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Elliptic operators, conormal derivatives and positive parts of functions (with an appendix by Haïm Brezis)

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Abstract

Haïm Brezis and Augusto Ponce introduced and studied several extensions of Kato's inequality, in particular Kato's inequalities up to the boundary involving the Laplacian and the normal derivative of the positive part of a $W^{1,1}$ function in a smooth domain [H. Brezis, A.C. Ponce, Kato's inequality when Δu is a measure, C. R. Acad. Sci. Paris Sér. I 338 (2004) 599–604; H. Brezis, A.C. Ponce, Kato's inequality up to the boundary, Commun. Contemp. Math. 10 (2008) 1217–1241]. Using potential theoretic methods we answer here some questions raised in [H. Brezis, A.C. Ponce, Kato's inequality up to the boundary, Commun. Contemp. Math. 10 (2008) 1217–1241] about the relations between the normal derivative of a function u and the normal derivative of its positive part u_+. The results apply to a large class of domains and elliptic operators in divergence form and finally an expression of the normal derivative of a function of u is given. In the final appendix, H. Brezis solves an old question of J. Serrin about pathological solutions of certain elliptic equations [J. Serrin, Pathological solutions of elliptic differential equations, Ann. Sc. Norm. Super. Pisa (3) 18 (1964) 385–387]. This is used in the paper to extend the first version of our main result.

Keywords: Second order elliptic equations; Potential theory; Boundary values problems; Weak solutions

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1. Introduction

Let U be an open subset of $\mathbb{R}^d$ and let u be a locally integrable function in U whose Laplacian Δu as a distribution is a locally integrable function in U. Kato's inequality [26] says that $\Delta(u_+)$ is a measure and that $\Delta(u_+) \geqslant \mathbf{1}_{\{u \geqslant 0\}} \Delta u$. Brezis and Ponce [9] have extended this result to the case where Δu is an arbitrary Radon measure in U. On another side, using methods of the fine potential theory, B. Fuglede has improved in [17] results of De la Vallée Poussin, M. Brelot and M.A. Grishin about the positivity of the trace on the set $\{u = 0\}$ of the measure Δu associated to a function u which is in U the difference of two subharmonic function—an important point being to define precisely the set $\{u = 0\}$. Fuglede's result also improves and extends Kato's inequality (see Section 5) but seems to have remained unnoticed by the followers of Kato's work.

In [10] H. Brezis and A. Ponce have introduced and studied forms of Kato's inequality up to the boundary. If U is a smooth bounded domain in $\mathbb{R}^d$, if $u \in W^{1,1}(U)$ is such that Δu is a finite Radon measure in U whose normal derivative $\partial_n u$—in some appropriate weak sense—is a measure on ∂U, it is shown in [10] that Δu_+, $\partial_n(u_+)$ are finite measures (in U and ∂U respectively) such that $\|\Delta u_+\| + \|\partial_n u_+\| \leqslant \|\Delta u\| + \|\partial_n u\|$ (where $\|.\|$ denotes the total mass). If moreover $\partial_n u \in L^1(\partial\Omega)$, then $\partial_n(u_+) \leqslant \mathbf{1}_{\{u>0\}} \partial_n u - \mathbf{1}_{\{u=0\}} (\partial_n u)_-$,—and even $\partial_n(u_+) = \mathbf{1}_{\{u>0\}} \partial_n u - \mathbf{1}_{\{u=0\}} (\partial_n u)_-$ if $u \in W^{2,1}(U)$—see [10] where other results on the normal derivative $\partial_n u_+$ are established.

The aim of this paper is to solve the questions in [10] about possible improvements of the above results (see [10, Section 1]). It will be also shown that these natural improvements hold in a quite general framework. Relying in particular on Fuglede's result mentioned above (see Theorem 5.1) and on a systematic use of the fine potential theory, we establish in particular (Theorem 6.1 in Section 6) the formula $\partial_n(u_+) = \mathbf{1}_{\{u>0\}} \partial_n u - \mathbf{1}_{\{u=0\}} (\partial_n u)_-$ assuming only that U is $C^{1,1}$ (or even $C^{1,\alpha}$, $\alpha > 0$), that $u \in W^{1,1}(U)$ and that Δu and $\partial_n u$ are finite measures in U and ∂U respectively. The assumption on U can be further relaxed if $u \in W^{1,2}(U)$. We will also show that the Laplacian can be replaced by a quite general second order uniformly elliptic operator in divergence form and that some other results of [10] can be extended to our framework. In Section 7, we prove as an application a formula giving the normal derivative $\partial_n f(u)$ for a class of functions f in $\mathbb{R}$. This is also a generalization of Theorem 6.1.

I am pleased to thank Haïm Brezis for attracting my attention and interest to the questions introduced by him and Augusto Ponce in [10] and for supplementing this paper by his recent solution of Serrin's conjecture in [32]. See Theorems A1.1 and A1.2 in Appendix A. The conjecture is about the non-existence of pathological solutions for certain elliptic equations (see Section 3 and Appendix A). It will be seen that Brezis results (announced in [8]) allow us to relax to a certain extent the required regularity assumptions in the paper's main results (see Section 6 and [4]).

2. The setting

We will consider a differentiable manifold M of class C^1, separable and of dimension $d \geqslant 2$, equipped with a second order elliptic operator of a type described below. It would in fact be more natural—but perhaps somewhat heavier—to consider Lipschitz manifolds.

We first state some natural definitions and simple facts needed in the sequel. The reader may just as well glance through this section and return to it when necessary.

2.1. Required Sobolev spaces and distributions

A density m in M is a positive Borel measure m on M such that for any chart $x : U \to \mathbb{R}^d$ of M, one has $dm = f(x)\,dx_1 \dots dx_d$ with $f > 0$ and continuous on $x(U)$; equivalently, for one (or for any) Riemannian C^0-metric g in M, $dm(z) = h(z)d\sigma_g(z)$ in M with $h > 0$ and continuous. Here and in the sequel σ_g denotes the Riemannian volume induced by g. A set $A \subset M$ is negligible if negligible w.r. to any density.

For U open in M, the Frechet space $L^1_{\mathrm{loc}}(U; m)$ is independent of the chosen density m and will be denoted $L^1_{\mathrm{loc}}(U)$. For $1 \leqslant p \leqslant \infty$ one defines similarly the Lebesgue spaces $\mathcal{L}^p_{\mathrm{loc}}(U)$ and $L^p_{\mathrm{loc}}(U)$. The subspace of functions $f \in L^p_{\mathrm{loc}}(U)$ with compact support in U is noted $L^p_c(U)$. If moreover, U is relatively compact in M, $L^p(U; m)$ does not depend on m and we set $L^p(U) = L^p(U; m)$.

One defines similarly, using local charts—or an arbitrary C^0-metric in M—the space of vector fields whose pth power is locally integrable in U ($1 \leqslant p \leqslant \infty$) and, for U relatively compact, the integrability of a vector field in U.

The Sobolev space $W^{1,p}_{\mathrm{loc}}(U)$, for U open in M and $1 \leqslant p \leqslant \infty$, is the space of functions $f \in L^p_{\mathrm{loc}}(U)$ such that for any chart $\varphi : U' \to V \subset \mathbb{R}^d$, $U' \subset U$, one has $f \circ \varphi^{-1} \in W^{1,p}_{\mathrm{loc}}(V)$. We define $W^{1,p}_c(U) := \{f \in W^{1,p}_{\mathrm{loc}}(U)\colon \operatorname{supp}(f) \text{ compact in } U\}$. The space $W^{1,\infty}_c(U) = \mathrm{Lip}_c(U)$ is the space of (locally) Lipschitz functions with compact support in U. This space is a natural space of test functions in U in our setting.

If g is a C^0-metric in M and if $f \in W^{1,p}_{\mathrm{loc}}(U)$ the gradient $\nabla_g f$ is well defined as a class (modulo negligible subsets of U) of locally integrable fields in U by the following property: for V open in $\mathbb{R}^d$ and $\phi : V \to W \subset U$ a C^1-diffeomorphism, one has $g(\nabla_g f[\phi(x)], v) = \nabla(f \circ \phi)(x).(D_x\phi)^{-1}v$, $v \in T_{\phi(x)}(M)$, for a.a. $x \in V$. The integrability of $\nabla_g f$ on a relatively compact subset of M is independent of the choice of g.

If Σ is an open subset of ∂U consisting of points having a neighborhood in which U is a C^1-smooth domain, $W^{1,p}_{\mathrm{loc}}(U \cup \Sigma)$ denotes the set of functions $f \in W^{1,p}_{\mathrm{loc}}(U)$ such that each point in Σ admits an open neighborhood V in M such that f and $\nabla_g f$ are of class L^p in $V \cap U$ (for any C^0-metric g in M). If U is relatively compact and C^1-smooth in M, we set $W^{1,p}(U) := W^{1,p}(\overline{U})$.

Distributions. The spaces of distributions that will be needed are the duals $\mathcal{D}'_1(U) = [C^1_c(U; \mathbb{R})]'$ for U open in M. The space $\mathcal{D}'_1(U)$ is the set of linear forms $\ell : C^1_c(U; \mathbb{R}) \to \mathbb{R}$ such that $\ell(f_n) \to 0$ for any sequence $\{f_n\}$ in $C^1_c(U; \mathbb{R})$ such that $\cup_{n \geqslant 1} \operatorname{supp}(f_n)$ is relatively compact in U and $\lim[\|f_n\|_\infty + \|\nabla_g f_n\|_{g,\infty}] = 0$ for some (or any) C^0-metric g in M. In fact we will mostly consider distributions $T \in \mathcal{D}'_1(U)$ in the form: $T(f) = \int_U f\,d\mu + \int_U g(V, \nabla f)\,d\sigma_g$, $\forall f \in C^1_c(U)$, where V is a locally integrable vector field in U and μ is a Radon measure in U.

2.2. Standard elliptic operators in M

We consider divergence form second order elliptic operators in M defined by a symmetric bilinear form $\beta : W^{1,2}_c(M) \times W^{1,2}_c(M) \to \mathbb{R}$ of the following type: for a certain C^0-metric g in M there exists a measurable section $A = A_g$ of $\mathrm{End}(T(M))$ such that (i) $A(x)$ is g-symmetric for all $x \in M$, (ii) A is locally uniformly bounded and accretive, i.e., for each compact subset K

of M there is a constant $C \geqslant 1$ for which $C^{-1} g_x(u,u) \leqslant g_x(A(x)u,u) \leqslant C g_x(u,u)$ when $x \in K$ and $u \in T_x(M)$, (iii) the form β is given by

$$\beta(u,v) = \int \big\langle A(\nabla_g u), \nabla_g v \big\rangle_g \, d\sigma_g \tag{2.1}$$

when $u, v \in W_c^{1,2}(M)$ ($\beta(u,v)$ is then also meaningful for say $u \in W_{\mathrm{loc}}^{1,1}(U)$, $v \in W_c^{1,\infty}(U)$ and U open in M).

If g_1 is another C^0-Riemannian metric in M, β admits a similar representation with respect to g_1. For if B is the continuous section of $\mathrm{End}(T(M))$ such that $g_1(\xi,\eta) = g(B\xi,\eta)$ for $\xi,\eta \in T(M)$, then $\nabla_{g_1}\varphi = B^{-1}(\nabla_g \varphi)$ for $\varphi \in W_c^{1,2}(M)$ and

$$\beta(u,v) = \int \big\langle A(B\nabla_{g_1} u), \nabla_g v \big\rangle_g \, d\sigma_g = \int \big\langle A(B\nabla_{g_1} u), \nabla_{g_1} v \big\rangle_{g_1} \, d\sigma_g. \tag{2.2}$$

Hence $\beta(u,v) = \int \langle A_1(\nabla_{g_1} u), \nabla_{g_1} v\rangle_{g_1} \, d\sigma_{g_1}$ with $A_1 = \frac{1}{J_{g_1}} A \circ B$ where the jacobian $J_{g_1} = \sqrt{\det B}$ is the density of σ_{g_1} with respect to σ_g.

Let us also notice that for a given metric g, the section A is unique (up to almost everywhere equality). If A' is another section representing β with respect to g and if $\tilde{A} = A - A'$ then for $u, v \in C_c^1(M)$ and $\varphi \in C^1(M)$, $\int \langle \tilde{A}\nabla u, \nabla v\rangle \varphi \, d\sigma = -\int \langle \tilde{A}\nabla \varphi, \nabla v\rangle u \, d\sigma = \int \langle \tilde{A}\nabla \varphi, \nabla u\rangle v \, d\sigma = -\int \langle \tilde{A}\nabla v, \nabla u\rangle \varphi \, d\sigma$ (using $u\varphi$, uv and $v\varphi$). Hence $\int \langle \tilde{A}\nabla u, \nabla v\rangle \varphi \, d\sigma = 0$. It follows that $\langle \tilde{A}\nabla u, \nabla v\rangle = 0$ a.e. Thus $\tilde{A} = 0$ a.e. in M.

The Dirichlet form β induces, for each open subset U of M, a map $\mathcal{L} : W_{\mathrm{loc}}^{1,1}(U) \to \mathcal{D}_1'(U)$ determined by the relations

$$\mathcal{L}(u)(v) = -\int_U \big\langle A(\nabla_g u), \nabla_g v \big\rangle_g \, d\sigma_g \tag{2.3}$$

for all $v \in C_c^1(U)$. These maps are independent of g and are local with the obvious meaning. They will be viewed as an elliptic operator $\mathcal{L}$ and in this paper such an operator will be called a *standard* elliptic operator in M. We will say that $\mathcal{L}$ is associated to β, or equivalently that β is the Dirichlet form associated to $\mathcal{L}$ and denote $\beta = \beta_{\mathcal{L}}$.

Remark 2.1. To give a meaning to $\mathcal{L}(u)$ as a function (for u sufficiently regular) the choice of a density m in M is required. This density determines canonical embeddings $\mathcal{L}_{\mathrm{loc}}^1(U) \to \mathcal{D}_1'(U)$ for each open subset U in M (by $f(\varphi) = \int f\varphi \, dm$ for $\varphi \in C_c^1(U)$) and $\mathcal{L}$ can be seen as the elliptic operator in divergence form which can be written

$$\mathcal{L}(u) := \frac{1}{\theta} \operatorname{div}_g(A\nabla_g u) \tag{2.4}$$

with respect to any given C^0-metric g, where $A = A_g$ is as in (2.1) and where θ is the density of m with respect to σ_g. By definition, for $f \in \mathcal{L}_{\mathrm{loc}}^1(U)$ and $u \in W_{\mathrm{loc}}^{1,1}(U)$, one has $\mathcal{L}u = f$ (in the weak sense) if and only if $\beta(u,v) = -\int f v \, dm$ for all $v \in C_c^1(U)$.

Remark 2.2. If M is moreover a C^2-differentiable manifold and if $A = A_g$ is locally Lipschitz in M, then for $u \in W^{2,1}_{\mathrm{loc}}(U)$, $h := \mathcal{L}(u) \in L^1_{\mathrm{loc}}(M)$ can be directly expressed through formula (2.4) which gives a meaning to h almost everywhere in U (for any fixed C^1-metric g).

Direct image by a diffeomorphism. Let $\Phi : M \to N$ be a C^1-diffeomorphism (or just a locally bilipschitz homeomorphism) between two (separable) C^1-manifolds and let $\mathcal{L}$ be a standard elliptic operator in M. The direct image $\Phi^*(\mathcal{L})$ is the standard elliptic operator $\mathcal{L}'$ in N associated to the Dirichlet form $\beta_{\mathcal{L}'}$ such that $\beta_{\mathcal{L}'}(f, g) = \beta_{\mathcal{L}}(f \circ \Phi, g \circ \Phi)$ for $f, g \in C^1_c(N)$. Equivalently for $f \in C^1_c(N)$, $\mathcal{L}'(f) = \Phi^*[\mathcal{L}(f \circ \Phi)]$ in $\mathcal{D}'_1(N)$, where for $S \in \mathcal{D}'_1(M)$ the direct image distribution $\Phi^*(S) \in \mathcal{D}'_1(N)$ is defined by the relations $\Phi^*(S)(f) = S(f \circ \Phi)$ for all $f \in C^1_c(N)$.

2.3. The conormal derivative on the boundary

Let U be an open subset of M. Using the procedure in [10] one may define the conormal derivative, with respect to $\mathcal{L}$, of a function $u \in W^{1,1}_{\mathrm{loc}}(U)$ as a distribution supported by ∂U—provided that u is sufficiently regular. Let g be a C^0-metric in M.

