ defined below appears in many questions involving maps in $W^{1,1}(\Omega; \mathbb{S}^1)$ (see e.g. [4,6,5,8] and [7]).

$$L(\mathbf{a}, \mathbf{M}) := \min\left\{ \sum_{\ell=1}^{n} |P_\ell - N_\ell| \;\middle|\; \begin{array}{l} P_\ell, N_\ell \in \{a_1, \ldots, a_k\} \bigcup \partial\Omega, \ell = 1, \ldots, n, \\ \text{and } \sum_{\ell=1}^{n} (\delta_{P_\ell} - \delta_{N_\ell}) = \sum_{j=1}^{n} M_j \delta_{a_j} \text{ in } \mathscr{D}'(\Omega) \end{array} \right\}. \tag{1.6}$$

Here, the points P_ℓ, N_ℓ need not be all distinct. With no loss of generality we may assume that $P_\ell \neq N_\ell$, $\forall\, \ell$, and then the integer n satisfies $n \leq \sum_{j=1}^{k} |M_j|$. Note that the measure $\sum_{\ell=1}^{n} (\delta_{P_\ell} - \delta_{N_\ell})$ does not "see" points in $\partial\Omega$, since equality in (1.6) holds in the sense of $\mathscr{D}'(\Omega)$. However, even in cases where the equality $\sum_{\ell=1}^{n} (\delta_{P_\ell} - \delta_{N_\ell}) = \sum_{j=1}^{n} M_j \delta_{a_j}$ can be achieved using only points $P_\ell, N_\ell \in \{a_1, \ldots, a_k\}$, it is sometimes advantageous to introduce artificial boundary points because they may lower the quantity $\sum_{\ell=1}^{n} |P_\ell - N_\ell|$ (see Example 3). If $(P_\ell, N_\ell)_{\ell=1}^{n}$ is a minimizer in (1.6), we say that the collection of segments (P_ℓ, N_ℓ), $\ell = 1, \ldots, n$, is a *minimal configuration* associated with $(\mathbf{a}, \mathbf{M})$.

Here are a few examples.

Example 1. When $a \in \Omega$ and $M = 1$, we have $L(a, M) = \operatorname{dist}(a, \partial\Omega)$. For any $N \in \partial\Omega$ such that $|a - N| = \operatorname{dist}(a, \partial\Omega)$, the segment (a, N) is a minimal configuration.

Example 2. Let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{M} = (+1, +1)$. Then $L(\mathbf{a}, \mathbf{M}) = \operatorname{dist}(a_1, \partial\Omega) + \operatorname{dist}(a_2, \partial\Omega)$. If $N_\ell \in \partial\Omega$ satisfies $|a_\ell - N_\ell| = \operatorname{dist}(a_\ell, \partial\Omega)$, $\ell = 1, 2$, then $((a_1, N_1), (a_2, N_2))$ is a minimal configuration.

Example 3. Let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{M} = (+1, -1)$. Then $L(\mathbf{a}, \mathbf{M}) = \min\{|a_1 - a_2|, \text{dist}(a_1, \partial\Omega) + \text{dist}(a_2, \partial\Omega)\}$. When $|a_1 - a_2| \le \text{dist}(a_1, \partial\Omega) + \text{dist}(a_2, \partial\Omega)$, a minimal configuration is (a_1, a_2). When $|a_1 - a_2| \ge \text{dist}(a_1, \partial\Omega) + \text{dist}(a_2, \partial\Omega)$, a minimal configuration is $((a_1, N_1), (P_2, a_2))$, with $N_1, P_2 \in \partial\Omega$ satisfying $|a_1 - N_1| = \text{dist}(a_1, \partial\Omega)$, respectively $|P_2 - a_2| = \text{dist}(a_2, \partial\Omega)$.

To each $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ we associate the number $\Sigma(u) \ge 0$ defined by

$$\Sigma(u) = \inf\left\{\int_\Omega |\nabla v|;\ v \in W^{1,1}(\Omega; \mathbb{S}^1),\ Jv = Ju \text{ in } \mathscr{D}'(\Omega)\right\}. \tag{1.7}$$

Roughly speaking, $\Sigma(u)$ measures the least $W^{1,1}$-energy required to produce $\mathbb{S}^1$-valued maps having the same singularities as u. The quantity $\Sigma(u)$ is ubiquitous in the analysis of the space $W^{1,1}(\Omega; \mathbb{S}^1)$ (see e.g. [5,7–9]). It is well-known (see [6] and [7]) that if $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ satisfies

$$Ju = \pi \sum_{j=1}^{k} M_j \delta_{a_j} \ \text{ in } \mathscr{D}'(\Omega) \tag{1.8}$$

for some pair $(\mathbf{a}, \mathbf{D})$, then

$$\Sigma(u) = 2\pi L(\mathbf{a}, \mathbf{M}). \tag{1.9}$$

Here are some natural questions related to (1.7):

Question 1. Is the infimum in (1.7) achieved?

Question 2. Characterize the minimizers if they exist.

Question 3. Analyze the asymptotic behavior of minimizing sequences for (1.7).

Questions 1 and 2 turn out to be much more delicate than expected. We have a full answer only in the case where u admits *one singularity*. In the case of multiple singularities, our results are incomplete. The bottom line is that the infimum in (1.7) is "rarely" achieved. On the other hand, we have a satisfactory answer to Question 3 in the most general situation.

Remark 1. Another natural approach consists of minimizing $\Sigma(u)$ in the space $BV(\Omega; \mathbb{S}^1)$, which is larger than $W^{1,1}(\Omega; \mathbb{S}^1)$. The main advantage of BV is that minimization problems are more likely to have solutions in BV than in $W^{1,1}$. Concerning our specific minimization problem, a first difficulty to overcome is the definition of Ju when u is merely BV. Clearly, the right-hand side of (1.1) is meaningless for such u. In [14], Ignat proposed several definitions of Ju with $u \in BV(\Omega; \mathbb{S}^1)$. They all coincide with (1.1) when $u \in W^{1,1}$, but do not preserve the algebraic properties (1.2)–(1.3), and in addition they are not stable under strong BV convergence. More significant for our problem, relaxing the functional setting to BV is not of much help: the analogous infimum in (1.7) is still not achieved, in general. We will illustrate this in Appendix B. We therefore restrict ourselves to the $W^{1,1}$ setting, technically simpler.

We first present our main results for one singularity, say $a = 0 \in \Omega$ with degree $M \ge 1$, so that $Ju = \pi M\, \delta_0$. In Theorems 1 and 2, we assume that $\text{dist}(0, \partial\Omega) = 1$ and denote by $\mathbb{D}$ the open unit disk centered at 0. Set

$$X := \{x \in \partial\Omega;\ |x| = 1\},\ Y := \{\theta \in [0, 2\pi];\ e^{\imath\theta} \in X\}. \tag{1.10}$$

Theorem 1. *The infimum is attained in (1.7) if and only if $|X| > 0$.*

If $|X| > 0$, then v is a minimizer in (1.7) if and only if v is locally constant in $\Omega \setminus \mathbb{D}$ and, in $\mathbb{D}$, we have $v(re^{\imath\theta}) = e^{\imath h(\theta)}$. Here, $h \in W^{1,1}((0,2\pi);\mathbb{R})$ satisfies

$$h \text{ is nondecreasing,} \tag{1.11}$$

$$h(2\pi) = h(0) + 2\pi M, \tag{1.12}$$

and

$$\operatorname{supp} h' \subset Y. \tag{1.13}$$

In the next result we discuss the asymptotic behavior of minimizing sequences in (1.7).

Theorem 2. *Let (v_n) be a minimizing sequence in (1.7). Then, up to a subsequence, we have, for some probability measure μ on $\mathbb{S}^1$ supported in X,*

$$\int_\Omega |\nabla v_n|\,\zeta \to 2\pi M \int_{\mathbb{S}^1} \left(\int_0^1 \zeta(rx)\,dr \right) d\mu(x),\ \forall\, \zeta \in C(\overline{\Omega}). \tag{1.14}$$

And conversely.

Remark 2. Let us assume, for simplicity, that $M = 1$ and that X contains only two points, b and c. Let (v_n) be a minimizing sequence in (1.7). Then, up to a subsequence,

$$|\nabla v_n|\,dx \rightharpoonup 2\pi\,\lambda \mathscr{H}^1 \llcorner_{[0,\,b]} + 2\pi\,(1-\lambda)\mathscr{H}^1 \llcorner_{[0,\,c]},$$

where λ could be *any* number in $[0,1]$. This is in sharp contrast with the situation analyzed by Brezis, Coron and Lieb in [6]. The minimization problem studied there is

$$\inf \left\{ \int_\Omega |\nabla v|^2;\ v : \Omega \to \mathbb{S}^2 \right\},$$

with $\Omega \subset \mathbb{R}^3$. The inf is taken over all maps v which are smooth outside the origin and have degree one around the origin. Assuming that the set

$$X = \{x \in \partial\Omega;\ |x| = \operatorname{dist}(0,\partial\Omega)\}$$

is finite, it is proved in [6] that, along a minimizing subsequence (v_n), we have $|\nabla v_n|^2\,dx \rightharpoonup 8\pi \mathscr{H}^1 \llcorner_{[0,\,b]}$, for some $b \in X$. The lack of quantization phenomenon for the $W^{1,1}$-norm was originally pointed out in [6] (see the comments following Theorem 8.2 and at the end of Appendix E).

The proofs of Theorems 1 and 2 are presented in Section 2.

We now turn to multiple singularities. We start with a simple case of non-existence.

Theorem 3. *Assume that there are only finitely or countably many minimal configurations associated with $(\mathbf{a},\mathbf{M})$. Then the infimum in (1.7)–(1.8) is not attained.*

More generally, assume that there exists a null set (for the Lebesgue measure) $Z \subset \Omega$ such that for every minimal configuration (P_ℓ, N_ℓ), $\ell = 1,\ldots,n$, associated with $(\mathbf{a},\mathbf{M})$ we have $\cup_{\ell=1}^n (P_\ell, N_\ell) \subset Z$. Then the infimum in (1.7)–(1.8) is not attained.

Remark 3. We return to the case of a single singularity, as in Theorem 1. Assume e.g. that $a = 0$, $M = 1$ and $\operatorname{dist}(0, \partial\Omega) = 1$. Then the minimal configurations are $(0, x)$, with $x \in X$ (here, X is as in (1.10)), and thus the smallest set Z containing all minimal configurations is $Z := \cup_{x\in X}(0, x)$. We may then rephrase the conclusion of Theorem 1 as follows. There exists a minimizer v in (1.7) if and only if Z has positive Lebesgue measure. In view of Theorem 3, it is tempting to conjecture that a statement similar to Theorem 1 holds in general. Roughly speaking, Theorem 3 asserts that, when the collection of all minimal configurations is a null set in Ω, there is no minimizer in (1.7). However, the converse is wrong; see Remark 5.

Here are some of our results concerning Question 3, i.e., the asymptotic behavior of minimizing sequences.

Theorem 4. *Assume that there are only finitely many minimal configurations $U_1, \dots, U_m$ associated with $(\mathbf{a}, \mathbf{M})$. Let (v_n) be a minimizing sequence in (1.7)–(1.8). Then, up to a subsequence,*

$$|\nabla v_n|\, dx \rightharpoonup 2\pi \sum_{j=1}^{m} \lambda_j \mathscr{H}^1 \llcorner U_j, \text{ with } \lambda_j \in [0, 1] \text{ and } \sum_{j=1}^{m} \lambda_j = 1.$$

And conversely.

The notation $\mathscr{H}^1 \llcorner U_j$ requires an explanation. Recall that a minimal configuration is a finite collection of segments (P_ℓ, N_ℓ). Then we define

$$\mathscr{H}^1 \llcorner U_j := \sum_{\ell} \mathscr{H}^1 \llcorner (P_\ell, N_\ell). \tag{1.15}$$

The proofs of Theorems 3 and 4 are given in Section 3. In fact, we will present in Section 3 far-reaching extensions of these results. Here is a typical example. Given a minimal configuration U associated with $(\mathbf{a}, \mathbf{M})$, we define $\mathscr{H}^1 \llcorner U$ via (1.15). We set

$$\mathscr{L}_{min} := \{Q = \mathscr{H}^1 \llcorner U;\ U \text{ is a minimal configuration associated with } (\mathbf{a}, \mathbf{M})\}. \tag{1.16}$$

Clearly, $\mathscr{L}_{min} \subset \mathscr{M}(\Omega) = [C_0(\Omega)]^*$ and $\|Q\|_{\mathscr{M}} = L(\mathbf{a}, \mathbf{M})$, $\forall\, Q \in \mathscr{L}_{min}$. Therefore, $\mathscr{L}_{min}$ is a bounded subset of $\mathscr{M}(\Omega)$. A basic property is

Lemma 1. *$\mathscr{L}_{min}$ is weakly-$*$ closed in $\mathscr{M}(\Omega)$, and thus $\mathscr{L}_{min}$ equipped with the weak-$*$ topology $\sigma(\mathscr{M}(\Omega), C_0(\Omega))$ is a compact metrizable space.*

We have the following

Theorem 5. *Let (v_n) be a minimizing sequence in (1.7)–(1.8). Then there exists a probability measure $\overline{\mu}$ on $\mathscr{L}_{min}$ such that up to a subsequence we have*

$$|\nabla v_n|\, dx \rightharpoonup 2\pi\, \overline{\kappa} \text{ in } \mathscr{M}(\Omega), \tag{1.17}$$

where

$$\langle \overline{\kappa}, \zeta \rangle := \int_{\mathscr{L}_{min}} \langle Q, \zeta \rangle \, d\overline{\mu}(Q),\ \forall\, \zeta \in C_0(\Omega). \tag{1.18}$$

Note that the right-hand side of (1.18) is well-defined, since $Q \mapsto \langle Q, \zeta \rangle$ is continuous on the compact $\mathscr{L}_{min}$.

We call the attention of the reader to the following

Open Problem. Find a tractable necessary and sufficient condition on $(\mathbf{a}, \mathbf{M})$ and Ω for the existence of a minimizer in (1.7)–(1.8).

In Section 4 we present some regularity results, e.g. continuity away from the points a_i, for the minimizers of (1.7)–(1.8), assuming they exist.

