ELLIPTIC EQUATIONS WITH CRITICAL EXPONENT ON SPHERICAL CAPS OF $\mathbf{S}^3$

By

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1 Introduction

We start with a problem on a geodesic ball B' centered at the North pole in $\mathbf{S}^3$:

$$\begin{cases} -\Delta_{\mathbf{S}^3} U = U^5 + \lambda U & \text{in } B', \quad (1.1a) \\ U > 0 & \text{in } B', \quad (1.1b) \\ U = 0 & \text{on } \partial B', \quad (1.1c) \end{cases}$$

where $\Delta_{\mathbf{S}^3}$ is the Laplace–Beltrami operator on B'. Let $\theta^* \in (0, \pi)$ be the radius of B', i.e., the geodesic distance of the North pole to $\partial B'$. The values $0 < \theta^* < \pi/2$ correspond to a spherical cap contained in the Northern hemisphere, $\theta^* = \pi/2$ corresponds to B' being the Northern hemisphere and the values $\pi/2 < \theta^* < \pi$ correspond to a spherical cap which covers the Northern hemisphere. Finally, $\theta^* = \pi$ corresponds to $B' = \mathbf{S}^3 \setminus \{\text{South pole}\}$.

Our main focus is to identify the range of values of the parameters θ^* and λ for which there exists a solution of Problem (1.1). Recall that a similar problem in $\mathbf{R}^3$ has been investigated in [6]:

$$\begin{cases} -\Delta_{\mathbf{R}^3} U = U^5 + \lambda U, \qquad U > 0 & \text{in } B_{R^*} \subset \mathbf{R}^3, \quad (1.2a) \\ U = 0 & \text{on } \partial B_{R^*}. \quad (1.2b) \end{cases}$$

The result established in [6] is the following:

Theorem BN. *Problem (1.2) has a solution if and only if*

$$\pi^2/(4(R^*)^2) < \lambda < \pi^2/((R^*)^2).$$

Moreover, the solution (after scaling) is a minimizer for

$$\operatorname{Inf}\Big\{ \int_{B_{R^*}} |\nabla u|^2\, dx - \lambda \int_{B_{R^*}} u^2\, dx \Big\},$$

subject to the constraint that $u \in H_0^1(B_{R^*})$ and

$$\int_{B_{R^*}} u^6\,dx = 1.$$

This solution was shown to be unique [15].

By contrast, we shall see that on the sphere S^3 the situation is quite different, especially for negative values of λ. There are three cases to be distinguished:

Case A: $3 + 4\lambda > 0$;

Case B: $3 + 4\lambda = 0$;

Case C: $3 + 4\lambda < 0$.

The special value $\lambda = -3/4$ is related to the fact that the conformal Laplacian on S^3 is $\Delta_{\mathbf{R}^3} - (3/4)$.

Case A has been extensively studied by Bandle, Benguria and Peletier (cf. [3] and [4]). The main result for this case resembles that of Theorem BN.

Theorem 1.1 ([3], [4]). *When* $3 + 4\lambda > 0$, *there exists a solution of Problem (1.1) if and only if*

(1.3) $$\lambda_1^* = (\pi^2 - 4(\theta^*)^2)/(4(\theta^*)^2) < \lambda < \lambda_1(-\Delta_{S^3}) = (\pi^2 - (\theta^*)^2)/(\theta^*)^2.$$

In fact, the solution can be obtained, after scaling, by minimization of the functional

(1.4) $$E(U) = \int_{B'} \left\{|\nabla U|^2 - \lambda U^2\right\}\,dy,$$

subject to the constraint

(1.5) $$\int_{B'} U^6\,dy = 1.$$

Remark 1.1. The range $3 + 4\lambda > 0$ is the one for which one can say that any solution must be radial by [12] applied to (2.3a-b). The solution is unique by [15].

In Case B, Problem (1.1) has no solutions for any θ^*. Here, equation (1.1a) can be reduced to $-\Delta_{\mathbf{R}^3} v = v^5$ in a ball of radius $R = \tan(\theta^*/2)$ in $\mathbf{R}^3$ (see Section 2); and the classical Pohozaev identity implies that this equation has no nontrivial solution on any ball.

Case C: $3 + 4\lambda < 0$.

This case is extremely interesting because the minimum of $E(U)$, defined in (1.4), subject to (1.5), is **never** achieved. As we shall see, non-minimizing solutions do exist for some range of values of the parameters θ^* and λ.

First we give a nonexistence result.

Theorem 1.2. *There are no solutions of Problem (1.1) if* $\theta^* \in (0, \pi/2]$ *and* $\lambda \leq -3/4$.

This theorem was proved in [3], using a result of Padilla [17] which states that if $\theta^* < \pi/2$, then any solution of Problem (1.1) must be radial. In Section 3, we give another proof, which does not require solutions to be radially symmetric.

Next, we turn to the question of existence.