Definition 2.3. Let $u \in W^{1,1}_{\mathrm{loc}}(U)$ be such that $\nabla_g u$ is integrable in a neighborhood of every point of ∂U and $\lambda := \mathcal{L}(u)$ is a signed Radon measure in U satisfying $|\lambda|(U \cap K) < \infty$ for every compact $K \subset M$. The conormal derivative $\partial_n u$ of u along ∂U is the distribution $\partial_n u \in \mathcal{D}'_1(M)$ defined by

$$(\partial_n u)(v) := \int_U v \, d\lambda + \int_U \big\langle A_g(\nabla_g u), \nabla v \big\rangle \, d\sigma_g \tag{2.5}$$

for $v \in C^1_c(M)$. This distribution is independent of the chosen metric g.

Obviously, (2.5) is also meaningful for $v \in \mathrm{Lip}_c(M)$ and $\partial_n(u)$ extends in a natural way to $\mathrm{Lip}_c(M)$. If U is C^1-smooth, it is clear that $\partial_n u$ depends solely on the C^1-structure of the manifold with boundary $\overline{U}$, the function u and the restriction $\mathcal{L}_{|U}$ (or $(\beta_{\mathcal{L}})_{|U}$).

Remark 2.4. It is easily checked that $\partial_n u$ is supported by ∂U and that the map $u \mapsto \partial_n u$ is local (if $u = 0$ in a neighborhood in U of $P \in \partial U$, then $\partial_n u$ vanishes in a neighborhood of P in M). Another important property is that if U is C^1-smooth then $\partial_n u$ is in fact a distribution on the submanifold ∂U (this was already noticed in the first version of [10]): that is $\partial_n(u)(v) = 0$ for $v \in C^1_c(M)$ vanishing in ∂U. For, in that case, $\partial_n(u)(v) = \partial_n u(v_N)$ if $v_N = \min(\max\{v, -\frac{1}{N}\}; \frac{1}{N})$ and

$$\int_U \big\langle A(\nabla u), (\nabla v_N) \big\rangle \, d\sigma_g = \int_{U \cap \{0 < |v| < \frac{1}{N}\}} \big\langle A(\nabla u), \nabla v \big\rangle \, d\sigma \to 0$$

for $N \to \infty$. It is well known that for $M = \mathbb{R}^d$, $L = \Delta$, U bounded and C^1-smooth and $u \in H^1(U) = W^{1,2}(U)$ then $\partial_n u \in H^{-1/2}(\partial U)$. See [10] for other examples.

Remark 2.5. Suppose that M is C^2, that g is a C^1 metric in M, that $A = A_g$ is locally Lipschitz and that U is C^1-smooth. Let ν denote the field of exterior g-normals along ∂U, let $n = n_A :=$

$A(\nu)$ the field of exterior A-conormals along ∂U and let ds denote the g-superficial measure in ∂U. One has then the classical formula (which easily follows from Stokes formula)

$$\int_U \mathcal{L}(u)v\,dm + \int_U \big\langle A(\nabla u), \nabla v\big\rangle\,d\sigma = \int_{\partial U} v D_n(u)\,ds \tag{2.6}$$

for $u \in \mathcal{C}_c^2(\overline{U})$ and $v \in \mathcal{C}_c^1(\overline{U})$ where $D_n u(z)$, $z \in \partial U$, is the derivative of u at z in the direction n (one may first assume that A is C^1-smooth and then use a limiting argument). So if u is the restriction to U of a $W^{2,\infty}_{\mathrm{loc}}(M)$ function (which implies that $u \in C^1(\overline{U})$) the distribution $\partial_n u$ is the measure with density $D_n(u)$ with respect to ds in ∂U.

3. Regular standard operators

J. Serrin [32] has shown that for a standard elliptic operator $\mathcal{L}$ in the C^1-manifold M, a solution $u \in W^{1,1}_{\mathrm{loc}}(U)$ of $\mathcal{L}(u) = 0$ in the weak sense (2.3) is not always an element of $W^{1,2}_{\mathrm{loc}}(U)$ and so is not in general a weak solution in the usual sense. It will be important for us to eliminate these so-called pathological solutions.

Definition 3.1. We will say that $\mathcal{L}$ (or $\beta = \beta_{\mathcal{L}}$) is regular if every $W^{1,1}_{\mathrm{loc}}$ solution u of $\mathcal{L}(u) = 0$ in a region U of M is necessarily an element of $W^{1,2}_{\mathrm{loc}}(U)$.

Clearly, regularity is a local property which is invariant under bilipschitz homeomorphisms. Classes of regular standard operators that will be useful in the sequel are described below. The following proposition was observed in [4, Lemme 4.1].

Proposition 3.2. *Suppose that M can be covered by open sets U_i, $i \in I$, equipped with bilipschitz homeomorphisms $\Phi_i : U_i \to V_i \subset \mathbb{R}^d$ such that the direct images forms $\beta_i(u, v) = \beta(u \circ \Phi_i, v \circ \Phi_i)$, $u, v \in C_c^1(V_i)$, admit Lipschitz coefficients, that is $\beta_i(u, v) = \sum_{\alpha,\beta} \int c^i_{\alpha,\beta} \partial_\alpha u \partial_\beta v\,dx$ where the $c^i_{\alpha,\beta}$ are locally Lipschitz in V_i. Then $\mathcal{L}$ is regular.*

In fact, as shown by Haïm Brezis in Appendix A, this proposition can be extended to the case where the coefficients $c^i_{\alpha,\beta}$ are only assumed to be Hölder continuous (Theorem A1.2 in Appendix A goes even further); this solves Serrin's conjecture in [32] (see also [20] for a partial solution) and will allow us to include in the main result Theorem 6.1 the case of $C^{1,\varepsilon}$-smooth domains with $0 < \varepsilon \leqslant 1$, and operators with C^ε-smooth coefficients. Our initial results took care of the case $\varepsilon = 1$ and only partially of the case $\varepsilon < 1$ (cf. Théorème 4.2 and the final remarks in part 4 of [4]). We will not expound here our proof of Proposition 3.2 since its methods are more or less explicitly contained in Brezis approach.

Another regularity criterion which relies on the previous one (and Brezis result when $\varepsilon < 1$) and which will be essential for us is given by the following statement. Here the symmetry of elliptic standard operators will be used.

Theorem 3.3. *Let $0 < \varepsilon \leqslant 1$ and let $\mathcal{L} = \sum_{1 \leqslant i,j \leqslant d} \partial_i(a_{ij}\partial_j)$ be a standard elliptic operator in the ball $B_R = B(0, R)$ of $\mathbb{R}^d$ (in particular $a_{ij} = a_{ji}$) with ε-Hölder continuous coefficients a_{ij} in $B_R^+ = B_R \cap \{x_d > 0\}$. Assume moreover that $\mathcal{L}$ is symmetric with respect to x_d (i.e., $a_{ij}(x', x_d) = a_{ij}(x', -x_d)$ if $1 \leqslant i, j < d$ or if $i = j = d$, and $a_{id}(x', x_d) = -a_{id}(x', -x_d)$ for $1 \leqslant i < d$). Then $\mathcal{L}$ is regular.*

The assumptions imply that—after perhaps modifying the coefficients on the hyperplane $\{x_d = 0\}$—the a_{ij} with $1 \leqslant i, j \leqslant d-1$ or $i = j = d$ are C^ε in B_R (these a_{ij} being even with respect to x_d). The point is that the a_{id}, $i < d$, may well be non-extendable by continuity on this hyperplane.

Proof. (a) To establish Theorem 3.3 it will be shown that there exists a small $R' \in (0, R)$ and a bilipschitz homeomorphism $\Psi : B_{R'} \to U$, $U \subset \mathbb{R}^d$, that commutes with the orthogonal symmetry $\sigma = \sigma_d$ with respect to the hyperplane $\{x_d = 0\}$, that is $C^{1,\varepsilon}$ on the manifold with boundary $\overline{B}_R^+ = B_R \cap \{x_d \geqslant 0\}$ and which is such that $\Psi^*(\mathcal{L})$ has C^ε coefficients in $\Psi(B_{R'})$—and not only in $\Psi(\overline{B}_{R'}^+)$. This will show, since $\Psi^*(\mathcal{L})$ is regular by Theorem A1.2, that $\mathcal{L}$ is regular in a neighborhood of 0. Similarly, $\mathcal{L}$ is regular in a neighborhood of each point in $B_R \cap \{x_d = 0\}$. Since $\mathcal{L}$ is clearly regular in $B_R \setminus \{x_d = 0\}$—using again Theorem A1.2—it follows that $\mathcal{L}$ is regular in B_R.

The map Ψ will be constructed in $\overline{B}_R^+ = B_R \cap \{x_d \geqslant 0\}$ by using the following procedure.

(b) *Construction of a class of diffeomorphism.* Let $V = (V_1, \dots, V_d)$ be a C^ε vector function in $\Sigma_R = \overline{B}_R \cap \{x_d = 0\}$ with $V_d = 1$. Let $\tilde{V}$ be a Whitney extension of V to $\mathbb{R}^d$ (Ref. [36]). Of course $\tilde{V}_d = 1$ and as is well known $\tilde{V}$ is C^∞ in $\mathbb{R}^d \setminus \Sigma_R$, C^ε in $\mathbb{R}^d$ and for each multi-index $\alpha \in \mathbb{N}^d$ one has an estimate in the form $|D^\alpha \tilde{V}(x)| \leqslant C_\alpha |x_d|^{\varepsilon - |\alpha|}$ for some positive constant C_α. In particular $\tilde{V}$ is C^ε in B_R.

Consider then the map

$$m \mapsto F(m) = m + m_d \tilde{V}^{(1)}(m)$$

in B_R, where $\tilde{V}^{(1)} := (\tilde{V}_1, \dots, \tilde{V}_{d-1}, 0)$.

Standard verifications show that F is $C^{1,\varepsilon}$-smooth in B_R. It suffices to show that $f_{jk}(x) := x_d \frac{\partial V_j}{\partial x_k}(x)$ is C^ε in B_R^+ which can be seen as follows. Consider $x, x' \in B_R^+$, $x'_d \leqslant x_d$, then $|f_{jk}(x) - f_{jk}(x')| \leqslant |f_{jk}(x)| + |f_{jk}(x')| \leqslant C(x_d^\varepsilon + x_d'^{\,\varepsilon}) \leqslant 2^{1+\varepsilon} C |x - x'|^\varepsilon$ if $|x - x'| \geqslant \frac{1}{2} x_d$. And if $|x - x'| \leqslant \frac{1}{2} x_d$ the mean value theorem and the above estimates of $|D^\alpha \tilde{V}|$ yield $|f_{jk}(x) - f_{jk}(x')| \leqslant C' x_d^{\varepsilon - 1} |x - x'| \leqslant C' 2^{\varepsilon - 1} |x - x'|^\varepsilon$.

Computing $D_0 F$ it is seen that the map F is even a local $C^{1,\varepsilon}$-diffeomorphism at 0 that fixes each point of Σ_R. Its differential maps the normal field $N := (0, 0, \dots, 0, 1)$ on Σ_R to the field V. So taking $\Phi := F^{-1}$ one gets a $C^{1,\varepsilon}$-diffeomorphism Φ on a ball $B_{R'}$, that fixes every point in $\Sigma_R \cap V$ and maps V to N in $B_{R'} \cap \Sigma_R$.

(c) *The construction of* Ψ. Let now Φ be a $C^{1,\varepsilon}$ diffeomorphism from $B_{R'}$ onto an open set $\Omega \subset B_R$, associated as above to a C^ε vector field V in B_R which is transverse to Σ_R (and to be chosen below). Let us denote Ψ the bilipschitz homeomorphism which coincides with Φ in $\overline{B}_{R'}^+ = B_{R'} \cap \{m; m_d \geqslant 0\}$ and is equal to $\sigma \circ \Phi \circ \sigma$ in $\overline{B}_{R'}^- = -\overline{B}_{R'}^+$.

The operator $\mathcal{L}' = \Psi^*(\mathcal{L})$ in $\Omega = \Psi(B_{R'})$ is—as is well known—easily computed. One has $\mathcal{L}'(u) = f$, $u \in W^{1,1}(\Omega)$, $f \in \mathcal{D}'_1(\Omega)$, if and only if $\mathcal{L}(u \circ \Psi) = \Psi^{*-1}(f)$, which means that for all $v \in C_c^1(\Omega)$

$$\sum_{i,j} \int_{B_{R'}} a_{ij}(x)(u \circ \Psi)_i(x)(v \circ \Psi)_j(x)\, dx = -f(v)$$

or

$$\sum_{\alpha,\beta}\int_{B_{R'}}\Big(\sum_{i,j}a_{ij}(x)\frac{\partial\Psi_\alpha}{\partial x_i}(x)\frac{\partial\Psi_\beta}{\partial x_j}(x)\Big)u_\alpha(\Psi)v_\beta(\Psi)\,dx=-f(v) \tag{3.1}$$

and

$$\sum_{\alpha,\beta}\int_{\Omega}\Big(\sum_{i,j}a_{ij}\big(\Psi^{-1}x'\big)\frac{\partial\Psi_\alpha}{\partial x_i}\big(\Psi^{-1}x'\big)\frac{\partial\Psi_\beta}{\partial x_j}\big(\Psi^{-1}x'\big)\Big)u_\alpha(x')v_\beta(x')\frac{1}{J(x')}\,dx'=-f(v) \tag{3.2}$$

where $J(x')=|\det\{D_{\Phi^{-1}(x')}\Psi\}|$. Recall that by assumption the a_{ij} are ε-Hölder continuous in $B_R^+=B_R\cap\{x_d>0\}$; let a_{ij}^+ denote the continuous extension of $a_{ij|B_R^+}$ to $\overline{B}_R^+$ and let $a_{\mathcal{L}}^+(x)$ denote the bilinear form associated to $\{a_{ij}(x)\}_{1\leqslant i,j\leqslant d}$.

The above computation shows that the elliptic operator $\mathcal{L}'$ has the required type if along $\Sigma_d=\{x_d=0\}\cap B_R$ the coefficients $b_{ij}:=a_{\mathcal{L}}^+(\nabla\Phi_i,\nabla\Phi_j)$ vanish when $1\leqslant i<d=j$. But, since Φ is in the form $\Phi(x)=x+x_d\,\tilde{V}^{(1)}(x)$, where V is a C^ε vector field (with $V_d=1$) in Σ_R, one has on Σ_d, $\nabla\Phi_i(x)=(0,\dots,0,1,0,\dots,0,V_i(x))$ (where 1 is the ith coordinate) for $i<d$ and $\nabla\Phi_d(x)=(0,\dots,0,1)$. Thus the condition to satisfy in order that $b_{i,d}$, with $1\leqslant i<d$, vanishes in Σ_d is that $a_{dd}^+(x)V_i(x)+a_{i,d}^+(x)=0$ for $x\in\Sigma_d$, or $V_i(x)=-\frac{a_{id}^+(x)}{a_{dd}^+(x)}$ in Σ_d. Now these relations for $1\leqslant i<d$ together with $V_d=1$ define a C^ε vector field V in B_R for which, by the previous calculations, the corresponding map Ψ has the desired property. □

Remark 3.4. There is a version of Theorem 3.3 for the case $\varepsilon=0$ (i.e., Hölder continuity is replaced by continuity). The conclusion being now that $\mathcal{L}$ is "weakly regular," that is every weak $\mathcal{L}$-solution $u\in W^{1,p}_{\mathrm{loc}}(U)$, $U\subset B_R$, with $p>1$ is in $W^{1,2}_{\mathrm{loc}}(U)$. The proof is similar to the proof of Theorem 3.3 above, using Theorem A1.1 instead of Theorem A1.2.

4. Potential theory

In this section we collect some known basic facts from potential theory. Let $\mathcal{L}$ be a standard elliptic operator defined on the C^1 manifold M. It is well known [34,23,24] that $\mathcal{L}$ defines a Brelot type potential theory (Refs. [7,22]) in M. The corresponding harmonic functions (or $\mathcal{L}$-harmonic functions) are the *continuous* representatives of weak solutions $u\in W^{1,2}_{\mathrm{loc}}(U)$ of $\mathcal{L}(u)=0$, U being an open subset of M (local charts reduce us to the more usual case where $M=\mathbb{R}^d$). More generally, an $\mathcal{L}$-local supersolution s in U (i.e., $s\in W^{1,2}_{\mathrm{loc}}(U)$ and such that $\int\langle A\nabla_g s,\nabla_g\varphi\rangle\,d\sigma_g\geqslant 0$ for all $\varphi\in W_c^{1,\infty}(U)$ with the notations of 2.2) admits a unique $\mathcal{L}$-superharmonic representative in U.