2. Proofs of Theorems 1 and 2

Proof of Theorem 1. We know that $\Sigma(u) = 2\pi L(0, M) = 2\pi M$ as in Example 1. The infimum in (1.7) is still $2\pi M$ if we replace Ω by $\mathbb{D}$. Assume that v is a minimizer in (1.7). Then

$$2\pi M = \int_\Omega |\nabla v| = \int_{\mathbb{D}} |\nabla v| + \int_{\Omega\setminus\mathbb{D}} |\nabla v| \geq 2\pi M + \int_{\Omega\setminus\mathbb{D}} |\nabla v|.$$

Consequently,

$$\int_{\mathbb{D}} |\nabla v| = 2\pi M \tag{2.1}$$

and

$$\int_{\Omega\setminus\mathbb{D}} |\nabla v| = 0, \tag{2.2}$$

so that $\nabla v = 0$ in $\Omega \setminus \mathbb{D}$ and thus

$$v \text{ is locally constant in } \Omega \setminus \overline{\mathbb{D}}. \tag{2.3}$$

Set, in polar coordinates,

$$\frac{\partial v}{\partial \tau} := \frac{1}{r}\frac{\partial v}{\partial \theta}.$$

Clearly, we have

$$\int_{\mathbb{D}} |\nabla v| = \int_{\mathbb{D}} \sqrt{\left|\frac{\partial v}{\partial r}\right|^2 + \left|\frac{\partial v}{\partial \tau}\right|^2} \geq \int_{\mathbb{D}} \left|\frac{\partial v}{\partial \tau}\right| = \int_{\mathbb{D}} \left|v \wedge \frac{\partial v}{\partial \tau}\right| = \int_0^1 \int_{C(0,r)} \left|v \wedge \frac{\partial v}{\partial \tau}\right| d\ell\, dr. \tag{2.4}$$

Here, $C(0, r)$ denotes the circle of radius r centered at 0. On the other hand, if $w \in W^{1,1}(\mathbb{S}^1; \mathbb{S}^1)$, then we have

$$\int_{\mathbb{S}^1} w \wedge \frac{\partial w}{\partial \tau}\, d\ell = 2\pi \deg w; \tag{2.5}$$

this is just another way of writing the classical Cauchy formula

$$\deg w = \frac{1}{2\imath\pi} \int_{\mathbb{S}^1} \frac{1}{w} \left(\frac{\partial w}{\partial \tau}\right) d\ell.$$

Rescaling (2.5) and returning to v we have

$$\int_{C(0,r)} v \wedge \frac{\partial v}{\partial \tau} d\ell = 2\pi\, \deg(v, C(0, r)) \text{ for a.e. } r \in (0, 1). \tag{2.6}$$

Assume temporarily that $v \in C(\mathbb{D} \setminus \{0\})$. By (1.5), we find that $\deg(v, C(0, r)) = M$ for every $r \in (0, 1)$. If we remove the continuity assumption, then we have the following result, whose proof is postponed to Section 4.

Lemma 2. *Let $v \in W^{1,1}(\mathbb{D};\mathbb{S}^1)$ be such that $Ju = \pi\, M\, \delta_0$. Then*

$$\deg(v, C(0,r)) = M \text{ for a.e. } r \in (0,1).$$

Proof of Theorem 1 *completed.* Combining Lemma 2 with (2.4) yields

$$v \wedge \frac{\partial v}{\partial \tau} \geq 0 \text{ a.e. on } \mathbb{D}$$

and

$$\frac{\partial v}{\partial r} = 0 \text{ a.e. on } \mathbb{D},$$

so that, on $\mathbb{D}$, v does not depend on r.

Therefore, we may write $v(re^{\imath\theta}) = e^{\imath h(\theta)}$ on $\mathbb{D}$, with $h \in W^{1,1}((0,2\pi);\mathbb{R})$ nondecreasing and satisfying $h(2\pi) = h(0)+2\pi M$. Since v is locally constant in $\Omega\setminus\overline{\mathbb{D}}$, we deduce that h is locally constant in $(0,2\pi)\setminus Y$, so that $\operatorname{supp} h' \subset Y$. Note that a function h with all the required properties exists if only if $|Y| > 0$. Conversely, it is easy to see that any function v locally constant in $\Omega\setminus\overline{\mathbb{D}}$, which can be written as $v(re^{\imath\theta}) = e^{\imath h(\theta)}$, where $h \in W^{1,1}((0,2\pi);\mathbb{R})$ satisfies (1.11)–(1.13), is a minimizer for (1.7). $\square$

Proof of Theorem 2. Theorem 2 is a special case of Theorems 5 and 7. However, since it requires some additional arguments to see that the three statements coincide in the specific setting of Theorem 2, we present a direct proof.

We start with some observations. A measure μ on $\mathbb{S}^1$ defines a (-1)-homogeneous measure $\widetilde{\mu}$ in Ω, having in polar coordinates the form

$$\widetilde{\mu}(\zeta) = \int_{\mathbb{S}^1}\int_0^1 \zeta(rx)\, dr d\mu(x),\ \forall\, \zeta \in C_0(\Omega).$$

Here, "(-1)-homogeneous" comes from the fact that, if μ has density f, then $\widetilde{\mu}$ has, in $\mathbb{D}$, density $\frac{1}{|x|} f(x/|x|)$, which is (-1)-homogeneous. In the special case where $\mu = \delta_b$ for some point $b \in \mathbb{S}^1$, we have $\widetilde{\mu} = \mathscr{H}^1 \llcorner [0,b]$.

The following facts are straightforward:

Fact 1. We have $\widetilde{\mu} = 0$ in $\Omega \setminus \mathbb{D}$ and $\widetilde{\mu}(\mathbb{D}) = \mu(\mathbb{S}^1)$.

Fact 2. If $\mu_n \rightharpoonup \mu$ in $\mathscr{M}(\mathbb{S}^1)$, then $\widetilde{\mu_n} \rightharpoonup \widetilde{\mu}$ in $\mathscr{M}(\Omega)$.

Fact 3. If $w \in W^{1,1}(\mathbb{D})$ is 0-homogeneous, then $|\nabla w|$ is (-1)-homogeneous in $\mathbb{D}$.

The main ingredient of the proof of Theorem 2 is the following

Lemma 3. *Let (v_n) be a minimizing sequence in* (1.7). *Let ν be such that, up to a subsequence, $|\nabla v_n|\, dx \rightharpoonup \nu$ in $\mathscr{M}(\Omega)$. Then $\nu = \widetilde{\xi}$ for some measure ξ on $\mathbb{S}^1$ with $\operatorname{supp} \xi \subset X$.*

Proof of Theorem 2 *completed.* Let (v_n) be a minimizing sequence in (1.7). As in the proof of Theorem 1, we have $\int_\Omega |\nabla v_n| \to 2\pi M$, and also $\int_{\mathbb{D}} |\nabla v_n| \geq 2\pi M$. Therefore, we must have $\int_{\Omega\setminus\mathbb{D}} |\nabla v_n| \to 0$ and $\int_{\mathbb{D}} |\nabla v_n| \to 2\pi M$. It follows that any ν as in the above lemma must satisfy $\nu(\mathbb{D}) = 2\pi M$. Using Fact 1, we thus have $\nu = 2\pi M\, \widetilde{\mu}$, with $\mu \in \mathscr{P}(\mathbb{S}^1)$ supported in X.

Conversely, let $\mu \in \mathscr{P}(\mathbb{S}^1)$ be supported in X. We may identify μ with an element in $\mathscr{P}(X)$. We want to construct a sequence such that (1.14) holds. It suffices to consider the case where $\mu = \sum_{j=1}^k \lambda_j \delta_{b_j}$, where $\lambda_j \in [0,1]$, $b_j \in X$, $\forall\, j = 1,\ldots,k$, $\sum_{j=1}^k \lambda_j = 1$.

Indeed, assuming this achieved, the case of a general $\mu \in \mathscr{P}(X)$ is settled as follows. Let

$$\mathscr{Q} := \{\nu \in \mathscr{M}(\Omega);\ \exists \text{ a sequence } (v_n) \text{ minimizing in (1.7) such that } |\nabla v_n|\, dx \rightharpoonup \nu\}.$$

By definition, $\mathscr{Q}$ is clearly weakly sequentially closed. Using Fact 2, if $\mu_n \rightharpoonup \mu$ in $\mathscr{M}(\mathbb{S}^1)$ and $2\pi\, \widetilde{\mu_n} \in \mathscr{Q}$, $\forall\, n$, then $2\pi\, \widetilde{\mu} \in \mathscr{Q}$. The above assumption is that $2\pi\, \widetilde{\mu} \in \mathscr{Q}$ when μ is a discrete probability on X. Since such probabilities are weakly sequentially dense in $\mathscr{P}(X)$, we find that $\mathscr{Q} \supset \{2\pi\, \widetilde{\mu};\ \mu \in \mathscr{P}(X)\}$, which is the desired converse.

We now return to the construction of (v_n) for a discrete measure $\mu = \sum_{j=1}^{k} \lambda_j\, \delta_{b_j}$ with

$$\lambda_j \in (0,1],\ \forall\, j = 1, \ldots, k,\ b_j \neq b_\ell,\ \forall\, j \neq \ell \text{ and } \sum_{j=1}^{k} \lambda_j = 1. \tag{2.7}$$

We may always take $M = 1$; the general case follows by choosing (v_n^M). Assuming e.g. that $b_1 = 1$, we order the points b_j in such a way that $b_j = e^{\imath\theta_j}$ with $0 = \theta_1 < \theta_2 < \cdots < \theta_k < 2\pi$.

We will use a variant of the dipole construction (see [6,7]). Given $\varepsilon > 0$ (sufficiently small) consider the cone in $\mathbb{R}^2$,

$$\widetilde{Q}_\varepsilon := \{(x_1, x_2);\ |x_1| < 2\varepsilon\, x_2,\ 0 < x_2 < 1/2\},$$

with height $1/2$ and base of length 2ε.

Next we consider the symmetry S with respect to the line $\{(x_1, 1/2);\ x_1 \in \mathbb{R}\}$, and set $Q_\varepsilon := \widetilde{Q}_\varepsilon \cup S(\widetilde{Q}_\varepsilon)$.

The set Q_ε will be used in the construction of a dipole with vertices 0 and $\imath$, as explained below. Given $0 < \lambda \le 1$, consider the functions $\widetilde{w}_\varepsilon : \widetilde{Q}_\varepsilon \to \mathbb{S}^1$, defined by

$$\widetilde{w}_\varepsilon(x_1, x_2) := e^{-\imath\pi\, \lambda\, x_1/(2\varepsilon\, x_2)}$$

and $w_\varepsilon : Q_\varepsilon \to \mathbb{S}^1$ defined by

$$w_\varepsilon(x) := \begin{cases} \widetilde{w}_\varepsilon(x), & \text{if } x \in \widetilde{Q}_\varepsilon \\ w_\varepsilon(Sx), & \text{if } x \in S(\widetilde{Q}_\varepsilon). \end{cases}$$

The standard dipole construction corresponds to $\lambda = 1$.

Set

$$\partial_+ Q_\varepsilon := (\partial Q_\varepsilon) \cap \{(x_1, x_2) \in \mathbb{R}^2;\ x_1 > 0\} \text{ and } \partial_- Q_\varepsilon := (\partial Q_\varepsilon) \cap \{(x_1, x_2) \in \mathbb{R}^2;\ x_1 < 0\}.$$

Note that

$$w_\varepsilon = \begin{cases} e^{-\imath\pi\, \lambda}, & \text{on } \partial_+ Q_\varepsilon \\ e^{\imath\pi\, \lambda}, & \text{on } \partial_- Q_\varepsilon, \end{cases}$$

so that w_ε is *not* constant on ∂Q_ε unless $\lambda = 1$. This is in sharp contrast with the standard dipole construction (where we can extend w_ε by a constant outside Q_ε). Here, w_ε jumps from $e^{-\imath\pi\, \lambda}$ to $e^{\imath\pi\, \lambda}$ when crossing Q_ε from $\partial_+ Q_\varepsilon$ to $\pi_- Q_\varepsilon$ (which corresponds to the standard orientation on a small circle centered at 0).

A straightforward calculation (see e.g. [6,7]) yields

$$\int_{Q_\varepsilon} |\nabla w_\varepsilon| \le 2\pi\, \lambda\, (1 + \varepsilon). \tag{2.8}$$

We place a first dipole w_ε on $Q_{1,\varepsilon}$ with vortices at 0 and b_1 (instead of 0 and $\imath$) and with $\lambda := \lambda_1$. We multiply the corresponding function w_ε by a factor $\xi_1 := e^{\imath\pi\, \lambda_1}$, so that the new function $w_{1,\varepsilon} := \xi_1\, w_\varepsilon$ jumps from 1 to $e^{2\imath\pi\, \lambda_1}$ when crossing $Q_{1,\varepsilon}$. Next we place a second dipole $Q_{2,\varepsilon}$ with vertices at 0 and b_2 and with

$\lambda := \lambda_2$. We multiply the corresponding function by a factor $\xi_2 := e^{\imath\pi\,(2\lambda_1+\lambda_2)}$, so that we obtain a function $w_{2,\varepsilon}$ defined in $Q_{2,\varepsilon}$ and satisfying

$$w_{2,\varepsilon\,|\partial_+ Q_{2,\varepsilon}} = w_{1,\varepsilon\,|\partial_- Q_{1,\varepsilon}} = e^{2\imath\pi\,\lambda_1}.$$

We proceed in the same manner until we reach $Q_{k,\varepsilon}$. Choosing $\xi_k := e^{\imath\pi\,(2\lambda_1+2\lambda_2+\cdots+2\lambda_{k-1}+\lambda_k)}$, we see that the new function $w_{k,\varepsilon}$ jumps when crossing $Q_{k,\varepsilon}$ from $e^{2\imath\pi\,(\lambda_1+\lambda_2+\cdots+\lambda_{k-1})}$ to

$$e^{2\imath\pi\,(\lambda_1+\lambda_2+\cdots+\lambda_k)} = e^{2\imath\pi} = 1 \text{ (by (2.7))},$$

which matches well with $w_{1,\varepsilon}$ since

$$w_{k,\varepsilon\,|\partial_- Q_{k,\varepsilon}} = w_{1,\varepsilon\,|\partial_+ Q_{1,\varepsilon}} = 1.$$

Next, we glue the functions $w_{j,\varepsilon}$ by taking appropriate constants on the components of $\mathbb{D}\setminus(\{0\}\cup\cup_{j=1}^k Q_{j,\varepsilon})$, in order to obtain a map v_ε continuous on $\mathbb{D}\setminus\{0\}$. Clearly, this map extends by continuity to $\overline{\mathbb{D}}\setminus(\{0\}\cup\cup_{j=1}^k\{b_j\})$. In addition, it satisfies $v_\varepsilon\in \mathrm{Lip}_{loc}(\overline{\mathbb{D}}\setminus(\{0\}\cup\cup_{j=1}^k\{b_j\}))$, $\deg(v_\varepsilon, C(0,r)) = 1$ for every $r<1$ and, by (2.8),

$$\int_{\mathbb{D}} |\nabla v_\varepsilon| \le 2\pi\,(1+\varepsilon).$$

Finally, we extend v_ε to Ω as follows. Let ω be a connected component of $\Omega\setminus\overline{\mathbb{D}}$. Then $(\partial\omega)\cap\mathbb{S}^1$ consists of an arc $\mathscr{A}$ on $\mathbb{S}^1$ whose interior is disjoint from X, and which connects two points in X. (Here, we use in an essential way the assumption that Ω is simply connected.) Therefore, v_ε is constant on $\mathscr{A}$ and we set $v_\varepsilon := v_{\varepsilon|\mathscr{A}}$ on ω.