Theorem 1.3. *Given any $\theta^* \in (\pi/2, \pi)$ there exist at least two non-constant solutions of Problem (1.1) for λ sufficiently large and negative.*

Remark 1.2. The solutions given in Theorem 1.3 are radial and **in this paper we concentrate on radial solutions.** However, we call attention to the fact that Padilla's result does not guarantee radial symmetry when $\theta^* > \pi/2$. In fact, we believe that there might be non-radial solutions in this range (see Remark 1.3).

We show in Section 2 that for large negative values of λ, Problem (1.1) can be reduced to a singular perturbation problem which enters in the theory developed in [1], [2] and these results can be applied to prove Theorem 1.3.

The following theorem is an improvement of Theorem 1.3 in that it gives the existence of an increasing number of solutions as $|\lambda|$ becomes larger and larger.

Theorem 1.4. *Given any $\theta^* \in (\pi/2, \pi)$ and any $k \geq 1$, there exists a constant $A_k > 0$ such that for $\lambda < -A_k$, Problem (1.1) has at least $2k$ solutions such that $U(\text{North pole}) \in (0, |\lambda|^{1/4})$.*

Remark 1.3. The solutions we construct in Theorems 1.3 and 1.4 are radial. In view of the result of [1] concerning the Morse index of the radial solutions computed in the full Sobolev space (including non-radial functions), it is reasonable to expect non-radial solutions bifurcating off the branches of radial solutions.

In Sections 6 and 7, we establish qualitative properties of these solutions (see Theorems 6.1 and 7.1).

In all the above results, we addressed the question of existence of solutions of Problem (1.1) for θ^* fixed in the interval $(\pi/2, \pi)$; we left $\lambda < -3/4$ free and proved the existence of solutions for λ sufficiently large negative.

It is natural to address a "reverse" question: fix $\lambda < -3/4$ and ask for which radii θ^* there exist solutions. Numerical evidence reported by Bandle and Benguria [3] suggests the existence of a curve $\theta = M_1(\lambda)$ defined for $\lambda \in (-\infty, -3/4)$ with the properties

$$\pi/2 < M_1(\lambda) < \pi \quad \text{for} \quad \lambda < -3/4 \qquad \text{and} \qquad M_1(-3/4) = \pi,$$

such that

$$\begin{aligned} \theta^* < M_1(\lambda) \quad &\Longrightarrow \quad \text{there exist no solutions,} \\ \theta^* > M_1(\lambda) \quad &\Longrightarrow \quad \text{there exists at least one solution.} \end{aligned}$$

Recently, this statement was partially established by Chen and Wei [9]: they proved that one solution exists if $\pi - \theta^*$ is sufficiently small (depending on λ). In Section 6, we use this result to establish a stronger result.

Theorem 1.5. *Given any $\lambda < -3/4$, there exist at least two solutions of Problem (1.1) if $\pi - \theta^*$ is sufficiently small (depending on λ).*

Assuming that $M_1(\lambda)$ exists, we can by Theorem 1.3 assert that

$$\lim_{\lambda \to -\infty} M_1(\lambda) = \pi/2.$$

In view of Theorem 1.3 and further numerical evidence, we now propose a more refined question concerning the solutions of Problem (1.1) with the property that $U(\text{North pole}) \in (0, |\lambda|^{1/4})$. Specifically, let

$$\lambda_n = -(1/4)(n^2 - 1), \qquad n = 2, 3, \ldots. \tag{1.6}$$

Open Problem 1.1. *Do there exist curves $\theta = M_k(\lambda)$ for $k = 1, 2, 3, \ldots$ defined on $(-\infty, \lambda_{2k})$ satisfying*

$$\pi/2 < M_k(\lambda) < \pi \quad \textit{for} \quad \lambda < \lambda_{2k} \qquad \textit{and} \qquad M_k(\lambda_{2k}) = \pi, \qquad k = 1, 2, \ldots,$$

such that

$$\theta^* < M_1(\lambda) \qquad \Longrightarrow \qquad \textit{there exist no solutions}$$

$$M_k(\lambda) < \theta^* < M_{k+1}(\lambda) \qquad \Longrightarrow \qquad \textit{there exist exactly } 2k \textit{ solutions}$$

and

$$\lim_{\lambda \to -\infty} M_k(\lambda) = \pi/2 \qquad \textit{for } k = 1, 2, 3, \ldots.$$

In Figure 1, we give a sketch of the regions in the (θ^*, λ)-plane, separated by the curves $M_k(\lambda)$, in which we expect different numbers of solutions.

In Figure 2, we show three pairs of solutions of Problem (1.1) when $\lambda = -15$ and $\theta^* = 3$.

It would be interesting to study the behavior of these solutions in the limit as $\theta^* \to \pi$. It seems that some converge to $u = 0$ on $\mathbf{S}^3 \setminus$ (Southpole), some converge to the constant $|\lambda|^{1/4}$, while others converge on $\mathbf{S}^3 \setminus$ (Southpole) to "ground states" described in (1.7) below.