B.1. *Green's function, potentials.* Cf. [34,23,24]. If U is an open subset in M where there exists an $\mathcal{L}$-superharmonic function which is non-constant in each component of U—we then say that U is admissible—, there exists an $\mathcal{L}$-Green's function $G=G_U^{\mathcal{L}}:U\times U\to\overline{\mathbb{R}}_+$ which is continuous, symmetric, finite off the diagonal and for every positive measure μ compactly supported in U, the function $G_\mu:=\int G(.,y)\,d\mu(y)$ is an $\mathcal{L}$-potential (i.e., it is $\mathcal{L}$-superharmonic and its greater $\mathcal{L}$-harmonic minorant vanishes in U). Moreover $G_\mu\in W^{1,r}_{\mathrm{loc}}(U)$ for $r<\frac{d}{d-1}$, $\mathcal{L}G_\mu=-\mu$

in the weak sense of [34] and $G\mu \in H^1_{\mathrm{loc}}(U \setminus \mathrm{supp}(\mu))$ [34,23]. One has also $\mathcal{L}[G_\mu] = -\mu$ in the weak sense (2.3). This easily follows from the approximation result [34, Théorème 9.2]. Every open subset of an admissible domain is admissible and if M is connected and not admissible, it is well known (Myrberg's theorem) that an open subset U is admissible if and only if $M \setminus U$ is not polar (see next paragraph). We will denote $\mathcal{P}(U)$ (resp. $\mathcal{S}(U)$) the set of all $\mathcal{L}$-potentials (resp. $\mathcal{L}$-superharmonic functions) in U.

B.2. *Local behavior of G, polar sets.* For every compact $K \subset U$ and every fixed C^0-metric g in M, there is an estimate: $c^{-1}[d_g(x,y)]^{2-d} \leqslant G(x;y) \leqslant c\, d_g(x,y)^{2-d}$ for $x, y \in K$ and a constant $c > 0$ (when $\dim(M) = 2$, $[d_g(x,y)]^{2-d}$ is to be replaced by $\log \frac{C}{d_g(x,y)}$, $C > 0$ sufficiently large). A polar set is a subset $A \subset M$ such that in a neighborhood of each of its points, A is contained in a set in the form $\{p = +\infty\}$ with p superharmonic in this neighborhood. Equivalently, A is polar in every local chart in the sense of classical potential theory [6,15].

B.3. *Thinness, fine topology.* Cf. [7,15,16]. *The set $A \subset M$ is thin at $a \in \overline{A} \setminus A$* if there exists $U \ni a$ open and $p \in \mathcal{P}(U)$ such that $p(a) < \liminf_{x \in A, x \to a} p(x)$. By definition, A is thin at every $a \notin \overline{A}$ and for $a \in A$, thinness at a is the same as thinness at a of $A \setminus \{a\}$. Using the estimates in B.2 one may show that thinness does not depend on the given standard operator $\mathcal{L}$ [23]. So thinness at a is the same as classical thinness in one (or all) local chart at a and can be characterized by the classical Wiener criterion [15]. One says that $V \subset M$ is a *fine neighborhood of a* if $a \in V$ and if V^c is thin at a. To this notion of neighborhood corresponds a topology called the *fine* topology and for which all $\mathcal{L}$-superharmonic functions are continuous. If $p = G\mu$ and $q = G\nu$ are two $\mathcal{L}$-potentials in M (assuming that M is admissible), then μ and ν coincide on the fine interior of the set $\{p = q\}$ (see Lemma 8.4). Also if $p = q$ almost everywhere (with respect to a density) in a finely open subset U, then $p = q$ everywhere in U since every finely open subset is non-negligible (cf. e.g. [6]).

B.4. *Balayage.* Let $p = G_\mu$ be a potential in the admissible open subset U of M (μ is the positive measure in U associated to p) and let $A \subset U$. Recall that the réduite R_p^A (with respect to U) is the infimum of all nonnegative $\mathcal{L}$-superharmonic functions in U that are larger than p in A; its lower semicontinuous regularization $\hat{R}_p^A$ is an $\mathcal{L}$-potential and is equal quasi-everywhere to R_p^A in U (cf. [22,24]). The measure $\mu^A = -\mathcal{L}(\hat{R}_p^A)$ associated to this potential is the swept-out of μ on A—with respect to U. It is known that $\mu^A = \int \varepsilon_x^A \, d\mu(x)$ where ε_x denotes the Dirac measure at x (in particular, $\mu \to \mu^A$ is linear). Also the swept-out measure ε_x^A is distinct from ε_x if and only if A is thin at x, and in this case ε_x^A does not charge polar sets. In fact for an arbitrary set $A \subset M$ and for $x \notin A$, the swept-out ε_x^A is concentrated on $\partial_f(A)$ the fine boundary of A (more precisely, on an ordinary K_σ subset of $\partial_f(A)$). Cf. [16], [15, pp. 183–186] or Theorem 8.3 in Section 8.

5. A precise form of Kato's inequality

This section is devoted to a precise form of Kato's inequality based on fine potential theory considerations and given by Bent Fuglede in [17]. Let us note that Brezis and Ponce [9] have independently obtained an extension of Kato's inequality for functions whose Laplacian is an arbitrary Radon measure. The reader should consult [17] and [9] for older related results.

Again M denotes a C^1 manifold and $\mathcal{L}$ is a standard elliptic operator in M. In all this section, except in—and after—the final Remark 5.8, we assume that $\mathcal{L}$ is regular.

Let $u \in W^{1,1}_{\mathrm{loc}}(M)$ be such that $-\mathcal{L}(u)$ (in the distribution sense (2.3) in M) is a Radon measure μ in M. In each relatively compact open subset ω of M, $v = u - G^{\mathcal{L}}_{\omega}(\mu_{|\omega})$ is $W^{1,1}$ such that $\mathcal{L}(v) = 0$ in ω so that v is $\mathcal{L}$-harmonic. Hence, u is (locally) equal almost everywhere to a difference of two $\mathcal{L}$-superharmonic functions and u admits a representative $\bar{u}$ which is finite and finely continuous outside a G_δ polar subset of M. This function $\bar{u}$ is unique up to modification on a polar subset. As in [17] let us precisely define (not only up to a polar subset) the set $\{u > 0\} \subset M$ as follows: a point $a \in M$ belongs to $\{u > 0\}$ if and only if the fine liminf at a of $\bar{u}$ is strictly positive—which is meaningful, a polar set being thin at every point. Since a nonempty finely open subset of M is non-negligible [6], we may more simply set

$$\{u > 0\} = \{a \in M\colon \exists \varepsilon > 0,\ \exists A \subset M \text{ thin at } a \text{ and s.t. } u \geqslant \varepsilon \text{ a.e. in } M \setminus A\} \tag{5.1}$$

where on the right-hand side u is seen as an element of $L^1_{\mathrm{loc}}(M)$. Clearly $\{u > 0\}$ is a finely open set which is disjoint from the finely open set $\{u < 0\} := \{-u > 0\}$. Moreover this set is Borel-measurable (a F_σ set, see [16] or Section 8). Its fine boundary $\partial_f\{u > 0\}$ is also Borel-measurable—more precisely a G_δ set (see Section 8).

We may now state the following result which is essentially contained in [17].

Theorem 5.1 *(A precise form of Kato's inequality). The distribution $\mathcal{L}(u_+)$ is a measure and $\mathcal{L}(u_+) = \mathbf{1}_{\{u>0\}}.\mathcal{L}(u) + \lambda_+$ in M where λ_+ is a positive measure concentrated on the finely closed set $\partial_f\{u > 0\}$ (recall that ∂_f means the fine boundary). Moreover, if M is $\mathcal{L}$-admissible and if $u = p - q$ with $p, q \in \mathcal{P}(M)$, the measure λ_+ is smaller than the swept-out on $\{u > 0\}^c$ of some positive measure in M supported by $\{u > 0\}$.*

We will give here a proof of this theorem which relies on the next lemma and is somewhat different from Fuglede's proof [17]. The following classical Fatou–Doob type property will be needed: if $p = G\mu$ and $q = G\nu$ are potentials in M (generated by the measures μ and ν) and if $A \subset M$ is a Borel polar set such that $\nu(A) = 0$, then p/q admits the fine limit $+\infty$ at μ-almost all points a of A ([15, p. 172], or see Theorem 8.1 in Section 8 below). A fact which contains the even more classical property that $\mathbf{1}_{\{p<\infty\}}.\mu$ charges no polar subset of M.

Lemma 5.2. *Let p_1, p_2 be $\mathcal{L}$-potentials in M such that $p_2 \leqslant p_1$, and let $u = p_1 - p_2$, $V = \{u > 0\}$ and $\mu_j = -\mathcal{L}(p_j)$, $j = 1, 2$. Then $\mathcal{L}(u) = \mathbf{1}_V.(\mu_2 - \mu_1) + ([\mathbf{1}_V.\mu_1]^{V^c} - [\mathbf{1}_V.\mu_2]^{V^c})$.*

Proof. We may assume that $\mu_1 \wedge \mu_2 = 0$ (after subtracting $\mu_1 \wedge \mu_2$ to each of μ_1 and μ_2). Since the p_j are finely continuous, we have $p_1 = p_2$ in $V := \{u > 0\}^c$ by the very definition of V. By the general property that have just been recalled, neither μ_1 nor μ_2 may charge a polar subset of V^c. Thus V^c is unthin at μ_j-almost all $a \in V^c$ (recall that the finely isolated points of V^c form a polar set) and, by the balayage properties, we have $\mathcal{L}(\hat{R}^{V^c}_{p_j}) = -\mathbf{1}_{V^c}.\mu_j - (\mathbf{1}_V.\mu_j)^{V^c}$ for $j = 1, 2$. Whence the equality in the statement on applying $\mathcal{L}$ to the equality $p_1 - p_2 = (p_1 - \hat{R}^{V^c}_{p_1}) - (p_2 - \hat{R}^{V^c}_{p_2})$. $\square$

Note the following particular case: if under the conditions of Lemma 5.2, one has $\mathbf{1}_V.\mu_2 \leqslant \mathbf{1}_V.\mu_1$ then $\mathcal{L}(u) = \mathbf{1}_{\{u>0\}}.\mathcal{L}(u) + \lambda_+$ where λ_+ is a positive measure concentrated on $\partial_f\{u > 0\}$ (in fact, the swept-out of $\nu = \mathbf{1}_V.(\mu_1 - \mu_2)$ on V^c, a measure which does not charge polar subsets).

Proposition 5.3. *Let p_1, p_2 be two $\mathcal{L}$-potentials in M such that $p_2 \leqslant p_1$ in M, let $u = p_1 - p_2$ and set $V = \{u > 0\}$. Then $\mathcal{L}(u) = \mathbf{1}_V.\mathcal{L}(u) + \lambda_+$ where λ_+ is a positive measure supported by $\partial_f V$ and smaller than the balayée on V^c of the positive measure $\mu_1 = -\mathbf{1}_V \mathcal{L}(p_1)$ (which is supported by V).*

We want to study the trace of $\mathcal{L}(u)$ on the finely closed set V^c, the last assertion being in fact ensured by Lemma 5.2. Set again $\mu_j = -\mathcal{L}(p_j)$ and assume—as possible—that $\mu_1 \wedge \mu_2 = 0$. We have already remarked in the beginning of the proof of Lemma 5.2 that the μ_j charge no polar subset of V^c. Similarly by the property reminded just before the statement of Lemma 5.2 we have $\nu(V^c \cap \{p_1 = +\infty\}) = 0$ for every positive measure ν in M such that $G\nu \leqslant p_1$.

Let $\varepsilon > 0$ be an arbitrary positive number and write $u = u \wedge \varepsilon + (u - \varepsilon)_+$. Observe that $u \wedge \varepsilon = p_1 \wedge (p_2 + \varepsilon) - p_2$ and $(u - \varepsilon)_+ = u - u \wedge \varepsilon = p_1 - p_1 \wedge (p_2 + \varepsilon)$ are also (almost everywhere equal to) differences of potentials. The measure $\mathcal{L}[(u - \varepsilon)_+]$ does not charge the finely open set $W = \{p_1 < p_2 + \varepsilon\}$ since $p_1 = p_1 \wedge (p_2 + \varepsilon)$ in W. Since moreover, $V^c \setminus W \subset \{p_1 = +\infty\}$, the measure $\mathcal{L}[(u - \varepsilon)_+]$ vanishes in V^c by the remarks in the above paragraph.

Consider then $w = u \wedge \varepsilon$. We have $\mathcal{L}(w) = \mathcal{L}(p_1 \wedge (p_2 + \varepsilon)) - \mathcal{L}(p_2)$ and thus $\mathcal{L}(w) \leqslant \mu_2$. Moreover in the finely open set $W_\varepsilon = \{p_1 > p_2 + \varepsilon\} \subset V$ we have $w = \varepsilon$ and hence $\mathcal{L}(w) = 0$ in W_ε. It is seen in that way that $\mathcal{L}(w) \leqslant \mathbf{1}_{V \setminus W_\varepsilon}.\mu_2$ in V.

Lemma 5.2 says now that $\mathbf{1}_{V^c}\mathcal{L}(u) = \mathbf{1}_{V^c}\mathcal{L}(w) \geqslant -[\mathbf{1}_V(\mathcal{L}(w))_+]^{V^c} \geqslant -[\mathbf{1}_{V \setminus W_\varepsilon}.\mu_2]^{V^c}$ (observe that $V = \{w > 0\}$). Since for $\varepsilon \downarrow 0$ the measure $\mathbf{1}_{V \setminus W_\varepsilon}.\mu_2$ decreases to zero, its swept-out also decreases to zero and hence $\mathbf{1}_{V^c}\mathcal{L}(u) \geqslant 0$.

Corollary 5.4. *Let U be open in M and let $u \in W^{1,1}_{\mathrm{loc}}(U)$, $u \geqslant 0$, be such that $\mathcal{L}(u)$ is a Radon measure in U. If $V := \{u > 0\}$ (in the sense of (5.1) in U) we have $\mathcal{L}u = \mathbf{1}_V.\mathcal{L}(u) + \lambda_+$ in U, where λ_+ is a positive measure in U supported by $U \cap \partial_f V$ (and hence singular w.r. to $\mathbf{1}_V.\mathcal{L}(u)$).*

Proof. Repeating the argument of [17], we observe that the required properties are local so that we may assume U to be $\mathcal{L}$-admissible and that $u = s_1 - s_2$ with $s_j \in \mathcal{S}_+(U)$; taking the réduites of s_j over large compact subsets of U, it is seen that without altering u in the neighborhood of a given point, we may also assume s_j to be a potential in U, $j = 1, 2$. We are then reduced to Proposition 5.3. □

The next observation also follows from Proposition 5.3.

Remark 5.5. Locally, the measure λ_+ in Corollary 5.4 is smaller than the swept-out on V^c of a positive measure supported by V. More precisely, if U_1, U_2 are relatively compact open subsets of U such that $\overline{U}_1 \subset U_2$, $\overline{U}_2 \subset U$, then $\mathbf{1}_{U_1}.\lambda_+ = \tau^{V^c \cap U_2}$ in U_1 where τ is a finite positive Borel measure supported by $V \cap U_2$, and the sweeping is made with respect to the ambient space U_2.

The following corollary will be used to extend to our setting inequalities due to Brezis and Ponce [10]. Recall that if μ is a Radon measure in an open subset U of M, we denote $\|\mu\|$ the total mass of $|\mu|$.

Corollary 5.6. *If $u \in W^{1,1}_c(U)$ is such that $\mathcal{L}(u)$ is a Radon measure in U, then $\|\mathcal{L}(u_+)\| \leqslant \|\mathcal{L}(u)\|$.*

Proof. Since u and u_+ are compactly supported we have $\int d\mathcal{L}(u) = 0$ and $\int d\mathcal{L}(u_+) = 0$ (write $\beta_{\mathcal{L}}(u,\varphi) = \beta_{\mathcal{L}}(u_+,\varphi) = 0$ for $\varphi \in C_c^1(M)$ with compact support in U and equal to 1 in a neighborhood of the support of u). Hence using the notations of Theorem 5.1 we have $\|\lambda_+\| = -\int_{\{u>0\}} d\mathcal{L}(u)$ and $|\int_{\{u>0\}^c} d\mathcal{L}(u)| = \|\lambda_+\|$. So $\|\mathcal{L}(u_+)\| = \int_{\{u>0\}} |d\mathcal{L}(u)| + \|\lambda_+\| \leqslant \int_{\{u>0\}} |d\mathcal{L}(u)| + |\int_{\{u>0\}^c} d\mathcal{L}(u)| \leqslant \|\mathcal{L}(u)\|$. $\square$

Let us close this section by two final observations. The first is independent of Proposition 5.3 and complements a remark made in the beginning of Section 5. It will be used in Section 6 (Remark 6.8).