Clearly, $v_n := v_{1/n}$, $n\ge n_0$, has all the required properties. □

Proof of Lemma 3. We start from the following version of (2.4):

$$\begin{aligned}2\pi\,M + o(1) &= \int_{\mathbb{D}} |\nabla v_n| = \int_{\mathbb{D}} \sqrt{\left|\frac{\partial v_n}{\partial r}\right|^2 + \left|\frac{\partial v_n}{\partial \tau}\right|^2} \ge \int_{\mathbb{D}} \left|\frac{\partial v_n}{\partial \tau}\right| = \int_{\mathbb{D}} \left|v_n\wedge\frac{\partial v_n}{\partial \tau}\right| \\ &= \int_0^1\int_{C(0,r)} \left|v_n\wedge\frac{\partial v_n}{\partial \tau}\right| d\ell\,dr \ge \int_0^1 \left|\int_{C(0,r)} v_n\wedge\frac{\partial v_n}{\partial \tau} d\ell\right| dr \ge 2\pi\,M.\end{aligned} \tag{2.9}$$

We find that the following hold:

$$\int_{\mathbb{D}} \left|\frac{\partial v_n}{\partial r}\right| \to 0,\ \int_{\Omega\setminus\mathbb{D}} |\nabla v_n| \to 0 \tag{2.10}$$

and

$$\int_{\mathbb{D}} \left(v_n\wedge\frac{\partial v_n}{\partial \tau}\right)^- \to 0. \tag{2.11}$$

Here, $t^- := \begin{cases}0, & \text{if } t\ge 0\\ -t, & \text{if } t<0\end{cases}$.

In particular, up to a subsequence we have, for a.e. $r\in(0,1)$,

$$\int_{C(0,r)} \left(v_n\wedge\frac{\partial v_n}{\partial \tau}\right)^- \to 0. \tag{2.12}$$

On the other hand, let us note the inequality

$$\int_{C(0,r)} |\nabla v_n|\,d\ell \ge 2\pi\,M,\ \text{ for a.e. } r\in(0,1); \tag{2.13}$$

this follows from Lemma 2 combined with the degree formula (2.6). By (2.9) and (2.13), we find that

$$\lim_{n\to\infty}\int_0^1 \left|\int_{C(0,r)} |\nabla v_n|\, d\ell - 2\pi\, M\right|\, dr = 0. \tag{2.14}$$

By (2.10) and (2.14), possibly up to a subsequence we have

$$(0,\infty) \ni r \mapsto \int_{C(0,r)\cap\Omega} |\nabla v_n| \quad \text{is dominated in } L^1((0,\infty)). \tag{2.15}$$

By similar calculations, we obtain

$$\int_{\mathbb{D}(0,r)} |\nabla v_n| \le \int_{\Omega} |\nabla v_n| - \int_{\mathbb{D}\setminus\mathbb{D}(0,r)} |\nabla v_n| \le 2\pi\, M\, r + o(1),\ \forall\, r \in (0,1), \tag{2.16}$$

and

$$\int_{\{x\in\overline{\Omega};\, |x|\ge 1-\varepsilon\}} |\nabla v_n| \le 2\pi\, M\, \varepsilon + o(1),\ \forall\, \varepsilon \in (0,1). \tag{2.17}$$

In view of (2.16)–(2.17), it suffices to prove the existence of a measure ξ on $\mathbb{S}^1$ such that

$$\int_{\Omega} |\nabla v_n|\, \zeta \to \int_{\mathbb{S}^1} \left(\int_0^1 \zeta(rx)\, dr\right)\, d\xi(x),\ \forall\, \zeta \in C_c^\infty(\mathbb{D}\setminus\{0\}) \tag{2.18}$$

and

$$\operatorname{supp}\xi \subset X. \tag{2.19}$$

Using a partition of unity in $\mathbb{D}\setminus\{0\}$, it suffices to establish (2.18) when ζ is supported in a set of the form

$$A := \left\{re^{\imath\theta};\, r_1 < r < r_2,\ \theta_1 < \theta < \theta_2\right\},\ 0 < r_1 < r_2 < 1,\ \theta_2 - \theta_1 < 2\pi.$$

Since A is simply connected and $Jv_n = 0$ in A, we may write, in A, $v_n = e^{\imath\varphi_n}$ with $\varphi_n \in W^{1,1}$. In terms of φ_n, (2.18) amounts to

$$\int_{\Omega} |\nabla \varphi_n|\, \zeta \to \int_{\mathbb{S}^1} \left(\int_{r_1}^{r_2} \zeta(rx)\, dr\right)\, d\xi(x),\ \forall\, \zeta \in C_c^\infty(A), \tag{2.20}$$

and it suffices to define ξ on the set $B := \left\{e^{\imath\theta};\, \theta_1 < \theta < \theta_2\right\}$. (Assuming that this has been done, by covering $\mathbb{S}^1$ with a finite number of B's we obtain a global object such that (2.18) holds.)

Consider some $R_n \in (r_1, r_2)$ such that:

1. The trace y_n of v_n on $C(0,R_n)$ belongs to $W^{1,1}$ and the sequence $(y_n(R_n\cdot))$ is bounded in $W^{1,1}(\mathbb{S}^1)$.
2. The trace ψ_n of φ_n on $C(0,R_n)\cap A$ belongs to $W^{1,1}$ and $e^{\imath\psi_n} = y_n$ in $C(0,R_n)\cap A$.
3. (2.12) holds with $r = R_n$.

Let ξ be a measure on B such that, possibly up to a subsequence, we have

$$\int_{\mathbb{S}^1} \left|\frac{\partial y_n}{\partial\tau}(R_n\omega)\right| g(\omega)\, d\ell(\omega) = \int_{\mathbb{S}^1} \left|\frac{\partial \psi_n}{\partial\tau}(R_n\omega)\right| g(\omega)\, d\ell(\omega) \to \int_{\mathbb{S}^1} g\, d\xi,\ \forall\, g \in C_c(B;\mathbb{R}). \tag{2.21}$$

We will prove that (2.18) holds for this ξ. In terms of phases, (2.10)–(2.15) imply

$$\int_A \left|\frac{\partial\varphi_n}{\partial r}\right| \to 0, \tag{2.22}$$

$$\int_{C(0,r)\cap A} \left(\frac{\partial\varphi_n}{\partial\tau}\right)^- \to 0 \text{ for a.e. } r \in (r_1, r_2) \tag{2.23}$$

and

$$(r_1, r_2) \ni r \mapsto \int_{C(0,r)\cap A} |\nabla \varphi_n| \quad \text{is dominated in } L^1((r_1, r_2)). \tag{2.24}$$

By (2.12) and (2.21), we also have

$$\int_{\mathbb{S}^1} \frac{\partial \psi_n}{\partial \tau}(R_n \omega)\, g(\omega)\, d\ell(\omega) \to \int_{\mathbb{S}^1} g\, d\xi, \ \forall\, g \in C_c(B; \mathbb{R}). \tag{2.25}$$

Using (2.22) and (2.24), for a.e. r in (r_1, r_2) we have

$$\varphi_n(r\cdot) - \psi_n(R_n \cdot) \to 0 \text{ in } L^1(B), \text{ uniformly in } r \in (r_1, r_2) \text{ as } n \to \infty \tag{2.26}$$

and thus (using (2.25) and (2.26))

$$\begin{aligned} -\int_A \varphi_n(x) \frac{\partial \zeta}{\partial \tau}(x)\, dx &= -\int_A \psi_n(R_n x/|x|) \frac{\partial \zeta}{\partial \tau}(x)\, dx + o(1) \\ &\to \int_B \int_{r_1}^{r_2} \zeta(rx)\, dr\, d\xi(x) \text{ as } n \to \infty, \ \forall\, \zeta \in C_c^\infty(A). \end{aligned} \tag{2.27}$$

Equivalently, we have

$$\int_A \frac{\partial \varphi_n}{\partial \tau} \zeta \to \int_{\mathbb{S}^1} \left(\int_{r_1}^{r_2} \zeta(rx)\, dr \right) d\xi(x), \ \forall\, \zeta \in C_c^\infty(A). \tag{2.28}$$

We obtain (2.20) from (2.22), (2.23), (2.24) and (2.28).

It remains to prove (2.19). In view of (2.28), this amounts to proving that, if the above B satisfies $\overline{B} \cap X = \varnothing$, then

$$\int_A \frac{\partial \varphi_n}{\partial \tau} \zeta \to 0, \ \forall\, \zeta \in C_c^\infty(A). \tag{2.29}$$

In turn, (2.29) is obtained via a slight modification of the above arguments. Let $\rho > 1$ be such that $\{rx;\, x \in B, 0 \le r < \rho\} \subset \Omega$. Existence of such ρ follows from the assumption $\overline{B} \cap X = \varnothing$. Consider now some $R_n \in (1, \rho)$ (instead of $R_n \in (r_1, r_2)$) such that the above properties 1–3 and (2.21) hold. Define ξ starting from these R_n's. This ξ will satisfy (2.27). By (2.10), we find that $\xi = 0$. $\square$

3. Proofs of Theorems 3–5

Since we will present generalizations of these theorems we need some preliminary material. We denote by $\mathscr{E}$ the following class of distributions:

$$\mathscr{E} := \left\{ T \in \mathscr{D}'(\Omega);\, T = \sum (\delta_{P_j} - \delta_{N_j}), \text{ with } P_j, N_j \in \overline{\Omega}, \ \sum |P_j - N_j| < \infty \right\}. \tag{3.1}$$

Up to a factor π, the class $\mathscr{E}$ characterizes all possible Jacobians of maps $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ [1,8,7] as explained below.

1. Let $u \in W^{1,1}(\Omega; \mathbb{S}^1)$. Then there exist points $P_j, N_j \in \overline{\Omega}$ such that

$$\sum |P_j - N_j| < \infty \tag{3.2}$$

and

$$Ju = \pi \sum (\delta_{P_j} - \delta_{N_j}) \text{ in } (W_0^{1,\infty}(\Omega))^* \text{ (and thus in } \mathscr{D}'(\Omega)). \tag{3.3}$$

2. Conversely, given $T \in \mathscr{E}$, there exists $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ such that $Ju = \pi T$.

3. In addition, we have

$$\Sigma(u) = 2\pi \inf \left\{ \sum |\widetilde{P}_j - \widetilde{N}_j|;\ \widetilde{P}_j, \widetilde{N}_j \text{ satisfy (3.2)–(3.3)} \right\}. \tag{3.4}$$

4. An equivalent formulation of (3.4) is the following. Define, for $T \in \mathscr{E}$, the Wasserstein norm

$$\|T\|_W := \sup \left\{ \langle T, \zeta\rangle;\ \zeta \in W_0^{1,\infty}(\Omega;\mathbb{R}), \|\nabla\zeta\|_{L^\infty} \le 1 \right\} < \infty. \tag{3.5}$$

(Alternatively, we could take, in (3.5), the sup over functions $\zeta \in C_c^\infty(\Omega;\mathbb{R})$.) Then

$$\Sigma(u) = 2\,\|Ju\|_W. \tag{3.6}$$

For such a general T, (generalized) minimal configurations need not exist [16], i.e., the inf in (3.4) need not be achieved. In this context, it is convenient to work with vector-valued measures. Following [7], we will consider *connections* and *minimal connections*. A *connection* $\mathscr{C}$ associated with $T \in \mathscr{E}$ is an $\mathbb{R}^2$-valued measure $\mathscr{C}$ on Ω satisfying:

$$\mathscr{C} := \sum \nu_i \mathscr{H}^1 \llcorner (S_i \cap \Omega), \tag{3.7}$$

where (S_i) is a countable collection of Borel subsets S_i of C^1 oriented curves $\gamma_i \subset \mathbb{R}^2$ with normals ν_i, such that $\sum \mathscr{H}^1(S_i \cap \Omega) < \infty$, and

$$\operatorname{curl} \mathscr{C} = T \text{ in } \mathscr{D}'(\Omega). \tag{3.8}$$

5. It turns out that $\Sigma(u)$ can also be computed starting from connections. More specifically, let $u \in W^{1,1}(\Omega;\mathbb{S}^1)$ and set $T := \frac{1}{\pi} Ju$. Then we have [7]

$$\inf\{\|\mathscr{C}\|_{\mathscr{M}};\ \mathscr{C} \text{ is a connection associated with } T\} = \|T\|_W = \frac{1}{2\pi}\Sigma(u). \tag{3.9}$$

6. The infimum in (3.9) is always achieved. A *minimal connection* is a minimizer in (3.9).
7. Let us return to the situation considered in the previous sections, where $T := \frac{1}{\pi} Ju$ is a finite sum of Dirac masses. Then $\|T\|_W = L(\mathbf{a}, \mathbf{M})$ (by (1.9) and (3.5)). Consider a minimal configuration $((P_\ell, N_\ell))_{\ell=1}^n$ associated with $(\mathbf{a}, \mathbf{M})$. Then $\mathscr{C} := \sum \nu_\ell \mathscr{H}^1 \llcorner (N_\ell, P_\ell)$ (with (N_ℓ, P_ℓ) the oriented segment from N_ℓ to P_ℓ) is a minimal connection [7]. And conversely.

In view of these considerations, Theorem 3 is an immediate consequence of the following.

Theorem 6. *Let $T \in \mathscr{E} \setminus \{0\}$. Assume that there are only finitely or countably many minimal connections associated with T.*

Then the infimum

$$\inf \left\{ \int_\Omega |\nabla v|;\ v \in W^{1,1}(\Omega;\mathbb{S}^1),\ Jv = \pi T \right\} \tag{3.10}$$

is not attained.

More generally, the same conclusion holds if we assume that there exists a null set (for the Lebesgue measure) $Z \subset \Omega$ such that $\operatorname{supp} \mathscr{C} \subset Z$ $\mathscr{H}^1$-a.e., for every minimal connection $\mathscr{C}$.

The proof of Theorem 6 relies on the coarea formula for Sobolev maps (see e.g. [12,15]) and a consequence of this formula. This key ingredient, Lemma 4, is a delicate result due to Alberti, Baldo and Orlandi ([1, Theorem 3.8], with roots in [2, Appendices A.5–A.8] and [11, 4.3]).

We first recall the Sobolev version of the coarea formula, following the presentation in [1, Sections 7.4, 7.5]. Let $v : \Omega \to \mathbb{S}^1$ be a Borel function such that $v \in W^{1,1}$. Let E be the set of points where v is not approximatively differentiable (for the definition of the approximate differential, see e.g. [3, Definition 3.70]). Then E is a null set (for the Lebesgue measure). For $\alpha \in \mathbb{S}^1$, we set

$$\mathscr{D}_\alpha = [v = \alpha] := \{x \in \Omega \setminus E;\ v(x) = \alpha\}.$$

Then there exists a full measure set $A \subset \mathbb{S}^1$ such that

$$\mathscr{D}_\alpha \text{ is } \mathscr{H}^1\text{-rectifiable, and } v \wedge \nabla v \neq 0\ \mathscr{H}^1\text{-a.e. on } \mathscr{D}_\alpha,\ \forall\, \alpha \in A. \tag{3.11}$$

In addition, the following "coarea formula" holds

$$\int_\Omega |\nabla v| = \int_{\mathbb{S}^1} \mathscr{H}^1(\mathscr{D}_\alpha)\, d\alpha. \tag{3.12}$$

More generally, we have the following

$$\int_\Omega g|\nabla v| = \int_{\mathbb{S}^1} \left(\int_{\mathscr{D}_\alpha} g\, d\mathscr{H}^1 \llcorner \mathscr{D}_\alpha \right) d\alpha. \tag{3.13}$$

Here, equality is valid for any non-negative Borel function g or for any Borel function g such that $g|\nabla v| \in L^1(\Omega)$.