Remark 1.4. The critical numbers λ_n are, up to a factor, the "radial" eigenvalues μ_n of $(-\Delta_{\mathbf{S}^3})$ in the whole of $\mathbf{S}^3$, which are given by

$$\mu_n = n^2 - 1, \qquad n = 2, 3, \ldots,$$

with associated eigenfunctions

$$\varphi_n(\theta) = \sin(n\theta)/\sin(\theta).$$

For n odd, the functions φ_n are symmetric with respect to $\theta = \pi/2$; and for n even, they are antisymmetric with respect to $\theta = \pi/2$. For further details, we refer to Section 4.

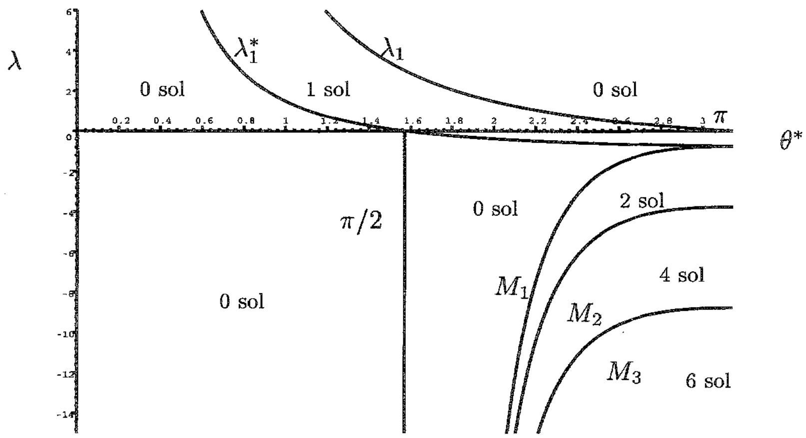

Figure 1. Regions in the (θ^*, λ)-plane with their number of solutions: proved in the region $\lambda \geq -3/4$ and expected in the region $\lambda < -3/4$.

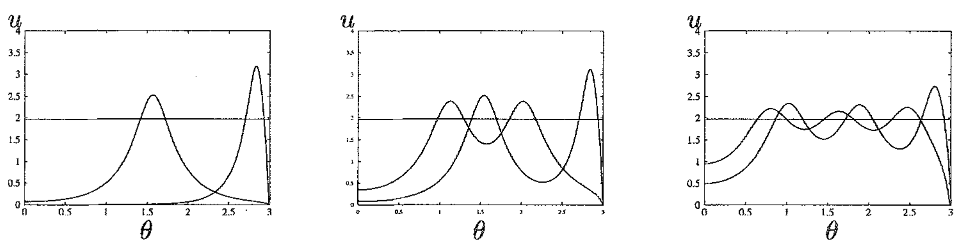

Figure 2. Pairs of 1-spike, 2-spike and 3-spike radial solutions $u(\theta)$ of Problem (1.1) for $\lambda = -15$ and $\theta^* = 3$.

A special role in the analysis of these solutions is played by a family of "ground states". They are solutions of the problem

$$\begin{cases} -\Delta_{\mathbf{S}^3} U = U^5 + \lambda U & \text{in } \mathbf{S}^3, \quad (1.7a) \\ U > 0 & \text{in } \mathbf{S}^3. \quad (1.7b) \end{cases}$$

These branches emanate from the constant solution $U_0 = |\lambda|^{1/4}$ of equation (1.1a) at the eigenvalues $\mu = \mu_{2m+1} = 4m(m+1)$, $m = 1, 2, \ldots$. We prove the following result about them.

Theorem 1.6. *Let $m \geq 1$, and let $\lambda < -m(m+1)$. Then for every $k \in \{1, 2, \ldots, m\}$, there exists (at least) one solution U_k of Problem (1.7), where $U_k = u_k(\theta)$ has the following properties:*

(a) *$u_k(\theta)$ has exactly k local maxima, or* **spikes** *on* $(0, \pi)$;

(b) $u_k(\pi - \theta) = u_k(\theta)$ *for* $0 < \theta < \pi$;

(c) $u_k(0) < |\lambda|^{1/4}$.

In Figure 3, we show a solution $u_k(\theta)$ such as discussed in Theorem 1.6 for $\lambda = -100$ with $k = 8$ local maxima.

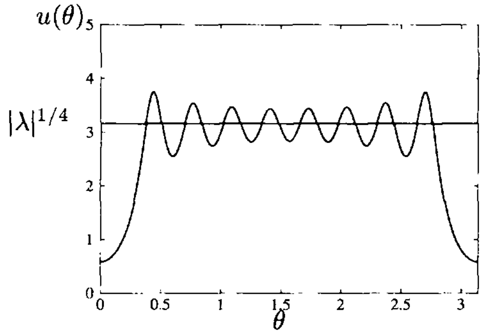

Figure 3. The 8-spike ground state solution $u_8(\theta)$ at $\lambda = -100$; notice the monotonicity of the maxima and the minima.