Remark 5.7. Let $u \in W^{1,1}_{\mathrm{loc}}(M)$ be such that $\mathcal{L}u$ is a measure and let $\lambda = |\mathcal{L}(u)|$. Observing that u is locally the difference of two $\mathcal{L}$-potentials and using the quotient limit theorem (Theorem 8.1) reminded before the statement of Lemma 5.2, it is seen that u admits a (non-necessarily finite) representative (or version) $\hat{u}$ that is finely continuous outside a polar and λ-negligible subset of M.

Remark 5.8. If the standard operator $\mathcal{L}$ is not assumed to be regular the results above (in particular Theorem 5.1) apply to every u which is locally a difference of two $\mathcal{L}$-superharmonic functions. This is in particular the case when $u \in W^{1,2}_{\mathrm{loc}}(M)$ and $\mathcal{L}(u)$ is a measure in M.

Indeed in every open and relatively compact subset ω of M, we have $u = w + h$ with $h \in W^{1,2}(\omega)$ satisfying $\mathcal{L}(h) = 0$ (so h is $\mathcal{L}$-harmonic) and $w \in W^{1,2}_0(\omega)$ is such that $\mathcal{L}w = -\mu$ in ω. Since $w \in W^{1,2}_0(\omega)$ we have also $\mathcal{L}(w) = -\mu$ in the weak sense of [34] and $w = G^{\mathcal{L}}_{\omega}(\mu) = G^{\mathcal{L}}_{\omega}(\mu_+) - G^{\mathcal{L}}_{\omega}(\mu_-)$.

Note also that thanks to Theorem A1.1 the first sentence in Remark 5.8 applies also if $u \in W^{1,p}(M)$ for some $p > 1$, $\mathcal{L}(u)$ is a measure and $\mathcal{L}$ has continuous coefficients in any local C^1 chart.

6. Kato's inequality up to the boundary

In this section we will first assume that M is a $C^{1,\alpha}$-manifold with $\alpha \in (0,1]$ and that the standard elliptic operator $\mathcal{L}$ has C^{α}-smooth coefficients. This means that in the representations (2.1), (2.3) of $\mathcal{L}$ with respect to a C^{α}-smooth metric g the section $A = A_g$ is locally Hölder continuous of exponent α. Equivalently at each point of M there is a chart in which the standard elliptic operator $\mathcal{L}$ is in the form $\mathcal{L} = \sum_{i,j} \partial_i(a_{ij}\,\partial_j)$ with C^{α} coefficients a_{ij}.

Let U be a $C^{1,\alpha}$ relatively compact open subset of M and let $u \in W^{1,1}_{\mathrm{loc}}(U)$ be such that $\mathcal{L}(u)$ is a Radon measure in U. We precisely define the set $\{u > 0\} \subset \overline{U}$ as in Section 5: if $\tilde{u}$ is a representative of u in U which is finely continuous outside some polar subset of U, $\{u > 0\}$ is the finely open subset of $\overline{U}$ of all point $a \in \overline{U}$ where $\tilde{u}$ admits a > 0 fine lower limit. In other words, considering u as an element of $L^1_{\mathrm{loc}}(U)$, $\{u > 0\}$ is the set of all points $a \in \overline{U}$ for which there exist $\varepsilon > 0$ and $A \subset U$ thin at a such that $u \geqslant \varepsilon$ a.e. in $U \setminus A$.

We may now state our main result. Two variants are given at the end of the section.

Theorem 6.1. *Under the above assumptions on M, U and $\mathcal{L}$, if $u \in W^{1,1}(U)$ is such that $\lambda := \mathcal{L}(u)$ and $\partial_n u$ are finite measures—in U and ∂U respectively—, then $\mathcal{L}(u_+)$ and $\partial_n(u_+)$ are also*

finite measures and $\partial_n(u_+) = \mathbf{1}_{\{u>0\}}.(\partial_n u) - \lambda_+$ *where* λ_+ *is a positive measure concentrated on* $\partial U \cap \partial_f^r\{u > 0\}$ *where* ∂_f^r *means the fine boundary in* $\overline{U}$. *More precisely we have*

$$\partial_n(u_+) = \mathbf{1}_{\{u>0\}}.(\partial_n u) - \mathbf{1}_{\{|u|>0\}^c}(\partial_n u)_- \tag{6.1}$$

Remark 6.2. In particular if $\partial_n u \in L^1(\partial U)$ then $\partial_n(u_+) \in L^1(\partial U)$ and we have the following equality in $L^1(\partial U)$: $\partial_n(u_+) = (\mathbf{1}_{\{u>0\}}.\partial_n u) - \mathbf{1}_{\{|u|>0\}^c}(\partial_n u)_-$. Thus Theorem 6.1 solves open problems 1 and 2 of [10, Section 1].[1]

Remark 6.3. The proof will also show that

$$\big\|\mathcal{L}(u_+)\big\| + \|\partial_n u_+\| \leqslant \big\|\mathcal{L}(u)\big\| + \|\partial_n u\| \tag{6.2}$$

which extends inequalities obtained by Brezis and Ponce in [10].

The proof of the first claim in Theorem 6.1 will be reduced to an application of Theorem 5.1.

We start with the following elementary lemma (which as well as the next lemma is valid in the general context of Section 2, that is when M and U are C^1 and $\mathcal{L}$ is an arbitrary standard operator in M). Let ω, ω' be two disjoint open subsets of M such that in the open region $B \subset M$, the set $\Sigma := B \setminus (\omega \cup \omega')$ is a C^1-hypersurface separating ω and ω'.

Lemma 6.4. *If* $v \in W^{1,1}_{\mathrm{loc}}(B)$ *is such that* $\mathcal{L}(v)$ *is a finite measure* μ *in* $B \setminus \Sigma$ *and if one denotes* $\partial_n(v)$, $\partial'_n(v)$ *the conormal derivatives (with respect to* $\mathcal{L}$*) of* $v_{|\omega\cap B}$ *(resp.* $v_{|\omega'\cap B}$*) along* Σ, *then* $\mathcal{L}(v) = \mu - (\partial_n(v) + \partial'_n(v))$ *in the sense of distributions in* B. *In particular the distribution* $\mathcal{L}(v)$ *is a measure in* B *if and only if* $\partial_n(v) + \partial'_n(v)$ *is a measure in* B *(supported a priori by* Σ*).*

Proof. If $\varphi \in C^1_c(B)$ is a test function in B, we have using the notations of Section 2:

$$\begin{aligned}
\mathcal{L}(v)(\varphi) &= -\int_B \langle A\nabla v, \nabla\varphi\rangle\, d\sigma \\
&= -\int_{\omega\cap B} \langle A\nabla v, \nabla\varphi\rangle\, d\sigma - \int_{\omega'\cap B} \langle A\nabla v, \nabla\varphi\rangle\, d\sigma \\
&= \Big(\int_{\omega\cap B} \varphi\, d\mu - \int_\Sigma \varphi\partial_n(v)\Big) + \Big(\int_{\omega'\cap B} \varphi\, d\mu - \int_\Sigma \varphi\partial'_n(v)\Big) \\
&= -\int_\Sigma \varphi\big(\partial_n(v) + \partial'_n(v)\big) + \int_B \varphi\, d\mu,
\end{aligned} \tag{6.3}$$

which is the desired result. □

[1] If $\hat{u}$ is the L^1 trace of u on ∂U, $\{u > 0\} \cap \partial U = \{\hat{u} > 0\} \cap \partial U$ (mod negligible sets in ∂U). Observing—see after Proposition 6.6—that we may assume $u := G^V\theta$ in U, with θ a finite measure in $V \supset \overline{U}$, the assertion easily follows.

We will use the following consequence of Lemma 6.4. In the sequel we say that a C^1 diffeomorphism $\Phi : V \to V'$ between two open subsets of M leaves $\mathcal{L}$ invariant if $\mathcal{L}_{|V'} = \Phi^*(\mathcal{L}_{|V})$ (see Section 2).

In the next lemma we maintain the assumptions and notations of Lemma 6.4. It is also assumed that $\omega \subset B$ and $\omega' \subset B$.

Lemma 6.5. *Let $\Phi : B \to B$ be an involutive C^1-diffeomorphism (so $\Phi \circ \Phi = Id_B$) applying ω onto ω', fixing every point of $\Sigma = \partial\omega \cap B$ and leaving $\mathcal{L}$ invariant. If $v \in W^{1,1}_{\mathrm{loc}}(\omega \cup \Sigma)$ is such that $\nu := \mathcal{L}(v)$ is a finite Radon measure in ω and if $\tilde{v}$ is the function in $W^{1,1}_{\mathrm{loc}}(B)$ obtained by extending v by symmetry (that is $\tilde{v}(x) = v(\Phi(x))$ when $x \in B \setminus \overline{\omega}$) we have*

$$\mathcal{L}(\tilde{v}) = \nu + \Phi^*(\nu) - 2\partial_n(v). \tag{6.4}$$

Here ν is considered as a finite measure in B supported by ω, $\Phi^*(\nu)$ is its direct image under Φ, and $\partial_n v$ is seen as a distribution in B (supported by Σ). In particular $\mathcal{L}(\tilde{u})$ is a measure in B if, and only if, $\partial_n(v)$ is a measure in Σ.

To check that $\tilde{v} \in W^{1,1}_{\mathrm{loc}}(B)$, one can, using local charts, reduce itself to the case where $B = M = \mathbb{R}^d$, $\omega = \{x_d < 0\}$, $\omega' = \{x_d > 0\}$, $\Sigma = \mathbb{R}^{d-1} \times \{0\}$ and where $\tilde{v}$ is compactly supported. It suffices then to observe that if $v = \lim v_j$ in $W^{1,1}(\omega)$, $v_j \in C^\infty(\overline{\omega})$, $\operatorname{supp} v_j \subset B(0, R)$ then $\tilde{v}_j \in W^{1,1}(\mathbb{R}^d)$ and $\|\tilde{v}_j - \tilde{v}_k\|_{W^{1,1}} = 2\|v_j - v_k\|_{W^{1,1}(\omega)}$. Thus $\tilde{v}$ is the limit of the sequence $\tilde{v}_j$ in $W^{1,1}(\mathbb{R}^d)$.

Set $v' = \tilde{v}_{|U'} = v \circ \Phi$. We have seen that $\mathcal{L}(v')$ coincides in ω' with the direct image measure $\nu' = \Phi(\nu)$ of ν under Φ. Moreover by Definition 2.3 of the conormal derivative, we have for $\psi \in C^1_c(B)$ and $\varphi = \psi \circ \Phi$,

$$\begin{aligned}
(\partial_n v')(\psi) &= \int_{\omega'} \psi \, d\nu' + \int_{\omega'} \big\langle A_g(\nabla_g v'), \nabla_g \psi \big\rangle \, d\sigma_g \\
&= \int_{\omega} \varphi \, d\nu + \int_{\omega} \big\langle A_g(\nabla_g v), \nabla_g \varphi \big\rangle \, d\sigma_g \\
&= (\partial_n v)(\varphi) \\
&= (\partial_n v)(\psi)
\end{aligned} \tag{6.5}$$

where we have used in the last line the fact that $\partial_n v_{|\partial B}$ vanishes on test functions which are null on $\partial\omega$—see Remark 2.4. Whence $\partial_n v' = \partial_n v$ in B and the statement follows from the previous Lemma 6.4.

Proof of Theorem 6.1. In most of what follows we will retain only the C^1 structures, and so use only the standard character of $\mathcal{L}$ (locally the "coefficients" of $\mathcal{L}$ are bounded measurable). We will return to the extra regularity assumptions ($C^{1,\alpha}$ regularity of M and U, and C^α regularity of the "coefficients" of $\mathcal{L}$) to establish Proposition 6.6 below; it is only there that they intervene and for a while it will be convenient to ignore them. Let us now proceed with the first step in the proof of Theorem 6.1.

First part. Let us introduce a compact C^1-manifold $\tilde{M}$ which is a double of the C^1-manifold with boundary $\overline{U}$: topologically it is obtained by gluing $\overline{U}$ with a copy $\overline{U'} = \overline{U} \times \{1\}$ of $\overline{U}$ by

identification of corresponding points of ∂U and $\partial U'$. It is provided with a natural bicontinuous symmetry $\Phi : \tilde{M} \to \tilde{M}$ such that $\Phi(x) = (x, 1)$, $\Phi(x, 1) = x$, $\forall x \in \overline{U}$.

We may then fix a C^1-differentiable structure on $\tilde{M}$ using the following known fact. There exists an open neighborhood V of ∂U in M and a C^1-diffeomorphism (of C^1-manifolds with boundaries) $\theta : V' = V \cap \overline{U} \to \partial U \times [0, 1)$; Whitney's theorem asserting the existence of a C^∞-structure on M compatible with its C^1-structure [25] reduces us to a classical property (I owe this argument to J.-B. Bost). If W denotes the open collar $V' \cup \Phi^{-1}(V')$ in $\tilde{M}$ and if s is the natural symmetry $(x, t) \to (x, -t)$ of the C^1-manifold $N = \partial U \times (-1, 1)$, there exists a unique C^1-structure on $\tilde{M}$ satisfying the following: (i) the map $\tilde{\theta} : W \to N$ equal to θ on V' and such that $\tilde{\theta} \circ \Phi = s \circ \tilde{\theta}$ is a C^1 diffeomorphism, (ii) this structure coincides with the initially given structure in U and $\Phi : U \to U'$ is a C^1-diffeomorphism.

For this C^1-structure in $\tilde{M}$, the initial $C^{1,\alpha}$-manifold with boundary $\overline{U}$ is a C^1 submanifold with boundary of $\tilde{M}$ and Φ is an involutive C^1-diffeomorphism of $\tilde{M}$ such that $\Phi \circ \Phi = Id_{\tilde{M}}$ and $\Phi(U) = U'$. There is not uniqueness in general of the C^1-structure that has been so obtained, but the induced Lipschitz structure is unique and much easier to define.

We may then fix a Φ invariant C^0-metric in $\tilde{M}$ (take any C^0-metric g in $\tilde{M}$ and set $g_0 = g + \Phi^*(g)$ for example). We may also extend $\mathcal{L}_{|U}$ to a Φ-invariant standard second order elliptic operator in $\tilde{M}$ (cf. Section 2.2): if $\mathcal{L}$ is associated to the section $A = A_g$ with respect to g_0 in U, it suffices to extend A to a Φ-invariant measurable section of $\mathrm{End}(T(M))$ with $A(x) = Id$ for $x \in \partial U$ (the values of A on ∂U are unimportant since ∂U is negligible in $\tilde{M}$).

We will now exploit the regularity assumptions of Theorem 6.1 to establish the following proposition.

Proposition 6.6. *The operator $\tilde{\mathcal{L}}$ is regular in $\tilde{M}$.*

Proof. It is plain that $\tilde{\mathcal{L}}$ is regular in U since $\mathcal{L}$ is regular and $\tilde{\mathcal{L}} = \mathcal{L}$ in U. And since $\tilde{\mathcal{L}} = \Phi^*(\tilde{\mathcal{L}})$ it is clear that $\tilde{\mathcal{L}}$ is regular in $\tilde{M} \setminus \Sigma$ where Σ denotes the boundary of U in $\tilde{M}$.

It remains to show that $\tilde{\mathcal{L}}$ is regular in a neighborhood of each point $m_0 \in \Sigma$. By assumption, since $\overline{U}$ is a $C^{1,\alpha}$-submanifold with boundary of M, there is a chart

$$\varphi : V \cap \overline{U} \to \overline{B_R^+} = B_R \cap \overline{\mathbb{R}}_+^d$$

which is $C^{1,\alpha}$ for the initial structure in U and transforms $\tilde{\mathcal{L}}_{|U \cap V}$ into a standard elliptic operator with α-Hölder continuous coefficients in B_R^+.

Extending φ by symmetry, one gets a bilipschitz homeomorphism $\tilde{\varphi} : V \to B_R$ transforming $\tilde{\mathcal{L}}$ into a standard elliptic operator L in B_R to which Proposition 3.3 applies. This operator L is thus regular in B_R and by regularity invariance under bilipschitz homeomorphism we see that $\tilde{\mathcal{L}}$ is regular in a neighborhood of m_0. $\square$

To establish Theorem 6.1, we thus may (and will) from now on assume that (i) $M = \tilde{M}$, U being seen as a C^1-open subset of the C^1-manifold $\tilde{M}$ and ∂U as its boundary in $\tilde{M}$, (ii) $\mathcal{L} = \tilde{\mathcal{L}}$ is regular Φ-invariant.

Proof of Theorem 6.1. Continuation. Denote by $\tilde{u}$ the extension by symmetry of u: $\tilde{u}(x) = u(\Phi(x))$ for $x \in U'$. Since Φ is a C^1-automorphism, Lemma 6.5 says that $\tilde{u} \in W^{1,1}_{\mathrm{loc}}(M)$ $(= W^{1,1}(M)$, M being now compact) and that $\mathcal{L}(\tilde{u})$ is a measure in M.