The next result is a reformulation of [1, Theorem 3.8]. (The latter is stated in terms of rectifiable currents. One "translates" it in terms of connections using the "dictionary" (3.18)–(3.20).)

Lemma 4. *Let $v \in W^{1,1}(\Omega; \mathbb{S}^1)$. Set $T := \frac{1}{\pi} Jv$ and $\mathscr{D}_\alpha := [v = \alpha]$, $\forall\, \alpha \in \mathbb{S}^1$. Then, for a.e. α,*

$$\mathscr{C}_\alpha := \frac{v \wedge \nabla v}{|v \wedge \nabla v|} \mathscr{H}^1 \llcorner \mathscr{D}_\alpha$$

is a connection associated with T.

Note that, in view of (3.11), the definition of $\mathscr{C}_\alpha$ makes sense for a.e. $\alpha \in \mathbb{S}^1$.

Proof of Theorem 6. We may assume that Z is a Borel set. Argue by contradiction and assume that v is a minimizer in (3.10), so that

$$\int_\Omega |\nabla v| = 2\pi \, \|T\|_W. \tag{3.14}$$

By Lemma 4 and (3.9), we have

$$\mathscr{H}^1(\mathscr{D}_\alpha) = \|\mathscr{C}_\alpha\|_{\mathscr{M}} \geq \|T\|_W, \text{ for a.e. } \alpha \in \mathbb{S}^1. \tag{3.15}$$

By (3.14), (3.15) and the coarea formula (3.12), we find that $\mathscr{C}_\alpha$ is a minimal connection, for a.e. $\alpha \in \mathbb{S}^1$. For any such α, we have

$$\mathscr{D}_\alpha = \operatorname{supp} \mathscr{C}_\alpha \subset Z\ \mathscr{H}^1 - \text{a.e.} \tag{3.16}$$

Let now $g := 1_Z$, so that

$$\text{for a.e. } \alpha \in \mathbb{S}^1,\ g = 1\ \mathscr{H}^1\text{-a.e. on } \mathscr{D}_\alpha. \tag{3.17}$$

Since $g = 0$ a.e., we find, using (3.17) and the coarea formula (3.13), that

$$0 = \int_\Omega g\, |\nabla v| = \int_{\mathbb{S}^1} \left(\int_{\mathscr{D}_\alpha} g\, d\mathscr{H}^1 \llcorner \mathscr{D}_\alpha \right) d\alpha = \int_{\mathbb{S}^1} \mathscr{H}^1(\mathscr{D}_\alpha)\, d\alpha = 2\pi \, \|T\|_W > 0.$$

This contradiction completes the proof of Theorem 6. $\square$

Remark 4. The idea of using the coarea formula to prove the inequality $\int_\Omega |\nabla v| \geq 2\pi \|T\|_W$ when v is smooth except at a finite number of singularities goes back to Almgren, Browder and Lieb [2].

Remark 5. The converse to Theorem 6 is wrong, even when T is a finite sum of Dirac masses. Actually, while the converse is true for one singularity (see Theorem 1), it is already wrong for two singularities. Indeed, consider a domain Ω and points $a_1, a_2 \in \Omega$ such that the following hold:
1. The set $X_1 := \{x \in \partial\Omega;\ |a_1 - x| = \operatorname{dist}(a_1, \partial\Omega)\}$ has positive length.
2. The set $X_2 := \{x \in \partial\Omega;\ |a_2 - x| = \operatorname{dist}(a_2, \partial\Omega)\}$ consists of a single point, say N_2.

Let $\mathbf{a} := (a_1, a_2)$ and $\mathbf{M} := (+1, +1)$, so that we have $T = \delta_{a_1} + \delta_{a_2}$. Then the minimal configurations are $((a_1, N_1), (a_2, N_2))$, with $N_1 \in X_1$. Therefore, any set Z as in Theorem 6 must contain $\cup_{N_1 \in X_1}(a_1, N_1)$, and thus has positive Lebesgue measure.

We claim that, for the above T, there exists no minimizer v in (3.10). Indeed, argue by contradiction. As in the proof of Theorem 6, for a.e. $\alpha \in \mathbb{S}^1$ the set $\mathscr{D}_\alpha$ is the support of a minimal connection. In our case, this means that, for such α, $\mathscr{D}_\alpha$ equals, $\mathscr{H}^1$-a.e., a set of the form $\mathscr{S}_{N_1} := (a_1, N_1) \cup (a_2, N_2)$. However, this is impossible since by definition $\mathscr{D}_\alpha \cap \mathscr{D}_\beta = \varnothing$, for a.e. $\alpha, \beta \in \mathbb{S}^1$ such that $\alpha \neq \beta$, while $\mathscr{H}^1(\mathscr{S}_{N_1} \cap \mathscr{S}_{\widetilde{N}_1}) > 0$, $\forall\, N_1, \widetilde{N}_1 \in X_1$.

Some preliminaries are needed in order to extend Theorems 4 and 5 to the case of a general T. These considerations may also be understood using tools from geometric measure theory. This would require working with rectifiable currents instead of connections. The two are related through the following straightforward equivalence:

$$\mathscr{C} \text{ is a connection associated with } T \iff \mathbf{C} := \mathscr{C}^\perp \text{ satisfies (3.19) below:} \tag{3.18}$$

$$\mathbf{C} \text{ is a finite mass integer multiplicity 1-rectifiable current such that } \partial\mathbf{C} = T. \tag{3.19}$$

(If $\mathscr{C} = (\mathscr{C}_x, \mathscr{C}_y)$, then $\mathscr{C}^\perp := (-\mathscr{C}_y, \mathscr{C}_x)$.)

In addition, we have

$$|\mathscr{C}| = |\mathbf{C}|. \tag{3.20}$$

We will take advantage of our specific situation and work only with connections and BV functions. This is reminiscent of the well-known observation of Hardt and Pitts [13] that in codimension one the theory of rectifiable currents is essentially equivalent to the theory of BV functions.

Fix some $T \in \mathscr{E}$, and define

$$\mathscr{K} := \{\mathscr{C};\ \mathscr{C} \text{ is a connection associated with } T\}. \tag{3.21}$$

Given $m \geq 0$, set

$$\mathscr{K}^m := \{\mathscr{C} \in \mathscr{K};\ \|\mathscr{C}\|_{\mathscr{M}} \leq m\}. \tag{3.22}$$

(In view of (3.9), $\mathscr{K}^m$ is non-empty if and only if $m \geq \|T\|_W$.)

By definition, $\mathscr{K}^m$ is a subset of the closed ball $\overline{B}(0, m)$ of $\mathscr{M}(\Omega; \mathbb{R}^2) = [C_0(\Omega; \mathbb{R}^2)]^*$, and thus has a metrizable topology inherited from the weak-$*$ topology $\sigma(\mathscr{M}(\Omega; \mathbb{R}^2), C_0(\Omega; \mathbb{R}^2))$ on $\overline{B}(0, m)$. We have the following vectorial analogue of Lemma 1 mentioned in the introduction.

Lemma 5. *$\mathscr{K}^m$ equipped with the weak-$*\sigma(\mathscr{M}(\Omega; \mathbb{R}^2), C_0(\Omega; \mathbb{R}^2))$ topology is a compact metrizable space.*

Although it is possible to give a direct proof of Lemma 5 (see Remark 7), it will be more pleasant to work with a distance, δ, defined globally on $\mathscr{K}$, such that the following holds.

Lemma 6. *The metric δ induces on $\mathscr{K}^m$ the $\sigma(\mathscr{M}(\Omega;\mathbb{R}^2), C_0(\Omega;\mathbb{R}^2))$ topology, $\forall\, m \geq 0$.*

The definition of δ we present below is reminiscent of the one of the flat norm [11, Section 4.1.19, p. 377]. Let $\mathscr{C}_1$, $\mathscr{C}_2 \in \mathscr{K}$. Then $\operatorname{curl}(\mathscr{C}_2 - \mathscr{C}_1) = 0$, and therefore there exists some $\psi \in BV(\Omega)$ such that $\mathscr{C}_2 = \mathscr{C}_1 + D\psi$. We define the distance

$$\delta(\mathscr{C}_1, \mathscr{C}_2) := \inf\left\{\|\psi\|_{L^1};\ \psi \in BV(\Omega),\ \mathscr{C}_2 = \mathscr{C}_1 + D\psi\right\}. \tag{3.23}$$

To start with, let us note the following simple

Lemma 7. *We have*

$$|\langle \mathscr{C}_2, \chi\rangle - \langle \mathscr{C}_1, \chi\rangle| \leq \delta(\mathscr{C}_1, \mathscr{C}_2)\, \|\operatorname{div}\chi\|_{L^\infty},\ \ \forall\, \mathscr{C}_1,\, \mathscr{C}_2 \in \mathscr{K},\ \ \forall\, \chi \in C_c^1(\Omega;\mathbb{R}^2).$$

Proof. This follows from the fact that, if $\mathscr{C}_2 = \mathscr{C}_1 + D\psi$, then

$$|\langle \mathscr{C}_2, \chi\rangle - \langle \mathscr{C}_1, \chi\rangle| = |\langle D\psi, \chi\rangle| = \left|\int_\Omega \psi\, \operatorname{div}\chi\right| \leq \|\psi\|_{L^1}\|\operatorname{div}\chi\|_{L^\infty}. \quad \square$$

Proof of Lemma 6. We have to prove that, for $\mathscr{C}_n, \mathscr{C} \in \mathscr{K}^m$, we have

$$\mathscr{C}_n \rightharpoonup \mathscr{C} \text{ in } \mathscr{M}(\Omega;\mathbb{R}^2) \iff \delta(\mathscr{C}_n, \mathscr{C}) \to 0. \tag{3.24}$$

Implication "$\Longleftarrow$" is clear, in view of Lemma 7. Conversely, write $\mathscr{C}_n = \mathscr{C} + D\psi_n$, with $\int_\Omega \psi_n = 0$. Then ψ_n is bounded in $BV(\Omega)$. Let $\psi \in BV(\Omega)$ be such that $\int_\Omega \psi = 0$ and, possibly up to a subsequence, $\psi_n \to \psi$ in $L^1(\Omega)$. Then

$$\int_\Omega \psi\, \operatorname{div}\chi = \lim_{n\to\infty}\int_\Omega \psi_n\, \operatorname{div}\chi = \lim_{n\to\infty}\langle \mathscr{C}_n - \mathscr{C}, \chi\rangle = 0,\ \forall\, \chi \in C_c^1(\Omega;\mathbb{R}^2). \tag{3.25}$$

It follows that $D\psi = 0$, and thus $\psi = 0$. Therefore, for the full original sequence (ψ_n), we have $\psi_n \to 0$ in $L^1(\Omega)$, so that $\delta(\mathscr{C}_n, \mathscr{C}) \to 0$. $\square$

In what follows, we endow $\mathscr{K}$ with the distance δ. Lemma 6 implies the following

Corollary 1. *The map $\mathscr{K} \ni \mathscr{C} \mapsto \|\mathscr{C}\|_{\mathscr{M}}$ is lower semi-continuous.*

The next statement is a variant of Federer's compactness theorem [11, Theorem 4.2.17, p. 414].

Lemma 8. *$(\mathscr{K}^m, \delta)$ is compact.*

The proof of Lemma 8 relies on the following

Lemma 9. *Fix some $\mathscr{C}_0 \in \mathscr{K}$. For any vector-valued distribution $F \in \mathscr{D}'(\Omega;\mathbb{R}^2)$, we have*

$$F \in \mathscr{K} \iff [F = \mathscr{C}_0 + D\psi \text{ for some } \psi \in BV(\Omega;\mathbb{Z})]. \tag{3.26}$$

Before proceeding to the proof of Lemma 9, let us recall a few facts concerning the fine structure of the distributional gradient $D\psi$ with $\psi \in BV(\Omega)$; see e.g. [3, Chapter 3]. If $\psi \in BV(\Omega;\mathbb{R})$, then the measure $D\psi$ can be (uniquely) written as a sum of an absolutely continuous part with respect to the Lebesgue measure,

$D^a\psi$, whose density is denoted $\nabla\psi$, a Cantor part $D^c\psi$ and a jump part $D^j\psi$ ([3, Definition 3.91]). With an abuse of notation, we write this decomposition as:

$$D\psi = \nabla\psi + D^c\psi + D^j\psi. \tag{3.27}$$

Assuming that ψ equals, outside its jump set ([3, Definition 3.67]), its approximate limit ([3, Definition 3.63]), we may write

$$D^j\psi = \sum_{i=0}^{\infty}(\psi^+ - \psi^-)\,\nu_i\,\mathscr{H}^1 \llcorner S_i. \tag{3.28}$$

Here, S_i are disjoint Borel subsets of C^1 curves γ_i, ν_i is a normal vector orienting γ_i, $u^\pm$ are the approximate side limits of ψ on γ_i ([3, Definition 3.67]) – the "+" and "−" sides are determined by ν_i – and $\sum_i \int_{S_i}|\psi^+ - \psi^-|\,d\mathscr{H}^1 < \infty$. With a more compact notation, if we set $S := \cup_i S_i$ and $\nu := \nu_i$ on S_i, then $D^j\psi = (\psi^+ - \psi^-)\,\nu\,\mathscr{H}^1 \llcorner S$, and S is the jump set of ψ.

We next recall Volpert's chain rule. If f is C^1 and Lipschitz and ψ is BV, then we have

$$D(f\circ\psi) = f'(\psi)\nabla\psi + f'(\psi)D^c\psi + \frac{f(\psi^+) - f(\psi^-)}{\psi^+ - \psi^-}D^j\psi. \tag{3.29}$$

Proof of Lemma 9. "$\Longrightarrow$" Let $F \in \mathscr{K}$ and $\psi_0 \in BV(\Omega;\mathbb{R})$ such that $F = \mathscr{C}_0 + D\psi_0$. Then $\nabla\psi_0 = 0$, $D^c\psi_0 = 0$ and $D^j\psi_0 = F - \mathscr{C}_0$. Taking the specific form of the elements in $\mathscr{K}$ into account, we have $\psi_0^+ - \psi_0^- \in \mathbb{Z}$ $\mathscr{H}^1$-a.e. on the jump set of ψ_0. Volpert's chain rule implies that $D[\sin(2\pi\,\psi_0)] = 0$ and $D[\cos(2\pi\,\psi_0)] = 0$, and thus $e^{\imath 2\pi\,\psi_0}$ is constant. It follows that, for some appropriate $C \in \mathbb{R}$, $\psi := \psi_0 + C$ satisfies $e^{\imath 2\pi\,\psi} = 1$, and this ψ satisfies the requirements of (3.26).