In Theorem 1.6, we have the restriction that solutions lie *below* $U_0 = |\lambda|^{1/4}$ on the North pole. We have convincing numerical evidence that there also exist solutions with values *above* U_0 at the North pole. This suggests the following

Open Problem 1.2. *Under the same assumptions as in Theorem 1.6, can one establish the existence of solutions* $\tilde{U}_k = \tilde{u}_k(\theta)$ *of Problem (1.7) with the properties*

(a) $\tilde{u}_k(\theta)$ *has exactly k local minima on* $(0, \pi)$;

(b) $\tilde{u}_k(\pi - \theta) = u_k(\theta)$ *for* $0 < \theta < \pi$;

(c) $\tilde{u}_k(0) > |\lambda|^{1/4}$*?*

In Theorem 1.6 and in Open Problem 1.2, we have considered *even* solutions, branching off the constant solution U_0 at the eigenvalues μ_{2m+1}. However, branches of solutions should also emanate from U_0 at the eigenvalues μ_{2m}.

Open Problem 1.3. *Let* $m \geq 1$ *and* $\lambda < -m^2 + (1/4)$. *Given any* $k \in \{1, \ldots, m\}$, *can one establish the existence of solutions* $\overline{U}_k = \overline{u}_k(\theta)$ *of Problem (1.7) with* $k - 1$ *local maxima (resp., minima) on* $(0, \pi)$*?*

When $m = 1$ and $\lambda < -3/4$, this open problem asks whether there exists an increasing or decreasing solution of Problem (1.7). We emphasize that even this case is still open. See also Section 8.

Remark 1.5. As we shall see, some of the proofs in this paper do not depend on the power 5 in equation (1.1a) being critical.

The plan of the paper is the following.

1. Introduction.

2. Preliminaries.
3. Nonexistence for $R^* \leq 1$ and $\lambda \leq -3/4$. Proof of Theorem 1.2.
4. Linearizing around the constant solution in $\mathbf{S}^3$.
5. Positive solutions on $\mathbf{S}^3$: Proof of Theorem 1.6.
6. Proof of Theorem 1.4; Part I: Single-spike solutions.
7. Proof of Theorem 1.4; Part II: Multi-spike solutions.
8. Further open problems.

After the Introduction, we introduce in Section 2 several equivalent formulations of Problem (1.1). Then, in Section 3, we prove a nonexistence result. In Section 4, we present a detailed study of the problem obtained by linearizing around the constant solution $U_0 = |\lambda|^{1/4}$ of Problem (1.7) when $\lambda < 0$. Then, in Section 5, we prove Theorem 1.6; and in Sections 6 and 7, we prove Theorems 1.4 and 1.5. Finally, in Section 8, we conclude with the formulation of some further open problems.

Some of our results were announced in [7].

2 Preliminaries

In this section, we present various changes of variables leading to different formulations of Problem (1.1). First, by stereographic projection onto the equator plane, we transform Problem (1.1) to a problem in $\mathbf{R}^3$, which can be more easily compared with the results stated in Theorem BN. We show that for large negative values of λ it is possible to formulate Problem (1.1) as a *singular perturbation problem*.

We concentrate on radial solutions. It is convenient for later purposes to work with the equation in three different variables: in $r = |x|$, in $\theta = 2\arctan(r)$, and in $t = (1/2)\{(1/r) - r\}$. In this last variable, equation (1.1a) transforms into a generalized *Emden–Fowler equation*.

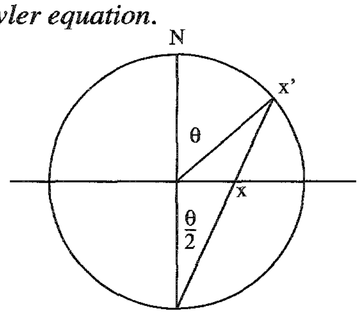

Figure 4. Stereographic projection.

Stereographic projection. Let $\Sigma : \mathbf{R}^3 \to S^3$ be the stereographic projection with vertex at the South pole (see Figure 4), and let $u(x) = U(\Sigma x)$, where $x \in \mathbf{R}^3$. Then equation (1.1a) becomes

$$(2.1) \qquad -(1/\rho^3)\operatorname{div}(\rho\nabla u) = \lambda u + u^5, \qquad \rho(x) = 2/(1+|x|^2), \qquad x \in B_{R^*},$$

where $R^* = \tan(\theta^*/2)$. For the proof of (2.1), we note that if $y = \Sigma x$, then

$$dy = \det(\mathrm{Jac}\Sigma)\, dx \qquad \text{and} \qquad \det(\mathrm{Jac}\Sigma) = \rho^3(x)$$