Combining now the precise form of Kato's inequality (Theorem 5.1) and Lemma 6.5, we will obtain the first assertion of Theorem 6.1. Indeed, $\mathcal{L}(\tilde{u})$ is a measure so $\tilde{u}_+ \in W^{1,1}_{\mathrm{loc}}(M)$ is

such that $\mathcal{L}(\tilde{u}_+)$ is a measure in M and one has $\mathcal{L}(\tilde{u}_+) = \mathbf{1}_{\{\tilde{u}>0\}}.\mathcal{L}(\tilde{u}) + \tilde{\lambda}_+$ where $\tilde{\lambda}_+$ is a positive Radon measure in M supported by $\partial_f(\tilde{u} > 0)$. Thus by Lemma 6.5 the distribution $\partial_n u_+$ is a Radon measure supported by ∂U and one has $\mathcal{L}(\tilde{u}) = \mathbf{1}_{U\cup\Phi(U)}\,\mathcal{L}(\tilde{u}) - 2\partial_n(u)$, $\mathcal{L}(\tilde{u}_+) = \mathbf{1}_{U\cup\Phi(U)}\mathcal{L}(\tilde{u}_+) - 2\partial_n(u_+)$. Passing to traces on ∂U we get $\partial_n(u_+) = \mathbf{1}_{\{u>0\}}.\partial_n(u) - \frac{1}{2}\mathbf{1}_{\partial U}\tilde{\lambda}_+$. This establishes the first claim of Theorem 6.1 with $\lambda_+ = \frac{1}{2}\mathbf{1}_{\partial U}\tilde{\lambda}_+$.

To prove the second claim in Theorem 6.1 let us first notice that by considering $-u$ we also have $\partial_n u_- = -\mathbf{1}_{\{-u>0\}}\partial_n u - \lambda_-$ where λ_- is a finite positive measure supported by $\partial U \cap \partial_f^r\{-u > 0\}$. Moreover we know from Theorem 5.1 that λ_+ is "locally" dominated by the swept-out on $\{\tilde{u} > 0\}^c$ of a finite positive measure supported by $\{\tilde{u} > 0\}$. More precisely (see Remark 5.5) for each $x_0 \in \partial U$ every admissible open neighborhood V of x_0 in M, λ_+ is near x_0 smaller than the swept-out (w.r. to V) on $\{\tilde{u} > 0\}^c \cap V$ of a positive measure concentrated in $\{\tilde{u} > 0\} \cap V$ and with compact support in V. Similarly λ_- is smaller in the vicinity of x_0 than the swept-out on $\{-\tilde{u} > 0\}^c \cap V$ of a positive measure concentrated in $\{-\tilde{u} > 0\} \cap V$ and compactly supported in V.

Proposition 6.9 stated and established below will show that $\lambda_+ \wedge \lambda_- = 0$. The second claim of Theorem 6.1 follows by observing that since $\partial_n u = \partial_n(u_+) - \partial_n(u_-) = -\lambda_+ + \lambda_-$ on the set $\partial U \cap \{|u| > 0\}^c$ one has $\lambda_+ = \mathbf{1}_{\{|u|>0\}^c}.[\partial_n u]_-$. The proof of Theorem 6.1 is then complete. □

Remark 6.7. Let us observe that at this stage Remark 6.3 easily follows from Remark 5.6—i.e., the case where u is compactly supported in U: indeed using the above and applying Lemma 6.5

$$\left\|\mathcal{L}(\tilde{u}_+)\right\| = 2\left\{\left\|\mathbf{1}_{U\cap\{u>0\}}\mathcal{L}(u)\right\| + \|\partial_n u_+\|\right\} \tag{6.6}$$

and similarly $\|\mathcal{L}(\tilde{u})\| = 2\{\|\mathbf{1}_U\mathcal{L}(u)\| + \|\partial_n u\|\}$. Whence the result since $\|\mathcal{L}(\tilde{u}_+)\| \leqslant \|\mathcal{L}(\tilde{u})\|$ by Remark 5.6.

Remark 6.8. In view of the next section, let us also notice that an application of Remark 5.7 to $\tilde{u}$ shows that the function u admits in $\overline{U}$ a (non-necessarily finite) finely continuous representative outside a Borel polar subset which is also negligible with respect to $|\partial_n u| + \mathbf{1}_U|\mathcal{L}(u)|$.

In order to work now with an admissible (with respect to $\mathcal{L}$) connected manifold we assume as we may that U is connected and consider from now on $M' = M \setminus (T_1 \cup T_2)$ where T_1 is a compact subset with nonempty interior in U and $T_2 = \Phi(T_1)$ is its symmetric image. The next proposition relies on the C^1-regularity of the hypersurface $\Sigma = \partial U$.

Proposition 6.9. *Let V, W be two finely open disjoint and Φ-invariant subsets of M' and let μ, ν be two finite positive measures supported by V and W respectively (i.e., $\mu^*(V^c) = \nu^*(W^c) = 0$). Let μ' (resp. ν') be the trace on $\Sigma = \partial U$ of the swept-out measure μ^{V^c} (resp. ν^{W^c}) in M'. Then $\mu' \wedge \nu' = 0$*

Proof. Adding to V the set of all points of M' where $M' \setminus V$ is thin and modifying similarly W we may assume that V and W are Borel sets (and even K_σ sets, cf. [6,15]). Arguing by contradiction and assuming that $\mu' \wedge \nu' \neq 0$ there exists a compact set $\mathbb{K} \subset \partial U \setminus V \cup W$ which does not separate M' and is such that the traces of μ' and ν' on $\mathbb{K}$ are non-vanishing mutually absolutely continuous positive measures.

Since $\mathbf{1}_{\mathbb{K}}.\mu'$ is smaller than the swept-out measure $\mu^{\mathbb{K}}$ in M' and since $\varepsilon_x^{\mathbb{K}}$ is the $\mathcal{L}$-harmonic measure of x in $M' \setminus \mathbb{K}$, we see that the harmonic measure class with respect to $M \setminus \mathbb{K}$ does not vanish and dominates the class of μ' (or ν') on $\mathbb{K}$.

To pursue, we will consider the Martin boundary of $\Omega := M' \setminus \mathbb{K}$ (w.r. to $\mathcal{L}$) and use several properties known in the case at hand ($\mathbb{K}$ contained in a Lipschitz hypersurface of M'). For the Martin boundary theory, the reader is referred to [29,31,15] and the exposition [3]. Recall that this theory associates to each admissible region Ω of M' a boundary $\widehat{\partial\Omega}$ whose main part consists of the "minimal" boundary points. Having fixed a reference point $x_0 \in \Omega$, to each minimal point $\zeta \in \widehat{\partial\Omega}$ corresponds on the one hand a unique positive $\mathcal{L}$-harmonic function K_ζ in Ω which is minimal and normalized at x_0, and on the other hand a notion of "minimal thinness" at ζ: $A \subset \Omega$ is minimally thin at ζ if $R_{K_\zeta}^A \not\equiv K_\zeta$ (the réduite is performed with respect to the domain Ω). A point $a \in \Omega$ is a pole of ζ if, for all $r > 0$, the set $\Omega \cap B(a, r)$ is not minimally thin at ζ, Ref. [31].

We will use here a variant of the following well-known property. Let F be a closed subset of Ω, let ν be the harmonic measure in $\Omega \setminus F$ of some point $a \in \Omega \setminus F$. Then ν-almost every point $z \in \partial\Omega$ is the unique pole of at least one minimal point $\zeta \in \widehat{\partial\Omega}$ such that F is minimally thin at ζ [31, p. 247 and Chapter V]. The simple variant we need is stated in the next lemma.

Lemma 6.10. *Let V be a finely open subset of M', let $\mathbb{K}$ be a compact subset of $\partial_f V$ not separating M', let μ be a finite positive measure supported by V and let μ' denote the trace on $\mathbb{K}$ of the swept-out measure (in M') of μ on V^c. Then with respect to $\Omega := M' \setminus \mathbb{K}$, μ'-almost all $x \in \mathbb{K}$ is the unique pole of at least one minimal point ζ in the Martin boundary of Ω such that $\Omega \setminus V$ is minimally thin at ζ.*

Let us sketch for the reader's convenience a proof of Lemma 6.10. It is easily seen that we may assume that V is relatively compact and by adding to V a polar subset that V^c is thin at no point of $V^c \cap \mathbb{K}^c$. Then V is an ordinary F_σ set (cf. [6] or [16]).

Let L be a compact subset of $\mathbb{K}$ such that $\mu'(L) > 0$. The function $x \mapsto u(x) := \varepsilon_x^{V^c}(L)$—sweeping with respect to M'—is subharmonic in $M' \setminus L$ (see Proposition 8.5 below). It vanishes quasi everywhere in $V^c \setminus L$ and $0 \leqslant u \leqslant 1$. Moreover $u \not\equiv 0$ since $\int_V u(x)\,d\mu(x) = \int_V \varepsilon_x^{V^c}(L)\,d\mu(x) = [\int_V \varepsilon_x^{V^c}\,d\mu(x)](L) = \mu'(L) > 0)$.

It follows that in $\Omega = M' \setminus \mathbb{K}$ the function $h(x) = \omega(x; L; \Omega)$ (the harmonic measure of x w.r. to Ω and L) is not stable by reduction—with respect to the domain Ω—on $V^c \cap \Omega$. Indeed by the maximum principle, we have $u \leqslant h$ in Ω (note that $h = \lim \downarrow s_n$ where the s_n are $\mathcal{L}$-positive superharmonic in Ω and $\liminf s_n \geqslant 1$ at every point of L) so $h - u$ is nonnegative $\mathcal{L}$-superharmonic, $h - u = h$ q.e. on $V^c \cap \Omega$ and $[\hat{R}_h^{\Omega\setminus V}]_\Omega \leqslant h - u$.

Now, if ω denotes the harmonic measure on the minimal Martin boundary of the fixed normalization point $x_0 \in \Omega$, we have $h = \int_{\pi^{-1}(L)} K_\zeta\,d\omega(\zeta)$ (here $\pi(\zeta)$ is the unique pole of ζ when it exists[2]). Thus $\hat{R}_h^{V^c} = \int_{\pi^{-1}(L)} \hat{R}_{K_\zeta}^{V^c}\,d\omega(\zeta)$. As $\hat{R}_h^{V^c} \not\equiv h$, the set A_L of all points ζ in $\pi^{-1}(L)$ where V^c is minimally thin has positive harmonic measure. In other words, the set of all points $x \in L$ such that x is the unique pole of at least one minimal point $\zeta \in \widehat{\partial\Omega}$ for which $\Omega \setminus V$ is minimally thin at ζ has > 0 harmonic measure. $\square$

[2] We need only the case where $\mathbb{K}$ is contained in a C^1 hypersurface. Then every Martin point over $\mathbb{K}$ has a well-defined pole [2].

Continuation of the proof of Proposition 6.9. In our case, the compact subset $\mathbb{K}$ is contained in a C^1-hypersurface and one has a rather precise description of the part X of the minimal Martin boundary of $M' \setminus \mathbb{K}$ lying above $\mathbb{K}$ ([2, Sections 7 and 8]—see generalizations in [1,5] and references therein). In particular (being of local nature the results in [2] extend to the setting of C^1 manifolds) there is a continuous projection π of X onto $\mathbb{K}$, which associates to each point $\zeta \in X$ its unique pole $x \in \mathbb{K}$, each point $x \in \mathbb{K}$ being a pole of one or two minimal points (compare also with the striking general result in [37] about triple points)—in the first case the point x is said to be simple and in the other case it is a double point. Moreover when x is a double point, Φ exchanges the two minimal points above x. Indeed the arguments in [12] show that from the Harnack boundary principle of [2] it follows that: (a) each sequence $\{x_n\}$ in $M' \setminus \mathbb{K}$ converging non-tangentially to some point $z \in \mathbb{K}$, admits only minimal points as cluster values on the Martin boundary $\widehat{\partial\Omega}$, (b) every minimal point ζ associated to $z \in \mathbb{K}$ is the limit of such a sequence. In particular for a connected subset $C \subset M' \setminus \partial U$ which is non-tangential for ∂U at $z \in \mathbb{K} \cap \overline{C}$, the cluster set $\overline{C} \cap \widehat{\partial\Omega}$ is reduced to one minimal boundary point.

We note here that the symmetry of the elliptic operator $\mathcal{L}$ is used again since the proof of the main result in [2, Section 7] (and final remark in Section 8) relies in an essential way on the symmetry of the elliptic operators under consideration.

We now deduce the following consequence using the invariance of $\mathcal{L}$ and V under Φ.

Consequence. If under the assumptions of Lemma 6.10 it is assumed moreover that V is Φ-symmetric and $\mathbb{K} \subset \partial U$, then for μ'-a.a. $x \in \mathbb{K}$, $M' \setminus V$ is minimally thin at each minimal point with pole x.

For if x is simple, the claim is already contained in Lemma 6.10, and if $x \in \mathbb{K}$ admits two corresponding minimal points in the Martin boundary of Ω and is such that $\Omega \setminus V$ is minimally thin w.r. to one of these points, $M' \setminus \Phi(V) = M' \setminus V$ is also minimally thin with respect to the other minimal point.

Conclusion. Proposition 6.9 is now established since for $\mu' \wedge \nu'$ almost all points $x \in \mathbb{K}$ the two subsets $M' \setminus V$ and $M' \setminus W$ are both minimally thin at each point in $\pi^{-1}(x)$ which is impossible. We have thus reached a contradiction. □

We now state a variant of Theorem 6.1, where using a stronger assumption on u, the problems related to the non-regularity of $\mathcal{L}$ vanish so that the smoothness assumptions on M and U can be notably relaxed.

Theorem 6.11. *Let M be a C^1-manifold, let $\mathcal{L}$ be a standard second order elliptic operator in M, and let U be a Lipschitz relatively compact open subset of M. If $u \in W^{1,2}(U)$ is such that $\mathcal{L}(u)$ and $\partial_n(u)$ are finite measures, $\mathcal{L}(u_+)$ and $\partial_n(u_+)$ are also finite measures and the following formula holds*

$$\partial_n(u_+) = \mathbf{1}_{\{u>0\}}.(\partial_n u) - \mathbf{1}_{\{|u|>0\}^c}(\partial_n u)_-. \tag{6.7}$$

Observe first that we may assume U to be C^1-smooth (assumptions and results are of local nature and invariant under bilipschitz homeomorphism). The point is then (see Remark 5.8) that for Ω open in M, an element $v \in W^{1,2}_{\mathrm{loc}}(\Omega)$ such that $\mu = -\mathcal{L}(v)$ is a Radon measure in Ω, can be written locally as the difference of two $\mathcal{L}$-superharmonic functions—even without assuming

that $\mathcal{L}$ is regular. This remark applies to u and u^+ and an inspection of the arguments used above shows that the proof of Theorem 6.1 extends to the case at hand (and can also be made simpler—Proposition 6.6 being now superfluous).

Remark 6.12. Let us also notice another variant of Theorem 6.1 which can be proved along the same lines (using now Remark 3.4 instead of Theorem 3.3) and which is based on Brezis improvement (Theorem A1.1 in Appendix A) of Hager and Ross result [20]. Let M, U and $\mathcal{L}$ be as above in Theorem 6.11, U being C^1-smooth and $\mathcal{L}$ having continuous coefficients. If $u \in W^{1,p}(U)$ with $p > 1$ is such that $\mathcal{L}(u)$ and $\partial_n(u)$ are finite measures, then the conclusions in Theorem 6.11 still hold.

Added in proofs. One may extend [10, Theorem 1.2] to our framework as follows. Assumptions and notations are as in the beginning of Section 6.

Proposition 6.13. *If $u \in W_0^{1,1}(U)$ is such that $\mathcal{L}(u)$ is a measure of finite total mass in U, then $\partial_n u$ is an absolutely continuous Radon measure on ∂U (i.e. $\partial_n u \in L^1(\partial U)$). Moreover* (i) $\|\mathcal{L}(u^+)\| \leqslant \|\mathcal{L}(u)\|$, (ii) $\|\partial_n u\| \leqslant \|\mathcal{L}(u)\|$ *and* (iii) *if $u \geqslant 0$, then $\partial_n u \leqslant 0$.*

Proof. Set $\mu := -\mathcal{L}(u)$ and denote G^U the $\mathcal{L}$-Green's function in U. We know [34] that $G^U\mu \in W_0^{1,1}(U)$ and therefore by the uniqueness principle Theorem A5.1 in Appendix A and [30, Chapter 5] we have that $u = G^U\mu$ (Lipschitz regularity for U suffices here).