"$\Longleftarrow$" Let $\psi \in BV(\Omega;\mathbb{Z})$ be such that $S = \mathscr{C}_0 + D\psi$. Since $\sin(2\pi\psi) = 0$, we have $\nabla\psi = 0$ and $D^c\psi = 0$ (by (3.29)). On the other hand, we have (possibly after performing a suitable decomposition of the S_i's and by changing, if necessary their orientations) $D^j\psi = \sum_{i=1}^{\infty} i\,\nu_i\,\mathscr{H}^1 \llcorner S_i$, with $\sum_{i=1}^{\infty} i\,\mathscr{H}^1(S_i) < \infty$. By repeating i times each S_i, we may rewrite this as $D^j\psi = \sum_{\ell=1}^{\infty}\nu_\ell\,\mathscr{H}^1 \llcorner \Sigma_\ell$, with $\sum_{\ell=1}^{\infty}\mathscr{H}^1(\Sigma_\ell) < \infty$. In view of the specific form of $\mathscr{C}_0$, we find that $F = D\,\psi - \mathscr{C}_0 \in \mathscr{K}$. □

Proof of Lemma 8. We may assume that $\mathscr{K}^m$ is non-empty. Let $(\mathscr{C}_n) \subset \mathscr{K}^m$. Fix some $\mathscr{C}_0 \in \mathscr{K}$, and write (using Lemma 9) $\mathscr{C}_n = \mathscr{C}_0 + D\,\psi_n$ for some $\psi_n \in BV(\Omega;\mathbb{Z})$. Without loss of generality, we may assume that $⨍_\Omega \psi_n \in [0,1)$. Thus (ψ_n) is bounded in $BV(\Omega)$. Let $\psi \in BV(\Omega)$ be such that an appropriate subsequence satisfies $\psi_{n_k} \to \psi$ in $L^1(\Omega)$ and a.e. Then ψ is $\mathbb{Z}$-valued. Set $\mathscr{C} := \mathscr{C}_0 + D\,\psi$, so that (again by Lemma 9) $\mathscr{C} \in \mathscr{K}$. By construction, we have $\delta(\mathscr{C}_{n_k}, \mathscr{C}) \to 0$ as $k \to \infty$. The fact that $\mathscr{C} \in \mathscr{K}^m$ follows from Corollary 1. □

Corollary 2. *Set*

$$\mathscr{K}_{min} := \{\mathscr{C} \in \mathscr{K};\ \|\mathscr{C}\|_{\mathscr{M}} = \|T\|_W\}. \tag{3.30}$$

Then $\mathscr{K}_{\min}$ is a non-empty set, and $(\mathscr{K}_{\min}, \delta)$ is compact.

Proof. Compactness follows from Lemma 8, non-emptiness from (3.9) and Corollary 1. □

Consider next a finite Borel measure μ on $\mathscr{K}^m$. By Lemma 6, the integral

$$\nu(\chi) := \int_{\mathscr{K}^m}\langle\mathscr{C}, \chi\rangle\, d\mu \tag{3.31}$$

makes sense for any $\chi \in C_0(\Omega;\mathbb{R}^2)$.

With a slight abuse of notation, we still denote $\overline{B}(0,m)$ a closed ball of $\mathscr{M}(\Omega;\mathbb{R})$. We have the following

Lemma 10. *Set $m_0 := \|T\|_W$. Then the map*

$$(\mathscr{K}_{min}, \delta) \ni \mathscr{C} \mapsto |\mathscr{C}| \in \overline{B}(0, m_0), \sigma(\mathscr{M}(\Omega;\mathbb{R}), C_0(\Omega;\mathbb{R}))$$

is continuous.

The proof of Lemma 10 uses the following standard result.

Lemma 11. *Let ν_n, ν be $\mathbb{R}^N$-valued finite Borel measures on a locally compact separable metric space X. If $\nu_n \rightharpoonup \nu$ and $\limsup_{n\to\infty}\|\nu_n\|_{\mathscr{M}} \le \|\nu\|_{\mathscr{M}}$, then $|\nu_n| \rightharpoonup |\nu|$.*

Proof of Lemma 11. Let μ be such that, possibly up to a subsequence still denoted (ν_n), we have $|\nu_n| \rightharpoonup \mu$. Then $\mu \ge |\nu|$ ([3, Proposition 1.62 (b)]. Combining this with $\mu(X) = \lim_{n\to\infty}|\nu_n|(X) \le |\nu|(X)$, we find that $\mu = |\nu|$. □

Proof of Lemma 10. Let $\mathscr{C}_n, \mathscr{C} \in \mathscr{K}_{min}$ be such that $\mathscr{C}_n \rightharpoonup \mathscr{C}$. Let κ be such that, possibly up to a subsequence, we have $|\mathscr{C}_n| \rightharpoonup \kappa$. We want prove that $\kappa = |\mathscr{C}|$. As in the proof of Lemma 11, we have $\kappa \ge |\mathscr{C}|$. On the other hand, we clearly have $\|\mathscr{C}_n\|_{\mathscr{M}} = \|T\|_W$, and thus $\|\kappa\|_{\mathscr{M}} = \|T\|_W = \|\mathscr{C}\|_{\mathscr{M}}$. We conclude via Lemma 11. □

In view of Lemma 10, if μ is a finite Borel measure on $\mathscr{K}_{min}$, then $\int_{\mathscr{K}_{min}} \langle|\mathscr{C}|, \zeta\rangle \, d\mu$ makes sense for any $\zeta \in C_0(\Omega;\mathbb{R})$. This allows us to give a meaning to (3.34).

Theorem 4 is a consequence of the following

Theorem 7. *Let $T \in \mathscr{E}$. Let (v_n) be a minimizing sequence in*

$$\inf\left\{\int_\Omega |\nabla v|;\ v \in W^{1,1}(\Omega;\mathbb{S}^1),\ Jv = \pi T\right\}. \tag{3.32}$$

Then there exists a probability measure μ on $\mathscr{K}_{min}$ such that, up to a subsequence, we have

$$\int_\Omega (v_n \wedge \nabla v_n)\cdot\chi \to 2\pi \int_{\mathscr{K}_{min}} \langle \mathscr{C}, \chi\rangle \, d\mu,\ \forall\, \chi \in C_0(\Omega;\mathbb{R}^2). \tag{3.33}$$

In addition, if we set

$$\langle \kappa, \zeta\rangle := \int_{\mathscr{K}_{min}} \langle |\mathscr{C}|, \zeta\rangle \, d\mu,\ \forall\, \zeta \in C_0(\Omega;\mathbb{R}), \tag{3.34}$$

then

$$|\nabla v_n| \, dx \rightharpoonup 2\pi\,\kappa. \tag{3.35}$$

And conversely.

Remark 6. As we will see in the proof below, in the special case where there are only finitely many minimal connections, $\mathscr{C}_1, \ldots, \mathscr{C}_m$, (3.33) and (3.35) become

$$v_n \wedge \nabla v_n \, dx \rightharpoonup 2\pi \sum_{j=1}^m \lambda_j \mathscr{C}_j \tag{3.36}$$

and

$$|\nabla v_n| \, dx \rightharpoonup 2\pi \sum_{j=1}^m \lambda_j |\mathscr{C}_j| \tag{3.37}$$

for some $\lambda_1, \ldots, \lambda_m \in [0, 1]$ such that $\sum_{j=1}^m \lambda_j = 1$.

Therefore, Theorem 4 is a special case of Theorem 7.

Proof of Theorem 7 and Remark 6. "$\Longrightarrow$" Given any minimizing sequence (v_n) in (3.32) and any $\varepsilon > 0$, we will construct a subsequence (v_{n_k}) and a discrete probability measure μ on $\mathscr{K}_{min}$, depending on the sequence (v_n) and on ε, such that

$$\left| \int_\Omega (v_{n_k} \wedge \nabla v_{n_k}) \cdot \chi - 2\pi \int_{\mathscr{K}_{min}} \langle \mathscr{C}, \chi \rangle \, d\mu \right| \le \varepsilon \, \|\chi\|_{W^{1,\infty}}, \ \forall \chi \in C^1_c(\Omega; \mathbb{R}^2). \tag{3.38}$$

Assuming this achieved, by a diagonal process we may construct probability measures μ_j, μ on $\mathscr{K}_{min}$ and a subsequence (v_{n_k}) such that

$$\left| \int_\Omega (v_{n_k} \wedge \nabla v_{n_k}) \cdot \chi - 2\pi \int_{\mathscr{K}_{min}} \langle \mathscr{C}, \chi \rangle \, d\mu_j \right| \le \frac{1}{j} \, \|\chi\|_{W^{1,\infty}}, \ \forall k \ge j \ge 1, \ \forall \chi \in C^1_c(\Omega; \mathbb{R}^2) \tag{3.39}$$

and $\mu_j \rightharpoonup \mu$. We find that this μ satisfies (3.33).

It remains to construct (v_{n_k}) and μ as in (3.38). Let (v_n) be a minimizing sequence in (3.32). We let $\mathscr{C}_{\alpha,n}$ (respectively $\mathscr{D}_{\alpha,n}$) denote the connections (respectively the level sets) associated with v_n as in Lemma 4.

By (3.9) and Lemma 4, for every n there exists a full measure set $A_n \subset \mathbb{S}^1$ such that

$$\mathscr{H}^1(\mathscr{D}_{\alpha,n}) = \|\mathscr{C}_{\alpha,n}\|_{\mathscr{M}} \ge \|T\|_W, \ \forall \alpha \in A_n. \tag{3.40}$$

On the other hand, using the coarea formula (3.12) and the fact that (v_n) is a minimizing sequence in (3.32), we find that

$$\int_\Omega |\nabla v_n| = \int_{\mathbb{S}^1} \mathscr{H}^1(\mathscr{D}_{\alpha,n}) \, d\alpha \to 2\pi \, \|T\|_W. \tag{3.41}$$

By (3.40) and (3.41), possibly up to a subsequence there exist some $f \in L^1(\mathbb{S}^1; [0, \infty))$ and a full measure set $A \subset \mathbb{S}^1$ such that

$$\mathscr{H}^1(\mathscr{D}_{\alpha,n}) \le f(\alpha), \ \forall n, \ \text{and } \mathscr{H}^1(\mathscr{D}_{\alpha,n}) \to \|T\|_W, \ \forall \alpha \in A. \tag{3.42}$$

By (3.40) and (3.42), we have

$$\|\mathscr{C}_{\alpha,n}\|_{\mathscr{M}} \le f(\alpha) \text{ and } \|\mathscr{C}_{\alpha,n}\|_{\mathscr{M}} \to \|T\|_W, \ \forall \alpha \in A. \tag{3.43}$$

Consider a finite family $(\mathscr{C}_i)_{i \in I} \subset \mathscr{K}_{min}$, depending on ε, such that the family of balls $(B(\mathscr{C}_i, \varepsilon/(2\pi)))_{i \in I}$ (for the distance δ) covers $\mathscr{K}_{min}$. Such a family exists, by Corollary 2. By Corollary 1, there exists some $t > 0$ such that

$$\|\mathscr{C}\|_{\mathscr{M}} \ge \|T\|_W + t, \ \forall \mathscr{C} \in \mathscr{K} \setminus \cup_{i \in I} B(\mathscr{C}_i, \varepsilon/(2\pi)). \tag{3.44}$$

Set, for $i \in I$,

$$A_{n,i,\varepsilon} := \{\alpha \in \mathbb{S}^1; \ \delta(\mathscr{C}_{\alpha,n}, \mathscr{C}_i) < \varepsilon/(2\pi)\}.$$

In view of Lemma 8 and of (3.43)–(3.44), we have

$$\left|\mathbb{S}^1 \setminus \cup_{i \in I} A_{n,i,\varepsilon}\right| \to 0 \text{ as } n \to \infty.$$

Possibly after passing to a subsequence, for simplicity still denoted (v_n), we may assume that

$$\int_{\mathbb{S}^1 \setminus \cup_{i \in I} A_{n,i,\varepsilon}} \|\mathscr{C}_{\alpha,n}\|_{\mathscr{M}} \le \frac{\varepsilon}{2}, \ \forall n, \tag{3.45}$$

$$|A_{n,i,\varepsilon}| \to 2\pi\lambda_i, \ \forall i \in I, \ \text{with } \lambda_i \in [0,1] \text{ and } \sum_{i \in I} \lambda_i = 1, \tag{3.46}$$

and

$$\sum_{i\in I}||A_{n,i,\varepsilon}|-2\pi\lambda_i| \le \frac{\varepsilon}{2\|T\|_W}. \tag{3.47}$$

Define $\mu := \sum_{i\in I}\lambda_i\delta_{\mathscr{C}_i}$. In view of Lemma 7 and (3.45)–(3.47), we obtain, using the coarea formula (3.13) with $g := \frac{(v_n\wedge\nabla v_n)\cdot\chi}{|\nabla v_n|}$:

$$\begin{aligned}\left|\int_\Omega (v_n\wedge\nabla v_n)\cdot\chi - 2\pi\int_{\mathscr{K}_{min}}\langle\mathscr{C},\chi\rangle\, d\mu\right| &= \left|\int_\Omega (v_n\wedge\nabla v_n)\cdot\chi - 2\pi\sum_{i\in I}\lambda_i\langle\mathscr{C}_i,\chi\rangle\right|\\ &= \left|\int_{\mathbb{S}^1}\langle\mathscr{C}_\alpha,\chi\rangle\, d\alpha - 2\pi\sum_{i\in I}\lambda_i\langle\mathscr{C}_i,\chi\rangle\right|\\ &\le \int_{\mathbb{S}^1\setminus\cup_{i\in I}A_{n,i,\varepsilon}}\|\mathscr{C}_{\alpha,n}\|_{\mathscr{M}}\|\chi\|_{L^\infty}\\ &\quad+\sum_{i\in I}\int_{A_{n,i,\varepsilon}}|\langle\mathscr{C}_{\alpha,n}-\mathscr{C}_i,\chi\rangle|\\ &\quad+\sum_{i\in I}||A_{n,i,\varepsilon}|-2\pi\lambda_i|\,\|\mathscr{C}_i\|_{\mathscr{M}}\|\chi\|_{L^\infty}\le\varepsilon\|\chi\|_{W^{1,\infty}}.\end{aligned} \tag{3.48}$$

This implies the validity of (3.38), and thus of (3.33).

We next derive (3.35) from (3.33) as follows. Set

$$\langle\nu,\chi\rangle := \int_{\mathscr{K}_{min}}\langle\mathscr{C},\chi\rangle,\ \forall\,\chi\in C_0(\Omega;\mathbb{R}^2) \tag{3.49}$$

(as in the right-hand side of (3.33)).

Since $Jv_n = \pi T$ and $v_n\wedge\nabla v_n \rightharpoonup 2\pi\,\nu$, we find that ν satisfies $\operatorname{curl}\nu = T$.

We next invoke the following improvement of (3.9) [7]:

$$\inf\{\|\xi\|_{\mathscr{M}};\ \operatorname{curl}\xi = T\} = \|T\|_W; \tag{3.50}$$

this implies that the quantity $\|T\|_W$ could be computed not only starting from connections, but also from general Borel measures satisfying $\operatorname{curl}\xi = T$.