and that

$$\int_{B'} |\nabla U|^2\, dy = \int_{B_{R^*}} |\nabla u|^2 \rho\, dx.$$

To bring the problem in line with Problem (1.2) we perform one more transformation and set

$$(2.2) \qquad v(x) = u(x)\sqrt{\rho(x)}.$$

Then Problem (1.1) becomes

$$\begin{cases} -\Delta v = \dfrac{3+4\lambda}{(1+|x|^2)^2} v + v^5, \qquad v > 0 & \text{in } B_{R^*}, \quad (2.3\text{a}) \\ v = 0 & \text{on } \partial B_{R^*}, \quad (2.3\text{b}) \end{cases}$$

and we see that whether $\lambda > -3/4$ or $\lambda < -3/4$ determines whether the coefficient of v is positive or negative.

Theorems 1.3 and 1.4 are concerned with values of λ which are large and negative. In this range, Problem (2.3) can be reformulated as a singular perturbation problem. Putting

$$w(\xi) = Av(R^*\xi), \qquad \text{with } A = |3+4\lambda|^{-1/4},$$

we obtain

$$\begin{cases} -\varepsilon^2\Delta w + V_{R^*}(\xi) w = w^5, \qquad w > 0 & \text{in } B_1, \quad (2.4\text{a}) \\ w = 0 & \text{on } \partial B_1, \quad (2.4\text{b}) \end{cases}$$

where

$$\varepsilon = \varepsilon_{R^*,\lambda} = 1/(R^*\sqrt{|3+4\lambda|}) \qquad \text{and} \qquad V_{R^*}(\xi) = 1/(\{1+(R^*)^2|\xi|^2\}^2).$$

We now assume radial symmetry. Returning to (2.1), we write $u(x) = u(r)$ with $r = |x|$. Equation (2.1) then becomes

$$\begin{cases} \left(\dfrac{r^2 u'}{1+r^2}\right)' + \dfrac{4r^2}{(1+r^2)^3}(u^5 + \lambda u) = 0, & 0 < r < R^*, \quad (2.5\text{a}) \\ u > 0, & 0 \le r < R^*, \quad (2.5\text{b}) \\ u'(0) = 0 \quad \text{and} \quad u(R^*) = 0. & \quad (2.5\text{c}) \end{cases}$$

Emden–Fowler formulation. We introduce the variables

$$2t = 1/r - r \qquad \text{and} \qquad y(t) = u(r), \qquad \text{and put} \quad 2T^* = 1/R^* - R^*. \tag{2.6}$$

Then Problem (2.5) transforms to

$$\begin{cases} y'' + \dfrac{1}{(1+t^2)^2}(y^5 + \lambda y) = 0 & T^* < t < \infty, \qquad (2.7a) \\ y > 0 & T^* < t < \infty, \qquad (2.7b) \\ y(T^*) = 0 \quad \text{and} \quad \lim\limits_{t\to\infty} y(t) \text{ exists.} & \qquad (2.7c) \end{cases}$$

Note that the North pole $r = 0$ is now mapped onto $t = \infty$, and the South pole $r = \infty$ onto $t = -\infty$. The equator $r = 1$ is located at $t = 0$, and the equation is invariant under the transformation $t \to -t$. Equation (2.7a) is a generalized Emden–Fowler equation in that, when we omit the 1 in the denominator and put $\lambda = 0$, then the equation becomes

$$y'' + t^{-4}y^5 = 0,$$

which is an example of the classical Emden–Fowler equation.

Geodesic coordinates. Let θ be the geodesic radial coordinate centered at the North pole (cf. Figure 4), and let $u(r) = z(\theta)$ be a function of θ only; then Problem (2.5) becomes

$$\begin{cases} -z'' - 2\coth(\theta)\, z' = z^5 + \lambda z, & 0 < \theta < \theta^*, \qquad (2.8a) \\ z > 0, & 0 < \theta < \theta^*, \qquad (2.8b) \\ z = 0, & \theta = \theta^*. \qquad (2.8c) \end{cases}$$

We see that this equation is invariant under the transformation $\theta \to \pi - \theta$. For convenience, we give here also two expressions relating θ and t:

$$t = \cot(\theta) \qquad \text{and} \qquad \sin^2(\theta) = 1/(1+t^2). \tag{2.9}$$

3 Nonexistence for $R^* \leq 1$ and $\lambda \leq -3/4$. Proof of Theorem 1.2

We establish a more general form of Theorem 1.2, and deal with the problem

$$\begin{cases} -\varepsilon^2 \Delta w + V(x)\, w = w^5 & \text{in } B_1, \qquad (3.1a) \\ w = 0 & \text{on } \partial B_1, \qquad (3.1b) \end{cases}$$

in which $V(x)$ is a smooth potential function and B_1 the unit ball in $\mathbf{R}^3$.