To prove the first claim we may assume that μ is positive. Then, if V is an open neighborhood of $\overline{U}$, writing $G^U\mu = G^V\mu - G^V\mu'$ where μ' is the swept-out of μ on $V \setminus U$ in V, it is easily checked using the definitions that $\partial_n u = -\mu'$.

Now μ is the limit of an increasing sequence $\{\mu_p\}$ of positive measures with compact supports in U and since $\|\mu_p - \mu\| \to 0$, $u_p = G^U\mu_p \to u$ in $W_0^{1,1}(U)$ and $\partial_n u_p$ decreases to $\partial_n u$ as $p \to \infty$. Since u_p is $C^{1,\alpha}$ in a neighborhood of ∂U in $\overline{U}$, $\partial_n u_p$ is absolutely continuous (and coincides with the standard conormal derivative if one fixes a C^α-metric in M). Hence $\partial_n u \in L^1(\partial U)$.

To prove (ii), write $u = G^U\mu_+ - G^U\mu_-$. Since $\partial_n u = -\mu'_+ + \mu'_-$, $\|\mu'_\pm\| \leqslant \|\mu_\pm\|$ we obtain (ii). Taking U as the ambient manifold, setting $W := \{u > 0\}$, $W' = \{-u > 0\}$, and using Theorem 5.1 and Lemma 5.2, we have $\mathcal{L}u_+ = -\mathbf{1}_W(\mu_+ - \mu_-) + \lambda$, with $\lambda = [\mathbf{1}_W\mu_+]^{W^c} - [\mathbf{1}_W\mu_-]^{W^c}$ and λ is positive and supported by $A = U \setminus (W \cup W')$. Using the similar formulas for u_- and the relation $u = u_+ - u_-$, we see that $\lambda = -\mathbf{1}_A(\mu_+ - \mu_-) + (-[\mathbf{1}_{W'}\mu_+]^{W'^c} + [\mathbf{1}_{W'}\mu_-]^{W'^c})$. Since sweeping-out decreases total masses, (i) easily follows.

Finally, if $u \geqslant 0$ in U, Theorem 5.1 and Lemma 5.2 yield $\mu'_+ - \mu'_- \geqslant 0$ (using the same notations as above). Whence (iii) (which can also be deduced from Theorem 6.1). □

7. An application and extension of Theorem 6.1

We return here to the assumptions of the beginning of Section 6. In particular U is a relatively compact $C^{1,\alpha}$-smooth open subset of M and $u \in W^{1,1}(U)$ satisfies the following conditions: (i) $\mathcal{L}(u)$ is a finite measure in U and (ii) $\partial_n(u)$ is a finite measure in ∂U. Recall (see Remark 6.8) that in $\overline{U}$, u admits a representative which is finely continuous outside a Borel polar subset N of $\overline{U}$, N being moreover negligible with respect to the measure $\lambda := \mathbf{1}_U|\mathcal{L}(u)| + |\partial_n u|$ (a Radon measure in $\overline{U}$). We fix such a representative which will still be denoted u and observe that up to

λ-negligible sets the usual sets $\{u > \theta\}$, $\theta \in \mathbb{R}$, coincide with the precise sets $\{(u - \theta) > 0\}$ as defined in Section 6. Similarly for the sets $\{u = \theta\} = \{|u - \theta| > 0\}^c$.

Let now $f : \mathbb{R} \to \mathbb{R}$ be a continuous function in $\mathbb{R}$ whose second derivative in the sense of distributions is a Radon measure in $\mathbb{R}$ with finite total mass. Thus the right and left derivatives f'_d and f'_g exist everywhere, have finite total variations and $\{t \in \mathbb{R};\ f'_g(t) \neq f'_d(t)\}$ is at most enumerable. Moreover by taking limits $f'_g(\pm\infty)$ and $f'_d(\pm\infty)$ will be considered as well-defined reals: $f'_d(+\infty) = f'_g(+\infty) = \lim_{t<\infty,\, t\to+\infty} f'_d(t) = \lim_{t<\infty,\, t\to+\infty} f'_g(t)$ and similarly for $f'_d(-\infty)$, $f'_g(-\infty)$.

We then have the following extension of Theorem 6.1.

Theorem 7.1. *The function $v = f(u)$ is an element of $W^{1,1}(U)$ and $\mathcal{L}(v)$ is a finite measure in U. Moreover, $\partial_n(v)$ is a finite measure and the following formula holds*:

$$\partial_n v = f'_g(u)\partial_n u - (\partial_n u)_-\big(f'_d(u) - f'_g(u)\big) = f'_d(u)\partial_n u - (\partial_n u)_+\big(f'_d(u) - f'_g(u)\big).$$

Here u is seen as defined and finely continuous (but not necessarily finite) outside a polar λ-negligible set in $\overline{U}$. The expressions in the last two members of the identity above are thus well-defined Radon measures in ∂U.

It is well known that $v \in W^{1,1}(U)$ and that $\nabla v = f'(u)\nabla u$, the gradient ∇u vanishing almost everywhere in $\{u \in A\}$ for any negligible subset A of $\mathbb{R}$. Let ν denote the finite measure such that $f'' = \nu$ in the distribution sense. By assumption $|\nu|(\mathbb{R}) < \infty$ and for $x \geqslant 0$:

$$\begin{aligned} f(x) &= f(0) + \int_0^x f'_d(t)\,dt = f(0) + f'_d(0)x + \int_0^x \Big[\int_{(0,t]} d\nu(\theta)\Big]\,dt \\ &= f(0) + f'_d(0)x + \int_{(0,x]} (x-\theta)\,d\nu(\theta) \\ &= f(0) + f'_d(0)x + \int_{(0,\infty)} (x-\theta)_+\,d\nu(\theta). \end{aligned} \tag{7.1}$$

With the similar formula for $x \leqslant 0$, one gets that for arbitrary $x \in \mathbb{R}$

$$\begin{aligned} f(x) = f(0) &+ f'_d(0)x_+ - f'_g(0)x_- \\ &+ \int_{(0,\infty)} (x-\theta)_+\,d\nu(\theta) + \int_{(-\infty,0)} (x-\theta)_-\,d\nu(\theta). \end{aligned} \tag{7.2}$$

It is then seen that $w := v - f(0) - f'_d(0)u_+ + f'_g(0)u_- \in W^{1,1}(U)$ is the vector integral in $W^{1,1}(U)$ given by the formula

$$w := v - f(0) - f'_d(0)u_+ + f'_g(0)u_- = \int_{\mathbb{R}\setminus\{0\}} u_\theta\,d\nu(\theta)$$

where $u_\theta = (u-\theta)_+$ for $\theta > 0$ and $u_\theta = (u-\theta)_-$ when $\theta < 0$. Note that the vector function $\theta \mapsto u_\theta$ from $\mathbb{R}$ into $W^{1,1}(U)$ is bounded continuous in $\mathbb{R}\setminus\{0\}$ and that the equality of w with the vector integral $\int_{\mathbb{R}\setminus\{0\}} u_\theta \, d\nu(\theta)$ can be checked on testing against functions in $L^\infty_c(U)$. Note also that $x \mapsto \int_{\mathbb{R}\setminus\{0\}} u_\theta(x)\, d\nu(\theta)$ gives directly a finely continuous representative of w in $\overline{U}$ outside a λ-negligible set.

As the measures $\mathcal{L}(u_\theta)$ have uniformly bounded total masses $\|\mathcal{L}(u_\theta\|$, it is easily checked (on using functions $\varphi \in C^1_c(U)$ as test functions) that $\mathcal{L}(w)$ is the measure $\int_{\theta\neq 0} \mathcal{L}(u_\theta)\, d\nu(\theta)$ in U. With the notations of Section 2 and a chosen C^0 metric g in M, one has for $\varphi \in C^1_c(M)$ the equality $\langle A_g \nabla_g w, \nabla_g \varphi\rangle = \int_{\mathbb{R}\setminus\{0\}} \langle A_g(\nabla_g u_\theta), \nabla_g\varphi\rangle\, d\nu(\theta)$ with, on the right-hand side, a vector integral in $L^1(U)$ (by the continuity of $v \mapsto \langle A_g(\nabla_g v), \nabla_g \varphi\rangle$ from $W^{1,1}(U)$ into $L^1(U)$). It then follows that $\partial_n w$ is the measure $\int_{\{\theta\neq 0\}} \partial_n(u_\theta)\, d\nu(\theta)$. So, setting $\lambda' = \partial_n u$, we have

$$\begin{aligned}
\partial_n w &= \int_{(0,\infty)} \{\mathbf{1}_{u>\theta}\lambda' - \mathbf{1}_{u=\theta}\lambda'_-\}\, d\nu(\theta) + \int_{(-\infty,0)} \{-\mathbf{1}_{u<\theta}\,\lambda' - \mathbf{1}_{u=\theta}\lambda'_+\}\, d\nu(\theta) \\
&= \mathbf{1}_{u>0}\big\{\big(f'_g(u) - f'_d(0)\big)\lambda' - \big(f'_d(u) - f'_g(u)\big)\lambda'_-\big\} \\
&\quad + \mathbf{1}_{u<0}\big\{-\big(f'_g(0) - f'_d(u)\big)\lambda' - \big(f'_d(u) - f'_g(u)\big)\lambda'_+\big\}. \qquad (7.3)
\end{aligned}$$

In this way we get that $\partial_n v = \mathbf{1}_{u=0}(-f'_d(0)\lambda'_- + f'_g(0)\lambda'_+) + \mathbf{1}_{u>0}\{f'_g(u)\lambda'_+ - f'_d(u)\lambda'_-\} + \mathbf{1}_{u<0}\{-f'_d(u)\lambda'_- + f'_g(u)\lambda'_+\}$. Finally

$$\partial_n\big(f(u)\big) = f'_g(u)\lambda'_+ - f'_d(u)\lambda'_- = f'_g(u)\partial_n(u) + \big(f'_g(u) - f'_d(u)\big)[\partial_n u]_-$$

and one has also that $\partial_n f(u) = f'_d(u)\partial_n u - (f'_d(u) - f'_g(u))[\partial_n u]_+$.

8. Annex

In this section we provide, for the reader's convenience, proofs of several well-known potential theoretic key facts which have been used above. Let M denote a C^1-manifold and let $\mathcal{L}$ be a standard second order elliptic operator in M. We assume that M is $\mathcal{L}$-admissible (there is a global Green's function G).

A.1. We start with the internal Fatou–Doob property mentioned after statement of Theorem 5.1.

Theorem 8.1. *(Cf. [15, p. 172].) Let p and q be $\mathcal{L}$-potentials in M with associated measures $\mu = -\mathcal{L}(p)$ and $\nu = -\mathcal{L}(q)$. If A is a Borel polar set which is ν-negligible, then* $\operatorname{fine\lim}_a \frac{p}{q} = +\infty$ *for μ-almost all $a \in A$.*

We want to show that for each $C > 0$, the finely closed set $F_C = \{p \leqslant Cq\}$ is thin at μ-almost all $a \in A$. We know that the set of points where F_C is not thin (this set is called the basis of F_C) is an ordinary G_δ. If F_C is unthin at each point of the compact set $K \subset A$, if $\{K_n\}$ is a decreasing sequence of compact neighborhoods of K shrinking to K in M, if $\mu' = \mathbf{1}_K \mu$ and if $p' = G(\mu')$, we have $p' = R_{p'}^{K_n\cap F_C} \leqslant C\hat{R}_q^{K_n\cap F_C} = C\int \hat{R}_{G_y}^{K_n\cap F_C}\, d\nu(y) \leqslant q$. Since for each $y \notin K$, $\hat{R}_{G_y}^{K_n}$ decreases to 0 outside K when $n \to \infty$, we see that $p'(x) = 0$ for all $x \in M\setminus K$ such that $q(x) < \infty$. This means that $p' \equiv 0$ and so $\mu(K) = 0$.

A.2. The finely open set $V := \{u > 0\}$ (Sections 5–7) is an F_σ set in M: for if B_n is the set of points where the set (defined up to a polar set) $\{u \leqslant \frac{1}{n}\}$ is unthin, the complement V^c is the intersection $\bigcap_{n\geqslant 1} B_n$. But B_n is a basis (B_n is equal to the set of points where it is unthin) and so is a G_δ set [6,15]. The fine boundary $\partial_f V = \partial_f\{u > 0\}$ is also a G_δ since $\partial_f V = \overline{\{u > 0\}}^f \setminus \{u > 0\}$ and $\overline{\{u > 0\}}^f$ is the basis set $\{u > 0\}$.

A.3. Next we consider balayage and start with the following simple lemma.

Lemma 8.2. *Let p, q be two nonnegative $\mathcal{L}$-superharmonic functions in the region U of M. If $p \geqslant q$ on the compact subset K of U, then $R_p^{K^c} \geqslant R_q^{K^c}$ in K.*

Proof. It is well known that there exists a strictly increasing sequence $\{q_n\}_{n\geqslant 0}$ of continuous functions in $\mathcal{S}_+(U)$ such that $q = \sup_{n\geqslant 1} q_n$. Then $\{p > q_n\}$ is an open set U_n containing K and taking U' open and such that $K \subset U' \subset \overline{U'} \Subset U_n$ the minimum principle gives that $R_p^{K^c} \geqslant R_{q_n}^{\overline{U'}^c}$ in U'. Letting U' decrease to K one gets $R_p^{K^c} \geqslant R_{q_n}^{K^c} = \hat{R}_{q_n}^{K^c}$ in K. Letting then n go to infinity the desired result follows. □

Theorem 8.3. *(See [15, pp. 183–186], [13].) If $A \subset M$ is thin at x the swept-out measure ε_x^A is concentrated on the fine boundary $\partial_f(A)$ of A.*

Replacing A by its basis, we may assume that A is a basis (in particular A and $\partial_f(A) = A \cap b(A^c)$ are G_δ sets). If $p \in \mathcal{P}(M)$ and $\mu = -\mathcal{L}(p)$ (so $p = G_\mu$), we have (for arbitrary $x \in M$) $R_{G_\mu}^A(x) = \int R_{G_x}^A\, d\mu = \int G_{\varepsilon_x^A}\, d\mu = \int G_\mu\, d\varepsilon_x^A$ and since $R_{G_\mu}^A$ is stable by reduction on A we obtain by replacing p by R_p^A that $\int G_\mu\, d\varepsilon_x^A = \int R_{G_\mu}^A\, d\varepsilon_x^A$. Taking for p a strict potential ([7], [15, p. 180], [16]) it is known that $M \setminus A = \{R_p^A < p\}$ and so we get $\varepsilon_x^A(M \setminus A) = 0$. Thus it have been shown that for any set $B \subset M$ thin at x, ε_x^B is supported by the fine closure of B.

It remains to see that for $x \notin A$, ε_x^A does not charge the fine interior V of a basis A. Set $p = R_{G_x}^A$. By Lemma 8.2 applied to p and G_x we have $p = R_p^{K^c}$ in M for every compact $K \subset A$. This means that ε_x^A is equal to its swept-out in K^c. So by the above ε_x^A does not charge the fine interior of K. This gives the desired result (since $V = \{q > R_q^A = \hat{R}_q^A\}$ if $q \in \mathcal{P}(M)$ is continuous and strict [15]).

Lemma 8.4. *Let p, q be two $\mathcal{L}$-superharmonic functions in the open subset U of M. If $p = q$ on a finely open subset V of U then the measures $\mathcal{L}(p)$ and $\mathcal{L}(q)$ coincide in V.*

Proof. By the assumptions $p = q$ on the finely open set $V' = \{x;\ V^c \text{ is thin at } x\}$ which contains V. Thus one may assume that V^c is a basis (in particular an ordinary G_δ set). Since the properties in the statement are of local nature we may also assume that p and q are $\mathcal{L}$-potentials in U.

By definition of the réduites $\hat{R}_p^V = R_p^V = R_q^V = \hat{R}_q^V$, so $\mathcal{L}(\hat{R}_p^V) = \mathcal{L}(\hat{R}_q^V)$. Since $(\varepsilon_x)^V = \varepsilon_x$ when $x \in V$ we obtain that $(\mathbf{1}_{V^c}.\mathcal{L}(p))^V + \mathbf{1}_V.\mathcal{L}(p) = (\mathbf{1}_{V^c}.\mathcal{L}(q))^V + \mathbf{1}_V.\mathcal{L}(q)$. But for $x \notin V$, the balayée $(\varepsilon_x)^V$ is supported by the fine boundary $\partial_f V$ of V and thus vanishes in V. Restricting to V in the previous relation we get $\mathbf{1}_V.\mathcal{L}(p) = \mathbf{1}_V.\mathcal{L}(q)$. □

A.4. Finally we turn to a property used in the proof of Lemma 6.10 and for which a reference seems difficult to locate.