Using (3.50) and the fact that (v_n) is a minimizing sequence in (3.32), we find that

$$\lim_{n\to\infty}\int_\Omega |v_n\wedge\nabla v_n| = \lim_{n\to\infty}\int_\Omega|\nabla v_n| = 2\pi\|T\|_W \le 2\pi\|\nu\|_{\mathscr{M}}. \tag{3.51}$$

Combining (3.33), Lemma 11 and (3.51), we obtain

$$|\nabla v_n|\, dx \rightharpoonup 2\pi\,|\nu|. \tag{3.52}$$

If we compare (3.35) with (3.52), formula (3.35) is a consequence of (3.52) and of the following

Lemma 12. *Let μ be a finite positive Borel measure on $\mathscr{K}_{min}$. Set*

$$\langle\nu,\chi\rangle = \langle\nu(\mu),\chi\rangle := \int_{\mathscr{K}_{min}}\langle\mathscr{C},\chi\rangle\, d\mu,\ \forall\,\chi\in C_0(\Omega;\mathbb{R}^2).$$

Then

$$\langle|\nu|,\zeta\rangle = \int_{\mathscr{K}_{min}}\langle|\mathscr{C}|,\zeta\rangle\, d\mu,\ \forall\,\zeta\in C_0(\Omega;\mathbb{R}). \tag{3.53}$$

Proof of Lemma 12. We may assume that μ is a probability. Let $\kappa = \kappa(\mu)$ denote the measure defined by right-hand side of (3.53). Set

$$\mathscr{S} := \{\mu \in \mathscr{P}(\mathscr{K}_{min});\ |\nu(\mu)| = \kappa(\mu)\}. \tag{3.54}$$

Then (3.53) amounts to

$$\mathscr{S} = \mathscr{P}(\mathscr{K}_{min}). \tag{3.55}$$

We first prove that discrete measures belong to $\mathscr{S}$. Equivalently, if $\mathscr{C}_j \in \mathscr{K}_{min}$ and $\lambda_j \in [0,1]$, $\forall\, j = 1, \ldots, k$, with $\sum_j \lambda_j = 1$, we have to prove that

$$\left|\sum_j \lambda_j \mathscr{C}_j\right| = \sum_j \lambda_j |\mathscr{C}_j|. \tag{3.56}$$

Inequality "$\le$" in (3.56) is clear. In order to obtain the opposite inequality, it suffices to prove that

$$\left\|\sum_j \lambda_j \mathscr{C}_j\right\|_{\mathscr{M}} \ge \sum_j \lambda_j \left\|\mathscr{C}_j\right\|_{\mathscr{M}}. \tag{3.57}$$

In turn, (3.57) follows from (3.50), since $\operatorname{curl}(\sum_j \lambda_j \mathscr{C}_j) = T$.

In order to complete the proof of (3.55), it suffices to prove that $\mathscr{S}$ is weakly-$*$ sequentially closed. Let $\mu_n, \mu \in \mathscr{P}(\mathscr{K}_{min})$ be such that $\mu_n \rightharpoonup \mu$ and $\mu_n \in \mathscr{S}$, $\forall\, n$. We clearly have $\nu(\mu_n) \rightharpoonup \nu(\mu)$ and $\kappa(\mu_n) \rightharpoonup \kappa(\mu)$. Assume that, possibly up to a subsequence, $|\nu(\mu_n)| \rightharpoonup \xi$. Then $\xi \ge \kappa(\mu)$. On the other hand, we have $\|\xi\|_{\mathscr{M}} = \|\kappa(\mu_n)\|_{\mathscr{M}} = \|T\|_W = \|\kappa(\mu)\|_{\mathscr{M}}$, and then Lemma 11 implies that $\xi = \kappa(\mu)$ and thus $\mu \in \mathscr{S}$. $\square$

Proof of Theorem 7 *completed.* During the proof of Lemma 12, we have established not only (3.35), but also (3.37) (while (3.36) is clear). In particular, we have justified Remark 6. Therefore, it remains to establish

"$\Longleftarrow$" As explained in the proof of the "$\Longleftarrow$" part of Theorem 2, it suffices to consider the case where μ is discrete, that is of the form $\mu = \sum_{j=1}^k \lambda_j \delta_{\mathscr{C}_j}$, with $\lambda_j \ge 0$, $\sum_{j=1}^k \lambda_j = 1$, $\mathscr{C}_j \in \mathscr{K}_{min}$, $\forall\, j = 1, \ldots, k$. We construct a sequence (v_n) satisfying (3.33), (3.35),

$$Jv_n = \pi T \text{ and } \int_\Omega |\nabla v_n| \to 2\pi \|T\|_W \tag{3.58}$$

in the case of two minimal connections, $\mathscr{C}_1$ and $\mathscr{C}_2$. The construction is exactly the same in case of more than two minimal connections.

Let $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ be such that $Ju = \pi T$. Then

$$\operatorname{curl} G = 0, \text{ where } G := u \wedge \nabla u - 2\pi(\lambda_1 \mathscr{C}_1 + \lambda_2 \mathscr{C}_2) \in \mathscr{M}(\Omega; \mathbb{R}^2).$$

Consider $\varphi \in BV(\Omega; \mathbb{R})$ such that $G = D\varphi$. By Lemma 14 in Appendix A, we may find a sequence $(\varphi_n) \subset C^\infty(\Omega) \cap W^{1,1}(\Omega)$ such that

$$\nabla \varphi_n \rightharpoonup D\varphi \text{ in } \mathscr{M}(\Omega) \tag{3.59}$$

and

$$\lim_{n\to\infty} \int_\Omega |u \wedge \nabla u - \nabla \varphi_n| = \|u \wedge \nabla u - D\varphi\|_{\mathscr{M}}. \tag{3.60}$$

We set $v_n := u\, e^{-\imath\varphi_n}$ and claim that the sequence (v_n) has all the required properties. Indeed, on the one hand we have

$$Jv_n = J(u\, e^{-\imath\varphi_n}) = Ju = \pi T.$$

By (3.60), we have

$$\int_\Omega |u \wedge \nabla u - \nabla\varphi_n| \to 2\pi\|\lambda_1\mathscr{C}_1 + \lambda_2\mathscr{C}_2\|_{\mathscr{M}} = 2\pi\|T\|_W; \tag{3.61}$$

for the latter equality, see (3.56).

We obtain (3.33), (3.35), (3.58) using (3.59), (3.61) and the identity $v_n \wedge \nabla v_n = u \wedge \nabla u - \nabla\varphi_n$. □

Remark 7. We present here a proof of Lemma 5 which does not rely on the distance δ. What we have to prove is that, if $\mathscr{C}_n \in \mathscr{K}$ and $\mathscr{C}_n \rightharpoonup \mu$, then $\mu \in \mathscr{K}$. In order to prove this, we rely on the following characterization of connections [7]. Fix some $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ such that $Ju = \pi T$. Then

$$[\mathscr{C} \in \mathscr{K}] \iff [\exists\, \psi \in BV(\Omega;\mathbb{R}) \text{ such that } u = e^{\imath\psi} \text{ and } D\psi = u \wedge \nabla u - 2\pi\,\mathscr{C}]. \tag{3.62}$$

With $\mathscr{C}_n$ as above, consider ψ_n corresponding to $\mathscr{C}_n$ as in (3.62). We may assume that $\int_\Omega \psi_n \in [0, 2\pi, \|\Omega\|)$, and then (ψ_n) is bounded in BV. Up to a subsequence, there exists some $\psi \in BV$ such that $\psi_n \to \psi$ in L^1 and a.e. Thus $u = e^{\imath\psi}$. On the other hand, we have $D\psi = u \wedge \nabla u - 2\pi\,\mu$. In view of (3.62), we find that $\mu \in \mathscr{C}$.

Proof of Lemma 1. In view of Corollary 2 and Lemma 10, it suffices to prove the following. If U is a minimal configuration and $\mathscr{C}$ is the corresponding minimal connection, then

$$|\mathscr{C}| = \mathscr{H}^1 \llcorner U. \tag{3.63}$$

Inequality "$\le$" is clear. On the other hand, we have

$$\|\mathscr{H}^1 \llcorner U\|_{\mathscr{M}} = L(\mathbf{a}, \mathbf{M}) = \|\mathscr{C}\|_{\mathscr{M}},$$

whence the equality in (3.63). □

Proof of Theorem 5. We will actually prove that the existence of μ as in (3.34)–(3.35) is equivalent to the existence of $\overline{\mu}$ as in (1.17)–(1.18). This is obtained as follows. For $\zeta \in C(\overline{\Omega})$, set

$$\varphi_\zeta : \mathscr{L}_{min} \to \mathbb{R},\ \varphi_\zeta(Q) := \langle Q, \zeta\rangle,\ \forall\, Q \in \mathscr{L}_{min},$$

and

$$\psi_\zeta : \mathscr{K}_{min} \to \mathbb{R},\ \psi_\zeta(\mathscr{C}) := \langle |\mathscr{C}|, \zeta\rangle, \forall\, \mathscr{C} \in \mathscr{K}_{min}.$$

Note that

$$[\varphi_\zeta = \varphi_{\tilde{\zeta}}] \iff [\psi_\zeta = \psi_{\tilde{\zeta}}] \tag{3.64}$$

and that

$$\|\varphi_\zeta\|_{L^\infty} = \|\psi_\zeta\|_{L^\infty}. \tag{3.65}$$

We continue as follows. Given $\mu \in \mathscr{P}(\mathscr{K}_{min})$ and $\kappa = \kappa(\mu)$ as in (3.34), set

$$\mathscr{T}(\varphi_\zeta) := \langle \kappa, \zeta \rangle = \int_{\mathscr{K}_{min}} \psi_\zeta \, d\mu.$$

By (3.64)–(3.65), $\mathscr{T}$ is well-defined, and clearly

$$|\mathscr{T}(\varphi_\zeta)| \le \|\psi_\zeta\|_{L^\infty} = \|\varphi_\zeta\|_{L^\infty}.$$

We find that $\mathscr{T}$ extends to a finite Borel measure $\overline{\mu}$ on $\mathscr{L}_{min}$, with $\|\overline{\mu}\|_{\mathscr{M}(\mathscr{L}_{min})} \le 1$. On the other hand, if we let $\zeta = 1$, we see that

$$L(\mathbf{a}, \mathbf{M})\, \overline{\mu}(\mathscr{L}_{min}) = L(\mathbf{a}, \mathbf{M}),$$

and thus $\overline{\mu}$ is a probability measure on $\mathscr{L}_{min}$ such that $\overline{\kappa}(\overline{\mu}) = \kappa(\mu)$.
The construction of μ starting from $\overline{\mu}$ follows the same lines. $\square$

4. Where continuity enters

In this section we consider two distinct directions where continuity enters.

4.1. Minimizing over continuous maps

Given a pair $(\mathbf{a}, \mathbf{M})$, we set

$$W(\mathbf{a}, \mathbf{M}) = \left\{ v \in W^{1,1}(\Omega; \mathbb{S}^1) \cap C(\Omega \setminus \{a_1, \ldots, a_k\});\ \deg(v, a_j) = M_j,\ \forall\, j = 1, \ldots, k \right\},$$

and

$$W^\infty(\mathbf{a}, \mathbf{M}) = \left\{ v \in W^{1,1}(\Omega; \mathbb{S}^1) \cap C^\infty(\overline{\Omega} \setminus \{a_1, \ldots, a_k\});\ v(z) = \left(\frac{z - a_j}{|z - a_j|} \right)^{M_j} \right. \\ \left. \text{near each } a_j,\ \forall\, j = 1, \ldots, k \right\}.$$

Clearly,

$$W^\infty(\mathbf{a}, \mathbf{M}) \subset W(\mathbf{a}, \mathbf{M}) \subset \left\{ v \in W^{1,1}(\Omega; \mathbb{S}^1);\ Jv = \pi \sum_{j=1}^k M_j \delta_{a_j} \right\}.$$

Set

$$\Lambda(\mathbf{a}, \mathbf{M}) = \inf_{v \in W(\mathbf{a}, \mathbf{M})} \int_\Omega |\nabla v|$$

and

$$\Lambda^\infty(\mathbf{a}, \mathbf{M}) = \inf_{v \in W^\infty(\mathbf{a}, \mathbf{M})} \int_\Omega |\nabla v|,$$

so that (with $L(\mathbf{a}, \mathbf{M})$ defined in (1.6))

$$L(\mathbf{a}, \mathbf{M}) \le \Lambda(\mathbf{a}, \mathbf{M}) \le \Lambda^\infty(\mathbf{a}, \mathbf{M}).$$

Theorem 8. *We have*

$$L(\mathbf{a},\mathbf{M}) = \Lambda(\mathbf{a},\mathbf{M}) = \Lambda^\infty(\mathbf{a},\mathbf{M}). \tag{4.1}$$

Clearly, Theorem 8 is an immediate consequence of the following

Lemma 13. *Given $v \in W^{1,1}(\Omega;\mathbb{S}^1)$ satisfying $Jv = \pi\sum_{j=1}^k M_j\delta_{a_j}$, there exists a sequence $(v_n) \subset W^\infty(\mathbf{a},\mathbf{M})$ such that $v_n \to v$ in $W^{1,1}$.*

Proof. Consider a reference map $v_0 \in W^\infty(\mathbf{a},\mathbf{M})$. Since $J(v\,\overline{v_0}) = 0$, we may write $v = v_0\, e^{\imath\varphi}$, where $\varphi \in W^{1,1}(\Omega;\mathbb{R})$. Consider a sequence $(\varphi_n) \subset C^\infty(\overline{\Omega};\mathbb{R})$ such that $\varphi_n \to \varphi$ in $W^{1,1}$ and $\varphi_n = 0$ near each a_j; this is possible since a point in the plane has zero $W^{1,1}$ capacity ([10, Section 4.7, Theorem 2]). Letting $v_n = v_0\, e^{\imath\varphi_n}$, we have $v_n \in W^\infty(\mathbf{a},\mathbf{M})$ and $v_n \to v$ in $W^{1,1}$. $\square$

Proof of Lemma 2. With $\Omega = \mathbb{D}$, let $(v_n) \subset W^\infty(0,M)$ be such that $v_n \to v$ in $W^{1,1}$. Possibly after passing to a subsequence, we may also assume that $v_n(r\cdot) \to v(r\cdot)$ in $W^{1,1}(\mathbb{S}^1)$ (and thus uniformly on $\mathbb{S}^1$) for a.e. $r \in (0,1)$. The conclusion of Lemma 2 follows from the stability of the degree under uniform convergence and the fact that $\deg(v_n, C(0,r)) = M$, $\forall\, n$, $\forall\, r \in (0,1)$. $\square$

4.2. Continuity of minimizers

Given a pair $(\mathbf{a},\mathbf{M})$, consider the minimization problem

$$\inf\left\{\int_\Omega |\nabla v|; v \in W^{1,1}(\Omega;\mathbb{S}^1), Jv = \pi\sum_j M_j\delta_{a_j}\right\}. \tag{4.2}$$

Theorem 9. *Assume that w is a minimizer in (4.2). (Warning: such a minimizer need not exist; see Theorems 1 and 3.) Then $w \in C(\Omega\setminus\{a_1,\ldots,a_k\})$.*

Proof. Let $\omega = D(b,R)$ be a disk in $\Omega\setminus\{a_1,\ldots,a_k\}$. Since $Jw = 0$ in $\mathscr{D}'(\omega)$, we may write, in ω, $w = e^{\imath\varphi}$ for some $\varphi \in W^{1,1}(\omega;\mathbb{R})$. For a.e. $r \in (R/2,R)$, we have $\varphi_{|C(b,r)} \in W^{1,1}(C(b,r))$, and thus φ is continuous on $C(b,r)$. On the other hand, φ is a minimizer of the problem

$$\inf\left\{\int_{D(b,r)} |\nabla\psi|;\ \psi \in W^{1,1}(D(b,r);\mathbb{R}) \text{ and } \psi = \varphi \text{ on } C(b,r)\right\}. \tag{4.3}$$

This follows from the fact that for every ψ as in (4.3), the map

$$v = \begin{cases} u, & \text{in } \Omega\setminus D(b,r)\\ e^{\imath\psi}, & \text{in } D(b,r)\end{cases}$$

is a competitor in (4.2).