Lemma 3.1. *Assume that for some $a = (a_1, a_2, a_3) \in B_1$,*

$$\sum_{i=1}^{3}(x_i - a_i)\frac{\partial V}{\partial x_i} + 2V > 0 \qquad \text{in } B_1 \tag{3.2}$$

and that w satisfies Problem (3.1) for some $\varepsilon > 0$. Then $w \equiv 0$.

Theorem 1.2 follows as a corollary of Lemma 3.1 when we take

$$V(r) = V_{R^*}(r) = 1/(\{1 + (R^*)^2 r^2\}^2)$$

and observe that

$$\sum_{i=1}^{3} x_i \frac{\partial V}{\partial x_i} + 2V = \frac{1}{r}(r^2 V)' > 0 \qquad \text{for } 0 < r \leq 1,$$

when $V(r) = V_{R^*}(r)$ and $R^* \leq 1$.

Proof of Lemma 3.1. Multiply equation (3.1a) by $\Sigma_{i=1}^{3}(x_i - a_i)(\partial w/\partial x_i)$ and integrate over B_1. This yields

$$\begin{aligned} &\frac{\varepsilon^2}{2}\int_{\partial B_1}\left(\frac{\partial w}{\partial n}\right)^2 [(x-a)\cdot n] + \frac{\varepsilon^2}{2}\int_{B_1}|\nabla w|^2 + \frac{3}{2}\int_{B_1}V(x)w^2 \\ &\qquad + \frac{1}{2}\int_{B_1}\left(\sum_{i=1}^{3}(x_i - a_i)\frac{\partial V}{\partial x_i}\right)w^2 = \frac{1}{2}\int_{B_1} w^6, \end{aligned} \tag{3.3}$$

where n denotes the normal to ∂B_1. Next, multiply equation (3.1a) by v to obtain

$$\varepsilon^2 \int_{B_1}|\nabla w|^2 + \int_{B_1}V(x)w^2 = \int_{B_1}w^6. \tag{3.4}$$

Now multiplying (3.4) by $1/2$ and then subtracting the result from (3.3), we obtain

$$\frac{\varepsilon^2}{2}\int_{\partial B_1}\left(\frac{\partial w}{\partial n}\right)^2[(x-a)\cdot n] + \frac{1}{2}\int_{B_1}\left(\sum_{i=1}^{3}(x_i - a_i)\frac{\partial V}{\partial x_i} + 2V\right)w^2 = 0. \tag{3.5}$$

Plainly, (3.5) and (3.2) imply that $w = 0$. □

The same argument shows that if $a = (a_1, a_2, a_3)$ is any point in B_1 and if

$$\sum_{i=1}^{3}(x_i - a_i)\frac{\partial V}{\partial x_i} + 2V > 0 \qquad \text{somewhere in } B_1,$$

then any solution of Problem (3.1) must be identically zero. This suggests the following.

Open Problem 3.1. Assume that $V \geq 0$ in B_1 and that for any $a \in B_1$,

$$\sum_{i=1}^{3}(x_i - a_i)\frac{\partial V}{\partial x_i} + 2V < 0 \qquad \text{somewhere in } B_1. \tag{3.6}$$

Is it true that for every ε sufficiently small, there is a solution of

$$\begin{cases} -\varepsilon^2 \Delta w + V(x)\, w = w^5, \quad w > 0 & \text{in } B_1, \quad (3.7a) \\ w = 0 & \text{on } \partial B_1\,? \quad (3.7b) \end{cases}$$

If the answer is negative, can one find a necessary and sufficient condition on V which guarantees the existence of a solution of Problem (3.7) for all ε sufficiently small?

4 Linearizing around the constant solution in $\mathbb{S}^3$

It is illuminating first to consider the analogue of equation (1.1a) in the whole of $\mathbb{S}^3$:

$$-\Delta_{\mathbb{S}^3} U = U^5 + \lambda U \qquad \text{in } \mathbb{S}^3. \tag{4.1}$$

It is evident that because $\lambda < 0$, the constant function

$$U_0 = |\lambda|^{1/4}$$

is a solution of (4.1). Linearizing equation (4.1) around this solution, i.e., taking $U = |\lambda|^{1/4} + \varepsilon\phi$ and dropping terms of $o(\varepsilon)$, we arrive at the equation

$$-\Delta_{\mathbb{S}^3}\phi = 4|\lambda|\phi \qquad \text{in } \mathbb{S}^3. \tag{4.2}$$

This leads to the search for nontrivial bounded solutions of (4.2), i.e., for eigenvalues of $-\Delta_{\mathbb{S}^3}$ on $\mathbb{S}^3$,

$$-\Delta_{\mathbb{S}^3}\phi = \mu\phi \qquad \text{in } \mathbb{S}^3. \tag{4.3}$$