Proposition 8.5. *Let $A \subset M$ and let L be a compact subset of A. Then $u(x) := \varepsilon_x^A(L)$ is an $\mathcal{L}$-subharmonic function in $M \setminus L$.*

Replacing A by its fine closure we may and will assume that A is finely closed. Let then $\pi := \hat{R}_1^L$ be the equilibrium potential of L in M, and let $\tilde{R}$ denote the réduite operator with respect to $U := M \setminus L$ (i.e. $\tilde{R}_s^B = \inf\{w \in \mathcal{S}_+(U);\ w \geqslant s \text{ in } B\}$). Then,

$$u(x) = \pi(x) - \tilde{R}_\pi^{A\cap U}(x), \quad x \in U, \tag{8.1}$$

which implies the result since $\tilde{R}_\pi^{A\cap U}$ is $\mathcal{L}$-superharmonic in U and π is $\mathcal{L}$-harmonic in U. To prove (8.1) we remark another formula about the reduites: if $s \in \mathcal{P}_c(M)$,

$$R_s^A = \hat{R}_s^L + \tilde{R}_{s-\hat{R}_s^L}^{A\cap U}, \quad x \in U, \tag{8.2}$$

which follows at once from Lemma 8.6 below. Now, (8.2) means that

$$\int s\, d\varepsilon_x^A = \int s\, d\varepsilon_x^L + \int s\, d\tilde{\varepsilon}_x^{A\cap U} - \int_U \left(\int s d\varepsilon_y^L \right) d\tilde{\varepsilon}_x^{A\cap U}(y), \quad x \in U, \tag{8.3}$$

which then also holds for $s \in \mathcal{P}_c(M) - \mathcal{P}_c(M)$. Since $\mathbf{1}_L$ is the limit of a decreasing sequence of such s_j (with supports shrinking to L), we get letting $j \to \infty$

$$\varepsilon_x^A(L) = \pi(x) - \int \pi\, d\tilde{\varepsilon}_x^{A\cap U}, \quad x \in U,$$

which is exactly (8.1).

Lemma 8.6. *Let $p \in \mathcal{P}(M)$ then $p - \hat{R}_p^L \in \mathcal{P}(U)$. If moreover $\mathcal{L}(p)$ is concentrated in the Borel set B and p is locally bounded, then $p - \hat{R}_p^L = \tilde{R}_{p-\hat{R}_p^L}^{B\cap U}$.*

Let h be nonnegative $\mathcal{L}$-harmonic in U and such that $h \leqslant p - \hat{R}_p^L$. Note again h its extension by zero outside U. Choosing $w \in \mathcal{S}_+(M)$ such that $w(x) = +\infty$ for all $x \in L$ where L is thin, clearly that $(h - \varepsilon w)_+$ is subharmonic in M and less than p (for every $\varepsilon > 0$). So $(h - \varepsilon w)_+ = 0$ and letting $\varepsilon \to 0$, $h = 0$ in U. This proves that $p - \hat{R}_p^L \in \mathcal{P}(U)$. The second claim follows then by the domination principle [6,15,13].

Appendix A. Solution of a conjecture by J. Serrin

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A1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, be a bounded domain and let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a weak solution of the equation

$$\sum_{i,j} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) = 0 \quad \text{in } \Omega, \tag{A1.1}$$

where the coefficients $a_{ij}(x)$ are bounded measurable and elliptic, i.e.,

$$\lambda|\xi|^2 \leqslant \sum_{i,j} a_{ij}(x)\xi_i\xi_j \leqslant \Lambda|\xi|^2, \quad x \in \Omega,\ \xi \in \mathbb{R}^N,$$

with $0 < \lambda \leqslant \Lambda < \infty$. A weak solution $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ satisfies, by definition,

$$\sum_{i,j} \int a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_j} = 0 \quad \forall \varphi \in C^1_c(\Omega), \tag{A1.2}$$

where the subscript c indicates compact support.

A celebrated result of E. DeGiorgi [14] asserts that if u is a weak solution of (A1.1) and moreover $u \in H^1_{\mathrm{loc}}(\Omega)$, then u is locally Hölder continuous, and in particular $u \in L^\infty_{\mathrm{loc}}(\Omega)$ (see also [35]). Subsequently J. Serrin produced in [32] a striking example showing that the assumption $u \in H^1_{\mathrm{loc}}(\Omega)$ is essential; more precisely, for every p, $1 < p < 2$, and all $N \geqslant 2$, he constructed an equation of the form (A1.1) which has a solution $u \in W^{1,p}(\Omega)$ and $u \notin L^\infty_{\mathrm{loc}}(\Omega)$. J. Serrin conjectured in [32] that if the coefficients a_{ij} are locally Hölder continuous, then any weak solution $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ of (A1.1) must be a "classical" solution, i.e., $u \in H^1_{\mathrm{loc}}(\Omega)$. Serrin's conjecture was established by R.A. Hager and J. Ross [20] provided u is a weak solution of class $W^{1,p}(\Omega)$ for some p with $1 < p < 2$.

We present here the solution of Serrin's conjecture in full generality, starting with $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, or even with $u \in BV_{\mathrm{loc}}(\Omega)$, i.e., $u \in L^1_{\mathrm{loc}}(\Omega)$ and its derivatives (in the sense of distributions) are measures.

The first result is an improvement of the theorem of Hager and Ross: instead of $a_{ij} \in C^{0,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$, we assume only $a_{ij} \in C^0(\bar{\Omega})$.

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Theorem A1.1. *Assume $a_{ij} \in C^0(\bar{\Omega})$ and $u \in W^{1,p}(\Omega)$ for some $p > 1$. If u is a weak solution of* (A1.1)*, then $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ for every $q < \infty$. Moreover*

$$\|u\|_{W^{1,q}(\omega)} \leqslant C\|u\|_{W^{1,p}(\Omega)},$$

for every $\omega \Subset \Omega$, where C depends only on N, λ, Λ, p, q, ω, Ω, and the modulus of continuity of a_{ij} on $\bar{\Omega}$.

Open problem 0. We do not know whether the conclusion of Theorem A1.1 holds in the two limiting cases: $p = 1$ and/or $q = \infty$. (The answer to both questions is positive if the coefficients a_{ij} are Dini continuous; see Theorem A1.2 below).

We now turn to Serrin's conjecture. Here we assume that the coefficients a_{ij} are Dini continuous in $\bar{\Omega}$, i.e., $a_{ij} \in C^0(\bar{\Omega})$, and

$$A(r) = \sum_{i,j} \sup_{x,y\in\Omega, |x-y|<r} \left|a_{ij}(x) - a_{ij}(y)\right|, \quad r > 0, \tag{A1.3}$$

satisfies

$$\int_0^1 \frac{A(r)}{r} dr < \infty. \tag{A1.4}$$

Theorem A1.2. *Assume that the coefficients a_{ij} are Dini continuous in $\bar{\Omega}$, and let $u \in BV(\Omega)$ be a weak solution of* (A1.1)*, then $u \in H^1_{\mathrm{loc}}(\Omega)$. Moreover*

$$\|u\|_{H^1(\omega)} \leqslant C\|u\|_{BV(\Omega)}, \tag{A1.5}$$

for every $\omega \Subset \Omega$, where C depends only on $N, \lambda, \Lambda, \omega, \Omega$, and the modulus of continuity of a_{ij} on $\bar{\Omega}$.

Remark 1. Surprisingly, the constant C in (A1.5) depends only on the modulus of continuity of a_{ij} in $\bar{\Omega}$, and *not* on the Dini modulus of continuity of a_{ij} in $\bar{\Omega}$. This suggests that Serrin's conjecture might be true assuming only the continuity of a_{ij} in $\bar{\Omega}$ (see Open problem 0 with $p = 1$).

Remark 2. Using Lemma A3.1 below we may assert that, under the assumptions of Theorem A1.2, $u \in C^1(\Omega)$. If the coefficients a_{ij} belong to $C^{0,\alpha}(\bar{\Omega})$, $0 < \alpha < 1$, one can further improve the conclusion of Theorem A1.2, namely $u \in C^{1,\alpha}(\bar{\omega})$ for every $\omega \Subset \Omega$. This is a consequence of the standard Schauder regularity theory for elliptic equations in divergence form with $C^{0,\alpha}$ coefficients (see e.g. [30, Theorem 5.5.3(b)], [19, Theorem 3.7], [18, Theorem 3.5], or [11, Theorem 2.6 in Chapter 9]). All the above results extend to elliptic systems.

Theorems A1.1 and A1.2 have been announced in [8].

A2. Proof of Theorem A1.1

We use a duality argument in conjunction with the following standard L^p-regularity property for elliptic equations in divergence form:

Lemma A2.1. *(See e.g. [30, Theorem 5.5.3(a)], or [11, Theorem 2.2 in Chapter 10].) Assume $a_{ij} \in C^0(\bar{\Omega})$ and $u \in H^1(\Omega)$ is a weak solution of*

$$\sum_{i,j} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) = \sum_j \frac{\partial}{\partial x_j} f_j \quad \text{in } \Omega, \tag{A2.1}$$

with $f_j \in L^r(\Omega)$ $\forall j$, and $r \in [2,\infty)$, then $u \in W^{1,r}_{\mathrm{loc}}(\Omega)$, and for $\omega \Subset \Omega$,

$$\|u\|_{W^{1,r}(\omega)} \leqslant C\left(\|u\|_{H^1(\Omega)} + \sum_j \|f_j\|_{L^r(\Omega)}\right)$$

where C depends on $N, \lambda, \Lambda, r, \omega, \Omega$, and the modulus of continuity of a_{ij}.

Proof of Theorem A1.1. We may always assume that Ω is a ball and that

$$1 < p < 2 < q. \tag{A2.2}$$

(When $p \geqslant 2$ we may apply Lemma A2.1 with $r = q$.) Let (f_j), $j = 1, 2, \ldots, N$, be given in $C_c^\infty(\Omega)$ with

$$\sum_j \|f_j\|_{L^{s'}(\Omega)} \leqslant 1 \tag{A2.3}$$

where

$$\frac{1}{s} + \frac{1}{s'} = 1, \qquad \frac{N}{N-1} < s \leqslant 2, \tag{A2.4}$$

and s will be chosen later.

Let $v \in H_0^1(\Omega)$ be the solution of

$$\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial v}{\partial x_j}\right) = \sum_j \frac{\partial}{\partial x_j} f_j \quad \text{in } \Omega. \tag{A2.5}$$

Clearly

$$\|v\|_{H^1(\Omega)} \leqslant C \sum_j \|f_j\|_{L^2(\Omega)} \leqslant C \tag{A2.6}$$

by (A2.3) and (A2.4). Moreover, by Lemma A2.1,

$$\|v\|_{W^{1,s'}(\omega)} \leqslant C, \tag{A2.7}$$

and $v \in W^{1,r}_{\mathrm{loc}}(\Omega)$ $\forall r < \infty$ (since $f_j \in C_c^\infty(\Omega)$).

From (A2.5) we have

$$\sum_{i,j}\int_{\Omega} a_{ij}\frac{\partial\varphi}{\partial x_i}\frac{\partial v}{\partial x_j}=\sum_{j}\int_{\Omega} f_j\frac{\partial\varphi}{\partial x_j}\quad \forall\varphi\in C_c^1(\Omega), \tag{A2.8}$$

and by density we see that (A2.8) also holds whenever $\varphi\in W_c^{1,t}(\Omega)$, for some $t>1$ (since $v\in W_{\mathrm{loc}}^{1,r}(\Omega)$, $\forall r<\infty$), but the value $t=1$ is not admissible since we do not know whether $v\in W_{\mathrm{loc}}^{1,\infty}(\Omega)$. Fix $\zeta\in C_c^\infty(\Omega)$ with $\zeta=1$ on ω. We may choose $\varphi=\zeta u$ in (A2.8) (here we use the assumption $u\in W^{1,p}(\Omega)$ and $p>1$). This yields

$$\sum_{i,j}\int_{\Omega} a_{ij}\left(\zeta\frac{\partial u}{\partial x_i}+u\frac{\partial\zeta}{\partial x_i}\right)\frac{\partial v}{\partial x_j}=\sum_{j}\int_{\Omega} f_j\left(\zeta\frac{\partial u}{\partial x_j}+u\frac{\partial\zeta}{\partial x_j}\right). \tag{A2.9}$$

On the other hand, by (A1.2), and a density argument we have

$$\sum_{i,j}\int_{\Omega} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial w}{\partial x_j}=0\quad \forall w\in W_c^{1,p'}(\Omega). \tag{A2.10}$$

Next we choose $w=\zeta v$ in (A2.10) (this w is admissible since $v\in W_{\mathrm{loc}}^{1,r}(\Omega)$ $\forall r<\infty$ and $p'<\infty$; here we use once more the assumption $p>1$). We obtain

$$\sum_{i,j}\int_{\Omega} a_{ij}\frac{\partial u}{\partial x_i}\left(\zeta\frac{\partial v}{\partial x_j}+v\frac{\partial\zeta}{\partial x_j}\right)=0. \tag{A2.11}$$

Comparing (A2.9) and (A2.11) we find

$$\begin{aligned}\sum_{j}\int_{\Omega}\zeta\frac{\partial u}{\partial x_j}f_j&=-\sum_{i,j}\int_{\Omega} a_{ij}\frac{\partial u}{\partial x_i}v\frac{\partial\zeta}{\partial x_j}+\sum_{i,j}\int_{\Omega} a_{ij}u\frac{\partial\zeta}{\partial x_i}\frac{\partial v}{\partial x_j}-\sum_{j}\int_{\Omega} f_j u\frac{\partial\zeta}{\partial x_j}\\&=I+II+III.\end{aligned} \tag{A2.12}$$

Recall that $p<2\leqslant N$ and, by the Sobolev embedding,

$$\|u\|_{L^{p^\star}(\Omega)}\leqslant C\|u\|_{W^{1,p}(\Omega)}, \tag{A2.13}$$

where

$$\frac{1}{p^\star}=\frac{1}{p}-\frac{1}{N}. \tag{A2.14}$$

Finally we choose $s\in(\frac{N}{N-1},2]$ according to the following dichotomy:

(a) *When $p^\star\leqslant 2$ we choose $s=p^\star$.*
Note that $p^\star>\frac{N}{N-1}$ because $p>1$. Then we have, since $s=p^\star$,

$$\frac{1}{p'}=1-\frac{1}{p}=\frac{1}{p^\star}+\frac{1}{s'}-\frac{1}{p}=\frac{1}{s'}-\frac{1}{N}, \tag{A2.15}$$

and by the Sobolev embedding we have

$$\|v\|_{L^{p'}(\omega)} \leqslant C\|v\|_{W^{1,s'}(\omega)}, \tag{A2.16}$$

which is valid since $s' = \frac{p^\star}{p^\star - 1} < N$ (because $p > 1$). Therefore, from (A2.7),

$$\|v\|_{L^{p'}(\omega)} \leqslant C \tag{A2.17}$$

and thus (with $\omega \supset \operatorname{Supp}\zeta$),

$$|I| \leqslant C\|u\|_{W^{1,p}(\Omega)}, \tag{A2.18}$$

where I is defined in (A2.12).

Next we have, by (A2.13) (with $s = p^\star$),

$$\begin{aligned} |II| &\leqslant C\|u\|_{W^{1,p}(\Omega)}\|v\|_{W^{1,s'}(\omega)} \\ &\leqslant C\|u\|_{W^{1,p}(\Omega)} \quad \text{by (A2.7).} \end{aligned} \tag{A2.19}$$

Finally

$$\begin{aligned} |III| &\leqslant C\sum_j \|f_j\|_{L^{s'}(\Omega)}\|u\|_{L^s(\Omega)} \\ &\leqslant C\|u\|_{W^{1,p}(\Omega)} \end{aligned} \tag{A2.20}$$

by (A2.13) and the choice $s = p^\star$.

Combining (A2.12), (A2.18), (A2.19) and (A2.20) yields

$$\left|\sum_j \int_\Omega \zeta \frac{\partial u}{\partial x_j} f_j\right| \leqslant C\|u\|_{W^{1,p}(\Omega)},$$

for every (f_j) in $C_c^\infty(\Omega)$ satisfying (A2.3) (where the constant C depends on ζ).

Therefore $\zeta \frac{\partial u}{\partial x_j} \in L^s(\Omega)$ and

$$\left\|\zeta \frac{\partial u}{\partial x_j}\right\|_{L^s(\Omega)} \leqslant C\|u\|_{W^{1,p}} \quad \forall j.$$

In particular, $u \in W^{1,p^\star}(\omega)$ with

$$\|u\|_{W^{1,p^\star}(\omega)} \leqslant C\|u\|_{W^{1,p}(\Omega)}. \tag{A2.21}$$

(b) *When* $p^\star > 2$ *we choose* $s = 2$.