We claim that actually φ is a minimizer of

$$\inf\left\{\|D\psi\|_{\mathscr{M}};\ \psi \in BV(D(b,r);\mathbb{R}) \text{ and } \psi = \varphi \text{ on } C(b,r)\right\}. \tag{4.4}$$

Indeed, by a result of Sternberg and Ziemer [18, Theorem 2.2], the minimality of φ in (4.4) is equivalent to

$$\int_{D(b,r)} |\nabla\varphi| \le \|\nabla\varphi + D\psi\|_{\mathscr{M}},\ \forall\psi \in BV_c(D(b,r);\mathbb{R}), \tag{4.5}$$

where the subscript c stands for compactly supported. In turn, (4.5) follows from the fact that φ minimizes (4.3) combined with the fact that for every ψ as in (4.5) there exists a sequence $(\psi_n) \subset C_c^\infty(D(b,r))$ such that $\psi_n \to \psi$ a.e. and

$$\int_{D(b,r)} |\nabla(\varphi + \psi_n)| \to \|\nabla\varphi + D\psi\|_{\mathscr{M}}$$

(see the proof of Lemma 14 in Appendix A).

We conclude by invoking the fact that, in a disk, a minimizer of (4.4) subject to a continuous Dirichlet boundary condition is continuous; this follows by combining [17, Theorems 3.5 and 3.7] with [18, Theorem 3.6]. $\square$

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Appendix A. An approximation lemma

The main result in this appendix is standard when $F = 0$; see e.g. [3, Theorem 3.9].

Lemma 14. *Let $F \in L^1(\Omega; \mathbb{R}^N)$ and $\varphi \in BV(\Omega)$. Then there exists a sequence $(\varphi_n) \subset C^\infty(\Omega) \cap W^{1,1}(\Omega)$ such that*

$$\varphi_n \to \varphi \text{ in } L^1(\Omega), \tag{A.1}$$

$$\nabla\varphi_n \rightharpoonup D\varphi \text{ in } \mathscr{M}(\Omega) \tag{A.2}$$

and

$$\lim_{n\to\infty} \int_\Omega |F - \nabla\varphi_n| = \|F - D\varphi\|_{\mathscr{M}}. \tag{A.3}$$

Proof. If $(\varphi_n) \subset W^{1,1}$ and $\varphi_n \to \varphi$ in L^1, then

$$\liminf_{n\to\infty} \int |F - \nabla\varphi_n| \geq \|F - D\varphi\|_{\mathscr{M}}.$$

Therefore, it suffices to find, for each $\varepsilon > 0$, a map $\psi = \psi_\varepsilon \in C^\infty(\Omega) \cap W^{1,1}(\Omega)$ such that $\|\varphi - \psi\|_{L^1(\Omega)} < \varepsilon$ and $\int |F - \nabla\psi| \leq \|F - D\varphi\|_{\mathscr{M}} + \varepsilon$. Then $\varphi_n := \psi_{1/n}$ has all the required properties.

Consider an exhaustion $\Omega = \cup_{j\geq 0}\Omega_j$ of Ω, with each Ω_j open and $\overline{\Omega}_j \Subset \Omega_{j+1}$. Let $U_0 := \Omega_0$ and, for $j \geq 1$, $U_j := \Omega_{j+1} \setminus \overline{\Omega}_{j-1}$. Thus $\Omega = \cup_{j\geq 0} U_j$ and each $x \in \Omega$ belongs to at most two U_j's. Let (ξ_j) be a partition of unity subordinated to the covering (U_j) of Ω. Let $\rho \in C_c^\infty(B_1(0))$ be a mollifier. With the sequence (ε_j) to be fixed later, we let $\psi := \sum_j \psi_j$, where $\psi_j := (\varphi\xi_j) * \rho_{\varepsilon_j}$.

We claim that, for an appropriate choice of ε_j, ψ has the required properties. To start with, if $\varepsilon_j < \operatorname{dist}(\operatorname{supp}\xi_j, \partial U_j)$, then $\psi_j \in C_c^\infty(U_j)$, and thus $\psi \in C^\infty(\Omega)$.

Next, if ε_j is sufficiently small, then $\|\psi_j - \varphi\xi_j\|_{L^1} < \frac{\varepsilon}{2^{j+1}}$, and thus $\|\psi - \varphi\|_{L^1} < \varepsilon$.

We continue by noting that

$$\begin{aligned}\nabla\psi &= \sum \nabla\psi_j = \sum (D(\varphi\xi_j)) * \rho_{\varepsilon_j} = \sum (\varphi\nabla\xi_j) * \rho_{\varepsilon_j} + \sum (\xi_j D\varphi) * \rho_{\varepsilon_j}\\ &= \underbrace{\sum [(\varphi\nabla\xi_j) * \rho_{\varepsilon_j} - \varphi\nabla\xi_j]}_{R} + \sum (\xi_j D\varphi) * \rho_{\varepsilon_j}\end{aligned}$$

(since $\sum \nabla\xi_j = \nabla\left(\sum \xi_j\right) = 0$.) We note that $\|R\|_{L^1} < \frac{\varepsilon}{2}$, provided the ε_j's are sufficiently small.

Similarly, we may write

$$F = \sum (\xi_j F) * \rho_{\varepsilon_j} + S,$$

where $\|S\|_{L^1} < \frac{\varepsilon}{2}$, provided the ε_j's are sufficiently small. Here, we use the fact that F belongs to L^1, and is not merely a measure.

Finally, for small ε_j we have

$$\begin{aligned}\int |F - \nabla\psi| &\le \sum \int |[\xi_j(F - D\varphi)] * \rho_{\varepsilon_j}| + \varepsilon \le \sum \|\xi_j(F - D\varphi)\|_{\mathscr{M}} + \varepsilon\\ &= \left\|\sum \xi_j(F - D\varphi)\right\|_{\mathscr{M}} + \varepsilon = \|F - D\varphi\|_{\mathscr{M}} + \varepsilon;\end{aligned}$$

in the above, the next to the last equality is a consequence of the following simple result. □

Lemma 15. *Let μ be an $\mathbb{R}^k$-valued Radon measure in Ω. If $\xi : \Omega \to \mathbb{R}_+$ is a non-negative Borel function, then $|\xi\mu| = \xi|\mu|$.*

In particular, if ξ_j are non-negative Borel functions, then

$$\left|\sum \xi_j\mu\right| = \sum \xi_j|\mu| \text{ and } \left\|\sum \xi_j\mu\right\|_{\mathscr{M}} = \sum \|\xi_j\mu\|_{\mathscr{M}}.$$

Proof. Let $\mu = F|\mu|$ be the polar decomposition of μ [3, Corollary 1.29]. Thus $|\mu|$ is a positive measure, and $F : \Omega \to \mathbb{S}^{k-1}$ is a Borel function. We have $\xi\mu = \xi F|\mu|$. Let B be a Borel subset of Ω. Then [3, Proposition 1.23]

$$|\xi\mu|(B) = \int_B |\xi F|\, d|\mu| = \int_B \xi|F|\, d|\mu| = \int_B \xi\, d|\mu| = (\xi|\mu|)(B). \quad \square$$

Appendix B. Minimization in BV

When $u \in BV(\Omega; \mathbb{S}^1)$, there is no obvious definition of Ju. We recall here the definition(s) proposed by Ignat [14]. In [14], one considers maps in $BV(\mathbb{S}^2; \mathbb{S}^1)$ instead of $BV(\Omega; \mathbb{S}^1)$, but the considerations there may be easily adapted to our setting.

Let, for $\beta \in \mathbb{R}$, $\mathrm{Arg}_\beta : \mathbb{S}^1 \to (\beta - \pi, \beta + \pi]$ be the determination of the argument given by $\mathrm{Arg}_\beta(e^{\imath\theta}) := \theta$, $\forall\, \theta \in (\beta - \pi, \beta + \pi]$, and set $\mathrm{Arg} := \mathrm{Arg}_0$.

Define $\rho_\beta : \mathbb{S}^1 \times \mathbb{S}^1 \to [-\pi, \pi]$,

$$\rho_\beta(z_1, z_2) := \begin{cases}\mathrm{Arg}(z_1/z_2), & \text{if } z_1 \ne -z_2\\ \mathrm{Arg}_\beta(z_1) - \mathrm{Arg}_\beta(z_2), & \text{if } z_1 = -z_2,\end{cases}$$

and set $\rho := \rho_0$.

Assuming that u equals its approximate limit outside its jump set, define

$$\langle J_\beta u, \zeta\rangle := -\frac{1}{2}\int_\Omega \nabla^\perp\zeta \cdot [u \wedge (\nabla u + D^c u)] - \frac{1}{2}\int_S \rho_\beta(u^+, u^-)\, \nu \cdot \nabla^\perp\zeta\, d\mathscr{H}^1,\ \forall\, \zeta \in C_c^1(\Omega; \mathbb{R}), \quad \text{(B.1)}$$

and set $J := J_0$. Here, $\nabla^\perp\zeta := (\nabla\zeta)^\perp = (-\partial\zeta/\partial x_2, \partial\zeta/\partial x_1)$.

Since the definition (1.1) is equivalent to

$$\langle Ju, \zeta\rangle = -\frac{1}{2}\int_{\Omega} \nabla^{\perp}\zeta\cdot[u\wedge\nabla u],\ \forall\,\zeta\in C^1_c(\Omega;\mathbb{R}), \tag{B.2}$$

we see that $J_\beta u$ coincides with Ju when $u\in W^{1,1}(\Omega;\mathbb{S}^1)$.

We next investigate in more details the properties of $J := J_0$ (following [14]); the case of an arbitrary β is similar.

1. The main result in [14] is that $\frac{1}{\pi}Ju\in\mathscr{E}$, $\forall\,u\in BV(\Omega;\mathbb{S}^1)$; here, $\mathscr{E}$ is the class defined in (3.1).
2. In connection with the study of J, the following "adapted energy" is natural:

$$u\mapsto\int_{\Omega}|Du|_{\mathbb{S}^1},\quad\text{with } |Du|_{\mathbb{S}^1} := |\nabla u| + |D^c u| + \operatorname{dist}_{\mathbb{S}^1}(u^+,u^-)\,\mathscr{H}^1 \llcorner S. \tag{B.3}$$

Here, $\operatorname{dist}_{\mathbb{S}^1}$ is the geodesic distance on $\mathbb{S}^1$.
3. Clearly, $\int_\Omega|Du|_{\mathbb{S}^1}\ge\int_\Omega|Du|$, since

$$|Du| = |\nabla u| + |D^c u| + |u^+ - u^-|\,\mathscr{H}^1 \llcorner S. \tag{B.4}$$

4. The interest of this new energy stems from the straightforward inequality

$$|\langle Ju,\zeta\rangle|\le\frac{1}{2}\|\nabla\zeta\|_{L^\infty}\int_\Omega|Du|_{\mathbb{S}^1},\ \forall\,\zeta\in C^1_c(\Omega;\mathbb{R}); \tag{B.5}$$

this uses the identity $|\rho(z_1,z_2)| = \operatorname{dist}_{\mathbb{S}^1}(z_1,z_2)$.
5. If we set $T := \frac{1}{\pi}Ju$, then (B.5) implies

$$\int_\Omega|Du|_{\mathbb{S}^1}\ge 2\pi\|T\|_W,\ \forall\,u\in BV(\Omega;\mathbb{S}^1)\text{ such that } Ju=\pi T. \tag{B.6}$$

By analoguey with (1.7), we consider the quantity

$$\Sigma_{\mathbb{S}^1}(u) := \inf\left\{\int_\Omega|Dv|_{\mathbb{S}^1};\ v\in BV(\Omega;\mathbb{S}^1),\ Jv = Ju\text{ in }\mathscr{D}'(\Omega)\right\} \tag{B.7}$$

and address the question of the existence of a minimizer in (B.7). In general, a minimizer does not exist. Indeed, assume that $0\in\Omega$ and that $\operatorname{dist}(0,\partial\Omega)=1$. Consider, as in (1.10), the non-empty set $X := \{x\in\partial\Omega;\ |x|=1\}$. Then we have the following partial analogue of Theorem 1.

Theorem 10. *Let $M\ge 1$ be an integer. Then*

$$\inf\left\{\int_\Omega|Dv|_{\mathbb{S}^1};\ v\in BV(\Omega;\mathbb{S}^1),\ Jv=\pi\,M\delta_0\ \textit{in}\ \mathscr{D}'(\Omega)\right\} = 2\pi\,M, \tag{B.8}$$

and the infimum in (B.8) is achieved if and only if $\#X\ge 2M+1$.

Proof. Equality in (B.8) is clear from (B.6) and (3.6).

"$\Longleftarrow$" Assume that $\#X\ge 2M+1$. Consider $b_1,\ldots,b_{2M+1}\in X$ mutually distinct points such that b_{j+1} is "after" b_j for the natural (counterclockwise) orientation of $\mathbb{S}^1$. Let $R_j := [0,b_j]$, $\forall\,j=1,\ldots,2M+1$, and set $R_{2d+2} := R_1$. For $j=1,\ldots,2M+1$, let U_j denote the domain delimited by R_j, R_{j+1} and $\partial\Omega$. Set $v := e^{\imath(j-1)p}$ in U_j, with $p := \frac{2\pi\,M}{2M+1}$. If we orientate R_j with the unit tangent vector to $\mathbb{S}^1$ at R_j (for the natural orientation on $\mathbb{S}^1$), then the jump of v across R_j is p. Using the fact that $p\in[0,\pi)$, we find that $\rho(v^+,v^-)\,\nu = p\nu$ on R_j. Since v has only pure jumps, formula (B.1) leads to

$$\langle Jv,\zeta\rangle = \frac{1}{2}(2M+1)\,p\,\zeta(0) = \langle\pi\,M\delta_0,\zeta\rangle,\ \forall\,\zeta\in C^1_c(\Omega).$$

Thus v is a competitor in (B.8). On the other hand, clearly

$$\int_\Omega |Dv|_{\mathbb{S}^1} = (2M+1)\,p = 2\pi\, M,$$

and thus v achieves the minimum in (B.8).