Thus, for radial solutions $\phi = \varphi(\theta)$, we seek solutions of the problem

$$\begin{cases} -\varphi'' - 2\cot(\theta)\,\varphi' = \mu\varphi \qquad \text{for } 0 < \theta < \pi, & (4.4a) \\ \varphi'(0) = 0. & (4.4b) \end{cases}$$

Since $\cot(\theta) \sim \theta^{-1}$ as $\theta \to 0$, for each initial value $\varphi(0)$, there exists a unique solution. We find that it is an appropriate multiple of the function

$$\varphi(\theta;\mu) = \frac{\sin(\theta\sqrt{1+\mu})}{\sin(\theta)}. \tag{4.5}$$

Plainly, for $\varphi(\theta;\mu)$ to be bounded on the whole interval $[0,\pi]$, we require that

$$\pi\sqrt{1+\mu} = n\pi \qquad \text{for some integer } n \geq 1.$$

This yields the following family of eigenvalues and eigenfunctions:

$$\mu_n = n^2 - 1 \qquad \text{and} \qquad \varphi_n(\theta) = \frac{\sin(n\theta)}{\sin(\theta)}, \qquad n \geq 2. \tag{4.6}$$

We see by inspection that φ_n has $n-1$ zeros on $(0,\pi)$. They are given by

$$\theta_k = (k/n)\pi, \qquad k = 1, 2, \ldots, n-1.$$

Note that φ_n is odd with respect to $\pi/2$ if n is even, and it is even with respect to $\pi/2$ if n is odd.

In Section 5, the critical points of φ_n play an important role. By inspection, we find that

(i) φ_{2k} has $k-1$ critical points on $(0,\pi/2)$ and $k-1$ critical points on $(\pi/2,\pi)$, where $k = 1, 2, \ldots$.

(ii) φ_{2k+1} has $k-1$ critical points on $(0,\pi/2)$ and $k-1$ critical points on $(\pi/2,\pi)$, as well as one critical point at $\theta = \pi/2$; $k = 1, 2, \ldots$.

We also observe that, thanks to the scaling factor $\sqrt{1+\mu}$ in the numerator of the expression (4.4) for $\varphi(\theta;\mu)$, we can say the following about the location of critical points.

Lemma 4.1. *Let $\mu > \mu_n$. Then the solution $\varphi(\theta;\mu)$ of Problem (4.3) has at least m critical points on $(0,\pi/2)$, where*

$$m = \frac{n}{2} - 1 \qquad \text{if } n \text{ is even},$$
$$m = \frac{n-1}{2} \qquad \text{if } n \text{ is odd}.$$

In Figure 5, we show $\varphi(\theta;\mu)$ for $\mu = \mu_5 = 24$ and for $\mu > \mu_5$.

Remark 4.1. General bifurcation theory yields the existence of a branch of solutions of (4.1), i.e., a pair $(\lambda(s), U(s))$ depending on a parameter s, defined in a neighborhood of $s = 0$, with $\lambda(0) = -\mu_n/4$ and $U(0) = |\lambda(0)|^{1/4}$ (cf. [11], [5], and [8]). At the first interesting value $\lambda = -\mu_2/4 = -3/4$, we have an explicit description of this branch. Specifically, it is given by

$$\lambda(s) = -\frac{3}{4}, \quad u(r;s) = \left(\frac{3}{4}\right)^{1/4}\left\{\frac{\alpha(s)(1+r^2)}{\alpha^2(s)+r^2}\right\}^{1/2}, \quad \alpha(s) = 1+s, \quad -1 < s < \infty,$$

where $u(r;s)$ is a solution of Problem (4.1) expressed in terms of the radial variable r in $\mathbb{R}^3$.

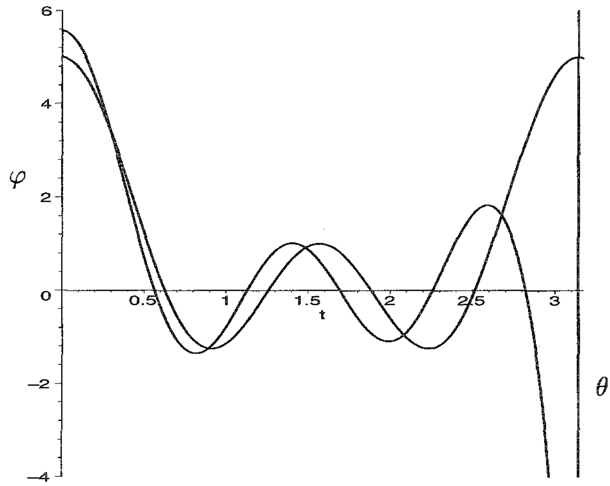

Figure 5. The function $\varphi(\theta;\mu)$ for $\mu = \mu_5 = 24$ and for $\mu = 30$.