Then

$$\|v\|_{H^1(\Omega)} \leqslant C \quad \text{by (A2.6)} \tag{A2.22}$$

and thus

$$\|v\|_{L^{p'}(\Omega)} \leqslant C \tag{A2.23}$$

since $p' < 2^\star$ when $N \geqslant 3$ (this is equivalent to $p^\star > 2$) and $p' < \infty$ when $N = 2$ (here we use once more the assumption $p > 1$).

From (A2.22) and (A2.23) we deduce that

$$|I| + |II| + |III| \leqslant C\|u\|_{W^{1,p}(\Omega)},$$

because

$$\|u\|_{L^2(\Omega)} \leqslant C\|u\|_{L^{p^\star}(\Omega)} \leqslant C\|u\|_{W^{1,p}(\Omega)}.$$

We now conclude as above that $u \in H^1(\omega)$ with

$$\|u\|_{H^1(\omega)} \leqslant C\|u\|_{W^{1,p}(\Omega)}. \tag{A2.24}$$

Iterating the preceding argument of case (a) in the dichotomy yields $u \in W^{1,p^\star}_{\mathrm{loc}}(\Omega)$, $u \in W^{1,p^{\star\star}}_{\mathrm{loc}}(\Omega)$, etc. until we reach the first value bigger than 2. At that point we use part (b) of the dichotomy.

Thus we have proved that any $u \in W^{1,p}(\Omega)$ with $1 < p < 2$ satisfying (A1.1), must belong to $H^1_{\mathrm{loc}}(\Omega)$ and

$$\|u\|_{H^1(\omega)} \leqslant C\|u\|_{W^{1,p}(\Omega)}.$$

Applying once more Lemma A2.1 with $f_j = 0\ \forall j$, gives $u \in W^{1,q}_{\mathrm{loc}}(\Omega)\ \forall q < \infty$ and

$$\|u\|_{W^{1,q}(\omega)} \leqslant C\|u\|_{W^{1,p}(\Omega)},$$

and the proof of the theorem is complete. □

A3. Proof of Theorem A1.2

For the proof of Theorem A1.2 we will need the following extension of the Schauder regularity theory for elliptic equations in divergence form with Dini continuous coefficients:

Lemma A3.1. *Assume $a_{ij} \in C^0(\bar{\Omega})$ satisfy* (A1.3)–(A1.4) *and let $u \in H^1(\Omega)$ be a weak solution of* (A2.1) *with $f_j \in C^\infty_c(\Omega)\ \forall j$, then $u \in C^1(\Omega)$.*

The conclusion of Lemma A3.1 comes with an estimate of the Dini modulus of continuity of Du involving the Dini modulus of continuity of a_{ij}. However we do *not* need such an estimate—we use only the qualitative form of Lemma A3.1; this explains Remark 1. It is not easy to find an early reference for Lemma A3.1. According to the experts (I am quoting M. Giaquinta), it was common knowledge in Pisa in the late 60s—the proof being based on Campanato's approach to Schauder estimates (as presented in [19], or [11]), combined with a result of S. Spanne (Corollary 1 in [33]). A complete proof may be found e.g. in [28, Theorem 5.1]. Y. Li [27] has

obtained a similar conclusion (also valid for systems) under weaker assumptions on the coefficients a_{ij}.

Proof of Theorem A1.2. We follow the same duality strategy as in the proof of Theorem A1.1. We start with a weak solution $u \in BV(\Omega)$ of (A1.1). We fix some $1 < s < \frac{N}{N-1}$, so that $2 \leqslant N < s' < \infty$. Let (f_j) in $C_c^\infty(\Omega)$ with

$$\sum_j \|f_j\|_{L^{s'}(\Omega)} \leqslant 1. \tag{A3.1}$$

Let $v \in H_0^1(\Omega)$ be the solution of (A2.5). By Lemma A3.1 we know that $v \in C^1(\Omega)$.

Clearly

$$\|v\|_{H^1(\Omega)} \leqslant C \sum_j \|f_j\|_{L^2(\Omega)} \leqslant C \quad \text{since } s' > 2, \tag{A3.2}$$

and by Lemma A2.1 (which uses only continuous coefficients a_{ij}) we have

$$\|v\|_{W^{1,s'}(\omega)} \leqslant C. \tag{A3.3}$$

On the other hand we know from the DeGiorgi–Stampacchia theory (which uses only bounded measurable coefficients a_{ij}) that $v \in L^\infty(\Omega)$ since $s' > N$; see e.g. [35] and [21]. Moreover

$$\|v\|_{L^\infty(\Omega)} \leqslant C \sum_j \|f_j\|_{L^{s'}(\Omega)} \leqslant C. \tag{A3.4}$$

By density we have

$$\sum_{i,j} \int_\Omega a_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial v}{\partial x_j} = \sum_j \int_\Omega f_j \frac{\partial \varphi}{\partial x_j} \quad \forall \varphi \in BV_c(\Omega), \tag{A3.5}$$

and we choose $\varphi = \zeta u$ in (A3.5) with ζ as in the proof of Theorem A1.1. This gives (A2.9).

On the other hand we may choose $\varphi = \zeta v$ in (A1.2). This gives (A2.11), which yields (A2.12).

Next we have

$$\begin{aligned}
|I| &\leqslant C\|u\|_{BV(\Omega)} \|v\|_{L^\infty(\Omega)} \leqslant C\|u\|_{BV(\Omega)} \quad \text{by (A3.3)},\\
|II| &\leqslant C\|u\|_{L^{N/N-1}(\Omega)} \|v\|_{W^{1,N}(\omega)}\\
&\leqslant C\|u\|_{L^{(N/N-1)}(\Omega)} \leqslant C\|u\|_{BV(\Omega)} \quad \text{by (A3.3) since } s' > N,\\
|III| &\leqslant C \sum_j \|u\|_{L^{N/N-1}(\Omega)} \|f_j\|_{L^N(\Omega)} \leqslant C\|u\|_{BV(\Omega)} \quad \text{by (A3.1) since } s' > N.
\end{aligned}$$

We conclude that

$$\left| \sum_j \int \zeta \frac{\partial u}{\partial x_j} f_j \right| \leqslant C\|u\|_{BV(\Omega)}$$

for every $f = (f_j)$ in $C_c^\infty(\Omega)$ satisfying (A3.1). Therefore $\zeta \frac{\partial u}{\partial x_j} \in L^s(\Omega)$ and, in particular, $u \in W^{1,s}(\omega)$ with

$$\|u\|_{W^{1,s}(\omega)} \leqslant C\|u\|_{BV(\Omega)}.$$

We may now apply Theorem A1.1 and conclude that $u \in H^1_{\mathrm{loc}}(\Omega)$ with

$$\|u\|_{H^1(\omega)} \leqslant C\|u\|_{BV(\Omega)}, \tag{A3.6}$$

where C depends only $N, \lambda, \Lambda, \omega, \Omega$ and the modulus of continuity of a_{ij} on $\bar{\Omega}$ (and *not* the Dini modulus of continuity of a_{ij}).

A4. More on the Open problem 0

As we already mentioned briefly in the Introduction there are several open problems related to Serrin's conjecture.

Open problem 1. Assume $a_{ij} \in C^0(\bar{\Omega})$ and $u \in W^{1,1}(\Omega)$ is a weak solution of (A1.1). Is it true that $u \in H^1_{\mathrm{loc}}(\Omega)$? Does one have an estimate of the form

$$\|u\|_{H^1(\omega)} \leqslant C\|u\|_{W^{1,1}(\Omega)}, \tag{A4.1}$$

for every $\omega \Subset \Omega$ where C depends only on $N, \lambda, \Lambda, \omega, \Omega$ and the modulus of continuity of a_{ij} on $\bar{\Omega}$? Same questions when $u \in BV(\Omega)$ instead of $W^{1,1}$.

Open problem 2. Assume $a_{ij} \in C^0(\bar{\Omega})$ and $u \in H^1(\Omega)$ is a weak solution of (A1.1). Is it true that $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ (resp. $u \in C^1(\Omega)$)? Does one have an estimate of the form

$$\|u\|_{W^{1,\infty}(\omega)} \leqslant C\|u\|_{H^1(\Omega)}, \tag{A4.2}$$

for every $\omega \Subset \Omega$ where C depends on $N, \lambda, \Lambda, \omega, \Omega$ and the modulus of continuity of a_{ij} on $\bar{\Omega}$?

Note that estimate (A1.5) in Theorem A1.2 gives some evidence in favor of a positive answer to Open problem 1. (We do not have any evidence in favor of a positive answer to Open problem 2.) A natural strategy in trying to solve Open problem 1 is to smooth the coefficients a_{ij} by a^ε_{ij} preserving the ellipticity and the modulus of continuity. Following the proof of Theorem A1.1 we have

$$\sum_{i,j} \int_\Omega a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial w}{\partial x_j} = 0 \quad \forall w \in C^1_c(\Omega). \tag{A4.3}$$

Fix some s such that $1 < s < \frac{N}{N-1}$, so that $2 \leqslant N < s' < \infty$. Let (f_j) in $C_c^\infty(\Omega)$ with

$$\sum_j \|f_j\|_{L^{s'}(\Omega)} \leqslant 1. \tag{A4.4}$$

Let $v^\varepsilon \in H_0^1(\Omega)$ be the solution of

$$\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}^\varepsilon \frac{\partial v^\varepsilon}{\partial x_j}\right) = \sum_j \frac{\partial}{\partial x_j} f_j. \tag{A4.5}$$

By Lemma A3.1 we know that $v^\varepsilon \in C^1(\Omega)$.

Clearly

$$\|v^\varepsilon\|_{H^1(\Omega)} \leqslant C. \tag{A4.6}$$

On the other hand

$$\|v^\varepsilon\|_{L^\infty(\Omega)} \leqslant C, \tag{A4.7}$$

and by Lemma A2.1

$$\|v^\varepsilon\|_{W^{1,s'}(\Omega)} \leqslant C \tag{A4.8}$$

since we have a uniform modulus of continuity for a_{ij}^ε.

Inserting $w = \zeta v^\varepsilon$ in (A4.3) yields

$$\sum_{i,j} \int_\Omega a_{ij} \frac{\partial u}{\partial x_i}\left(\zeta \frac{\partial v^\varepsilon}{\partial x_j} + v^\varepsilon \frac{\partial \zeta}{\partial x_j}\right) = 0. \tag{A4.9}$$

From (A4.5) we have

$$\sum_{i,j} \int_\Omega a_{ij}^\varepsilon \frac{\partial v^\varepsilon}{\partial x_j}\left(\zeta \frac{\partial u}{\partial x_i} + u \frac{\partial \zeta}{\partial x_i}\right) = \sum_j \int_\Omega f_j \left(\zeta \frac{\partial u}{\partial x_j} + u \frac{\partial \zeta}{\partial x_j}\right). \tag{A4.10}$$

Comparing (A4.9) and (A4.10) we obtain

$$\begin{aligned}
\sum_j \int_\Omega \zeta \frac{\partial u}{\partial x_j} f_j &= -\sum_{i,j} \int_\Omega a_{ij} \frac{\partial u}{\partial x_i} v^\varepsilon \frac{\partial \zeta}{\partial x_j} + \sum_{i,j} \int a_{ij}^\varepsilon u \frac{\partial \zeta}{\partial x_i} \frac{\partial v^\varepsilon}{\partial x_j} \\
&\quad - \sum_j \int_\Omega f_j u \frac{\partial \zeta}{\partial x_j} + \sum_{i,j} \int_\Omega (a_{ij}^\varepsilon - a_{ij}) \zeta \frac{\partial v^\varepsilon}{\partial x_j} \frac{\partial u}{\partial x_i} \\
&= I + II + III + IV.
\end{aligned} \tag{A4.11}$$

Following the proof in Section A.3 we see that

$$|I| + |II| + |III| \leqslant C\|u\|_{W^{1,1}} \tag{A4.12}$$

with C independent of ε.

The only natural estimate of $|IV|$ is

$$|IV| \leqslant \left\|a_{ij}^\varepsilon - a_{ij}\right\|_{L^\infty(\Omega)} \|u\|_{W^{1,1}(\Omega)} \left\|v^\varepsilon\right\|_{W^{1,\infty}(\omega)}.$$

If we knew that

$$\left\| v^{\varepsilon} \right\|_{W^{1,\infty}(\omega)} \leqslant C \tag{A4.13}$$

with a constant C independent of ε (but depending on the norm of $\|f_j\|_{C^k}$, k large) we would be able to pass to the limit as $\varepsilon \to 0$, then deduce that $u \in W^{1,s}(\omega)$ and proceed as in Section 3. Unfortunately the bound (A4.13) seems out of reach and closely related to Open problem 2.

A5. Questions of uniqueness for weak solutions

All the questions discussed above are naturally linked to the problem of uniqueness of weak solutions.

Assume first that the coefficients a_{ij} are only bounded and measurable. Let $u \in H_0^1(\Omega)$ satisfy (on a smooth domain Ω)

$$\sum_{i,j} \int_{\Omega} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} = 0 \quad \forall \varphi \in C_c^1(\Omega), \tag{A5.1}$$

then, clearly $u = 0$.

The same conclusion *need not be true* if we assume only $u \in W_0^{1,p}(\Omega)$ for some $p < 2$. This fact is closely related to Serrin's phenomenon. Indeed consider e.g. in $\mathbb{R}^2$ the function

$$U = x_1 r^{-1-\varepsilon}$$

constructed in [32]. Then U satisfies (A5.1) in $\Omega = B_1$ with $a_{ij} = \delta_{ij} + (a-1)\frac{x_i x_j}{r^2}$, $a = 1/\varepsilon^2$ and $U \in W^{1,p}(\Omega)$ $\forall p < 2/(1+\varepsilon)$ and $U \notin H_0^1(\Omega)$. Let $V = \zeta U$ where $\zeta \in C_c^\infty(B_1)$ with $\zeta = 1$ near 0. Clearly we have

$$\sum_{i,j} \int_{\Omega} a_{ij} \frac{\partial V}{\partial x_i} \frac{\partial \varphi}{\partial x_j} = \int_{\Omega} F\varphi \quad \forall \varphi \in C_c^1(\Omega), \tag{A5.2}$$

where $F = -\sum_{i,j} a_{ij} \frac{\partial U}{\partial x_i} \frac{\partial \zeta}{\partial x_j} - \sum_{i,j} \frac{\partial}{\partial x_j}(a_{ij} U \frac{\partial \zeta}{\partial x_i})$. Note that $F \in C^\infty(\bar{\Omega})$ (since $\zeta = 1$ near 0).

Let $\widetilde{V} \in H_0^1(\Omega)$ be the solution of

$$\sum_{i,j} \int_{\Omega} a_{ij} \frac{\partial \tilde{V}}{\partial x_i} \frac{\partial \varphi}{\partial x_j} = \int_{\Omega} F\varphi \quad \forall \varphi \in H_0^1(\Omega). \tag{A5.3}$$

Then $u = V - \widetilde{V} \in W_0^{1,p}(\Omega)$ $\forall p < 2/(1+\varepsilon)$ and satisfies (A5.1) but $u \not\equiv 0$ since $V \notin H_0^1(\Omega)$.

However we have:

Theorem A5.1. *Assume $a_{ij} \in C^0(\bar{\Omega})$ and $u \in W_0^{1,p}(\Omega)$ for some $p > 1$. If u is a weak solution of* (A5.1)*, then $u \equiv 0$. The same conclusion holds if $u \in W_0^{1,1}(\Omega)$ provided the coefficients a_{ij} are Dini continuous on $\bar{\Omega}$.*

Here is a question reminiscent of Open problem 1:

Open problem 3. Assume $a_{ij} \in C^0(\bar{\Omega})$ and let $u \in W_0^{1,1}(\Omega)$ satisfy (A5.1). Is it true that $u \equiv 0$?

Theorem A5.1 is an immediate consequence of the following:

Lemma A5.2. *(See e.g. [30, Theorem 5.5.5'], [28, Theorem 5.1], or [27].) Assume $a_{ij} \in C^0(\bar{\Omega})$ and let $v \in H_0^1(\Omega)$ be the weak solution of*

$$\sum_{i,j} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial v}{\partial x_i}\right) = F \quad \text{in } \Omega, \tag{A5.4}$$

with $F \in C_c^\infty(\Omega)$, then $v \in W^{1,r}(\Omega)$, for every $r < \infty$. If the coefficients a_{ij} are Dini continuous on $\bar{\Omega}$, then $v \in C^1(\bar{\Omega})$.

Added in proofs. The answers to Open problems 1, 2 and 3 are negative. Interesting examples have been constructed by T. Jin, V. Maz'ya and J. Van Schaftingen (paper in preparation).

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