"$\Longrightarrow$" Let $\zeta_0(x) = (1-|x|)_+$, and let $\Upsilon(x) := \frac{x^\perp}{|x|}$, so that $-\nabla^\perp \zeta_0 = 1_{\mathbb{D}}\, \Upsilon$. We claim that the following formula holds for any $T \in \mathscr{E}$ and $v \in BV(\Omega; \mathbb{S}^1)$ such that $Jv = \pi T$:

$$\pi\langle T, \zeta_0\rangle = \langle Jv, \zeta_0\rangle = \frac{1}{2}\int_{\mathbb{D}} v \wedge (\,\Upsilon \cdot \nabla v + \Upsilon \cdot D^c v) + \frac{1}{2}\int_{S\cap\mathbb{D}} \rho(v^+, v^-)\; \Upsilon \cdot \nu\, d\mathscr{H}^1. \tag{B.9}$$

Let us note that, although the quantity $\langle Jv, \zeta\rangle$ was defined only for $\zeta \in C^1_c(\Omega)$, it makes sense more generally for $\zeta \in W^{1,\infty}_0(\Omega)$, since $T \in \mathscr{E}$.

Equality (B.9) is obtaining by a straightforward limiting procedure starting from the formula of $\langle Jv, \zeta\rangle$, with $\zeta(x) := f(|x|)$ and $f \in C^1_c([0,1))$ such that $f = 1$ near 0.

If, in addition v is a competitor in (B.8), then $T = \pi\, M\, \delta_0$, and thus (B.9) reads

$$\int_{\mathbb{D}} v \wedge (\,\Upsilon \cdot \nabla v + \Upsilon \cdot D^c v) + \int_{S\cap\mathbb{D}} \rho(v^+, v^-)\; \Upsilon \cdot \nu\, d\mathscr{H}^1 = 2\pi\, M. \tag{B.10}$$

Assume now that v is a minimizer in (B.8). Then

$$\int_\Omega [|\nabla v| + |D^c v|] + \int_S |\rho(v^+, v^-)|\, d\mathscr{H}^1 = 2\pi\, M. \tag{B.11}$$

Comparing (B.10) and (B.11), we find that

$$v \text{ is locally constant in } \Omega \setminus \overline{\mathbb{D}} \text{ and } \mathscr{H}^1(S \setminus \mathbb{D}) = 0. \tag{B.12}$$

We next invoke the following straightforward consequence of the polar decomposition (see the proof of Lemma 15).

Lemma 16. *Let $\mu \in \mathscr{M}(\Omega\,; \mathbb{R}^N)$. If $G, H : \Omega \to \mathbb{R}^N$ are Borel vector fields such that $|G| = 1$ $|\mu|$-a.e., $G \cdot H = 0$ $|\mu|$-a.e. and if $\int_\Omega G \cdot \mu = \int_\Omega |\mu|$, then $H \cdot \mu = 0$.*

Proof of Lemma 16. Let $\mu = F\,|\mu|$ be the polar decomposition of μ. Then

$$0 = \int_\Omega [|\mu| - G\cdot\mu] \geq \int_\Omega [|\mu| - |G\cdot\mu|] = \int_\Omega (1 - |F\cdot G|)\,|\mu| \geq 0.$$

It follows that $F \parallel G$ $|\mu|$-a.e., and thus $F \perp H$ $|\mu|$-a.e., whence the conclusion. □

Proof of Theorem 10 continued. Set

$$\mu := v \wedge \nabla v + v \wedge D^c v + \rho(v^+, v^-)\,\nu\,\mathscr{H}^1 \llcorner (S\cap\mathbb{D}).$$

Let us note that

$$|\mu| = |\nabla v| + |D^c v| + |\rho(v^+, v^-)|\,\mathscr{H}^1 \llcorner (S\cap\mathbb{D}). \tag{B.13}$$

Applying Lemma 16 in $\mathbb{D}$ for this μ, with $G := \Upsilon$ and $H(x) := x/|x|$, and using (B.13), we find that $\frac{\partial v}{\partial r} = 0$ in $\mathbb{D}$. By (B.12), this implies that

$$\frac{\partial v}{\partial r} = 0 \text{ in } \Omega. \tag{B.14}$$

Assume next that X is finite (for otherwise the implication we want to prove is clear). Let us write, with points enumerated in the natural order on $\mathbb{S}^1$, $X = \{b_1, \dots, b_k\}$. Set $R_j := [0, b_j]$, $b_{k+1} := b_1$ and $R_{k+1} := R_1$. Then (B.12) and(B.14) imply that v is locally constant in the set $U := \Omega \setminus \cup_{j=1}^k R_k$. Let U_j be the component of U delimited by R_j, R_{j+1} and the arc on $\partial\Omega$ from b_j to b_{j+1}, $\forall\, j = 1, \dots, k$. Let α_j be the value of v on U_j, and $\alpha_{k+1} := \alpha_1$. On the one hand, we have (using (B.1))

$$2\pi\, M\, \delta_0 = 2Jv = \sum_{j=1}^{k} \rho(\alpha_{j+1}, \alpha_j)\, \delta_0. \tag{B.15}$$

On the other hand, v is a minimizer in (B.8), and thus

$$2\pi\, M = \sum_{j=1}^{k} |\rho(\alpha_{j+1}, \alpha_j)|. \tag{B.16}$$

Using the fact that $|\rho(z_1, z_2)| \le \pi$, we find from (B.16) that $k \ge 2M$. We actually claim that we cannot have $k = 2M$ (whence the desired implication). Indeed, if $k = 2M$ and (B.15) and (B.16) hold, then $\rho(\alpha_j, \alpha_{j+1}) = \pi$, $\forall\, j = 1, \dots, 2M$. This is equivalent to

$$\alpha_{j+1} = -\alpha_j,\ \operatorname{Im}\, \alpha_{j+1} \ge 0 \text{ and } \alpha_j \ne -1. \tag{B.17}$$

However, (B.17) cannot hold simultaneously for j and $j + 1$. This contradiction completes the proof of the theorem. $\square$

Remark 8. Let us take a closer look at what we have proved. Without any assumption on X, we have obtained that any minimizer v (if it exists) satisfies:

$$v \text{ is locally constant in } \Omega \setminus \cup_{x\in X}[0, x],\ v(x) = w(x/|x|) \text{ in } \overline{\mathbb{D}},\ \text{with } w \in BV(\mathbb{S}^1; \mathbb{S}^1). \tag{B.18}$$

With more work, we may prove the following exact counterpart of Theorem 1, that we state here without proof.

Theorem 11. *v is a minimizer in (B.8) if and only if v is as in (B.18), where w has the following form: $w(e^{\imath\theta}) = e^{\imath h(\theta)}$, $\forall\, \theta \in [\beta, \beta + 2\pi]$, for some $\beta \in \mathbb{R}$ and some $h \in BV([\beta, \beta + 2\pi])$ such that:*

1. *h is continuous at its endpoints β and $\beta + 2\pi$.*
2. *$h(\beta + 2\pi) = h(\beta) + 2\pi\, M$.*
3. *h is non-decreasing.*
4. *h is locally constant in $\{\theta \in (\beta, \beta + 2\pi);\ e^{\imath\theta} \notin X\}$.*
5. *If θ is a jump point of h, then $h(\theta+) - h(\theta-) \le \pi$.*
6. *If, in addition $h(\theta+) - h(\theta-) = \pi$, then $\rho(e^{\imath h(\theta+)}, e^{\imath h(\theta-)}) = \pi$.*

The situation is even worse if we consider the standard energy $\int_\Omega |Dv|$. Indeed, with the same assumptions on Ω as above, we have the following result.

Theorem 12. *Let $M \ge 1$ be an integer. Then*

$$\inf\left\{\int_\Omega |Dv|;\ v \in BV(\Omega; \mathbb{S}^1),\ Jv = \pi\, M\delta_0 \text{ in } \mathscr{D}'(\Omega)\right\} = 4\, M, \tag{B.19}$$

and the infimum in (B.19) is never achieved.

Proof. We first establish "$\le$" in (B.19). We assume, with no loss of generality, that $1 \in X$. For sufficiently small ε and for $j = 0, \dots, 2M$, let $b_j = b_{j,\varepsilon} = t_j\, e^{\imath j \varepsilon}$ be the first intersection point with $\partial\Omega$ of the half-line $\{t\, e^{\imath j\varepsilon};\ t > 0\}$. Then $t_0 = 1$ and $t_j \to 1$ as $\varepsilon \to 0$. Let $R_j := [0, b_j]$, $j = 0, \dots, 2M$, $R_{2M+1} := R_0$, $b_{2M+1} := b_0 = 1$. Let U_j be the domain delimited by R_j, R_{j+1} and the arc of $\partial\Omega$ from b_j to b_{j+1} (with the natural orientation on $\partial\Omega$), $\forall\, j = 0, \dots, 2M$. Let $\alpha_j = \alpha_{j,\varepsilon} := e^{\imath j\,(\pi-\varepsilon)}$, $j = 0, \dots, 2M$, and $\alpha_{2M+1} := \alpha_0 = 1$. Define $v = v_\varepsilon$ by $v := \alpha_j$ in U_j, $j = 0, \dots, 2M$. When $\varepsilon < \frac{\pi}{2\,M}$, we have (using (B.1))

$$2Jv = \sum_{j=0}^{2M} \rho(\alpha_{j+1}, \alpha_j)\, \delta_0 = (2\,M\,(\pi - \varepsilon) + 2\,M\,\varepsilon)\ \delta_0 = 2\pi\, M\, \delta_0,$$

and thus v is a competitor in (B.19). On the other hand, we have

$$\int_\Omega |Dv| = \sum_{j=0}^{2M} |\alpha_{j+1} - \alpha_j| = 2\,M\,|1 + e^{-\imath\varepsilon}| + |1 - e^{-\imath 2\,M\,\varepsilon}| \to 4\,M \text{ as } \varepsilon \to 0,$$

whence "$\le$" in (B.19).

For the reverse inequality, consider an arbitrary competitor v in (B.19). Using

$$|z_1 - z_2| \ge \frac{2}{\pi}\rho(z_1, z_2),\ \forall\, z_1, z_2 \in \mathbb{S}^1,\ \text{with equality iff } z_1 = \pm z_2, \tag{B.20}$$

we find that

$$|Dv| \ge \frac{2}{\pi}|Dv|_{\mathbb{S}^1},\ \text{with equality iff } \nabla v = 0,\ D^c v = 0 \text{ and } v^+ = -v^-\ \mathscr{H}^1\text{-a.e.} \tag{B.21}$$

By (B.21), we find that

$$\int_\Omega |Dv| \ge \frac{2}{\pi}\int_\Omega |Dv|_{\mathbb{S}^1} \ge 4\,M, \tag{B.22}$$

and equality requires that $\nabla v = 0$, $D^c v = 0$ and that v is a minimizer in (B.8). In particular, this implies "$\ge$" in (B.19).

By the above, the proof of Theorem 10 and (B.18), we find that, if v is a minimizer in (B.19), then in $\mathbb{D}$ we have $v(x) = w(x/|x|)$, where $w \in BV(\mathbb{S}^1; \mathbb{S}^1)$ has only pure jumps part (and thus is a step function). Moreover, equality in (B.22) requires that at any jump point $x \in \mathbb{S}^1$ of w we have $w(x+) = -w(x-)$, and therefore $\rho(w(x+), w(x-)) = \pm\pi$. In addition, since v is a minimizer in (B.8), we have $\rho(w(x+), w(x-)) > 0$ at any jump point x (see (B.15) and (B.16)). However, this is impossible (see the argument after (B.17)). This implies that there exists no minimizer for (B.19). $\square$

References

[1] G. Alberti, S. Baldo, G. Orlandi, Functions with prescribed singularities, J. Eur. Math. Soc. (JEMS) 5 (3) (2003) 275–311.

[2] F. Almgren, W. Browder, E.H. Lieb, Co-area, liquid crystals, and minimal surfaces, in: Partial Differential Equations (Tianjin, 1986), in: Lecture Notes in Math., vol. 1306, Springer, Berlin, 1988, pp. 1–22.

[3] L. Ambrosio, N. Fusco, D. Pallara, Functions of bounded variation and free discontinuity problems, in: Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 2000.

[4] F. Bethuel, H. Brezis, J.-M. Coron, Relaxed energies for harmonic maps, in: Variational Methods (Paris, 1988), in: Progr. Nonlinear Differential Equations Appl., vol. 4, Birkhäuser Boston, Boston, MA, 1990, pp. 37–52.

[5] J. Bourgain, H. Brezis, P. Mironescu, $H^{1/2}$ maps with values into the circle: minimal connections, lifting, and the Ginzburg-Landau equation, Publ. Math. Inst. Hautes Études Sci. 99 (2004) 1–115.

[6] H. Brezis, J.-M. Coron, E.H. Lieb, Harmonic maps with defects, Comm. Math. Phys. 107 (4) (1986) 649–705.

[7] H. Brezis, P. Mironescu, Sobolev Maps with Values Into the Circle, Birkhäuser, 2018 (in preparation).

[8] H. Brezis, P. Mironescu, A. Ponce, $W^{1,1}$-maps with values into S^1, in: In Geometric Analysis of PDE and Several Complex Variables, in: Contemp. Math., vol. 368, Amer. Math. Soc., Providence, RI, 2005, pp. 69–100.

[9] H. Brezis, P. Mironescu, I. Shafrir, Distances between classes in $W^{1,1}(\Omega;\mathbb{S}^1)$, Calc. Var. Partial Differential Equations 57 (1) (2018) 14.

[10] L.C. Evans, R.F. Gariepy, Measure theory and fine properties of functions, in: Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[11] H. Federer, Geometric measure theory, in: Die Grundlehren Der Mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969.

[12] P. Hajlasz, Sobolev mappings, co-area formula and related topics, in: Proceedings on Analysis and Geometry (Russian) (Novosibirsk Akademgorodok, 1999), Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., Novosibirsk, 2000, pp. 227–254.

[13] R.M. Hardt, J.T. Pitts, Solving Plateau's problem for hypersurfaces without the compactness theorem for integral currents, in: Geometric Measure Theory and the Calculus of Variations (Arcata, Calif., 1984), in: Proc. Sympos. Pure Math., vol. 44, Amer. Math. Soc., Providence, RI, 1986, pp. 255–259.

[14] R. Ignat, The space $BV(S^2, S^1)$: minimal connection and optimal lifting, Ann. Inst. H. Poincaré Anal. Non Linéaire 22 (3) (2005) 283–302.

[15] J. Malý, D. Swanson, W.P. Ziemer, The co-area formula for Sobolev mappings, Trans. Amer. Math. Soc. 355 (2) (2003) 477–492.

[16] A.C. Ponce, On the distributions of the form $\sum_i(\delta_{p_i} - \delta_{n_i})$, C. R. Math. Acad. Sci. Paris 336 (7) (2003) 571–576.

[17] P. Sternberg, G. Williams, W.P. Ziemer, Existence, uniqueness, and regularity for functions of least gradient, J. Reine Angew. Math. 430 (1992) 35–60.

[18] P. Sternberg, W.P. Ziemer, The Dirichlet problem for functions of least gradient, in: Degenerate Diffusions (Minneapolis, MN, 1991), in: IMA Math. Appl., vol. 47, Springer, New York, 1993, pp. 197–214.