Next, by analogy with (4.3), we consider the eigenvalue problem for $-\Delta_{\mathbb{S}^3}$ on a ball $B' = B'_{\theta^*} = \{0 < \theta < \theta^*\}$, where $\theta^* \in (0,\pi)$, with Dirichlet boundary condition

$$\begin{cases} -\Delta_{\mathbb{S}^3}\phi = \mu\phi & \text{in } B'_{\theta^*}, \quad (4.7a)\\ \phi = 0 & \text{on } \partial B'_{\theta^*}. \quad (4.7b)\end{cases}$$

We find the principal radial eigenfunction $\phi_1^* = \varphi_1^*(\theta)$ and the corresponding principal eigenvalue μ_1^* by adjusting μ so that θ^* coincides with the first zero of $\varphi(\theta;\mu)$, i.e.,

$$\varphi(\theta;\mu_1^*) > 0 \quad \text{for } 0 \le \theta < \theta^* \qquad \text{and} \qquad \varphi(\theta^*;\mu_1^*) = 0.$$

This means that we require that

$$\theta^*\sqrt{1+\mu_1^*} = \pi.$$

This yields for the principal eigenvalue and eigenfunction

$$\mu_1^* = \frac{\pi^2 - (\theta^*)^2}{(\theta^*)^2} \qquad \text{and} \qquad \varphi_1^*(\theta) = \frac{\sin\left(\frac{\pi}{\theta^*}\theta\right)}{\sin(\theta)}. \tag{4.8}$$

Similarly, higher eigenvalues and eigenfunctions are found by putting

$$\theta^*\sqrt{1+\mu_j^*} = j\pi, \qquad j = 1,2,\ldots.$$

In this manner, we obtain

$$\mu_j^* = \frac{j^2\pi^2 - (\theta^*)^2}{(\theta^*)^2} \qquad \text{and} \qquad \varphi_j^*(\theta) = \frac{\sin\left(\frac{j\pi}{\theta^*}\theta\right)}{\sin(\theta)}, \qquad j = 1,2,\ldots. \tag{4.9}$$

5 Positive solutions on S^3: Proof of Theorem 1.6

In this section, we establish that for every $n \geq 1$ and every $\lambda < -n(n+1)$, there exist n classical solutions U_j, $j = 1, 2, \ldots, n$, of the problem

$$\begin{cases} -\Delta_{\mathrm{S}^3} U = U^5 + \lambda U & \text{on } \mathrm{S}^3, \quad (5.1a) \\ U > 0 & \text{on } \mathrm{S}^3. \quad (5.1b) \end{cases}$$

They are of the form $U_j = u_j(\theta)$ and have the following properties:

(a) $$u_j(\theta) = u_j(\pi - \theta), \qquad 0 < \theta < \pi,$$

i.e., they are symmetric with respect to the equator $\theta = \pi/2$;

(b) $u_k(0) \in (0, |\lambda|^{1/4})$;

(c) given any $1 \leq j \leq n$, the solution u_j has precisely k local maxima and $j-1$ local minima on the interval $(0, \pi)$.

We emphasize here that numerical evidence suggests that there are also solutions for which $U(0) > |\lambda|^{1/4}$ (see Open Problem 1.2). Here the focus is on those endowed with property (b).

The construction of these solutions is carried out by means of a shooting technique based on ideas developed in [18], [19]. This method uses a continuation argument in which the movement of critical points is closely followed as the shooting parameter varies.

The analysis is best carried out in terms of the Emden–Fowler formulation given in Section 2. In this formulation, Problem (5.1) becomes

$$\begin{cases} y'' + \dfrac{1}{(1+t^2)^2}(y^5 - |\lambda| y) = 0, & -\infty < t < \infty, \quad (5.2a) \\ y > 0, & -\infty < t < \infty, \quad (5.2b) \\ \lim\limits_{t \to \pm\infty} y(t) \text{ exists.} & \quad (5.2c) \end{cases}$$

Note that the finite limits of $y(t)$ as $t \to \pm\infty$ correspond to the value of U at the North pole ($t = \infty$) and at the South pole ($t = -\infty$).

For convenience, we scale y so that the constant solution becomes unity. Thus, we write

$$y(t) = |\lambda|^{1/4} \overline{y}(t).$$

Then Problem (5.2) becomes

$$\begin{cases} y'' + |\lambda| a(t) f(y) = 0, & -\infty < t < \infty, \quad (5.3a) \\ y > 0, & -\infty < t < \infty, \quad (5.3b) \\ \lim\limits_{t \to \pm\infty} y(t) \text{ exists,} & \quad (5.3c) \end{cases}$$

where we have omitted the overbar again and written

$$a(t) = \frac{1}{(1+t^2)^2} \qquad \text{and} \qquad f(y) = y^5 - y. \tag{5.4}$$

To construct solutions of Problem (5.3), we consider the "initial value problem"

$$\begin{cases} y'' + |\lambda| a(t) f(y) = 0, & (5.5a) \\ y(t) \to \gamma \qquad \text{as } t \to +\infty. & (5.5b) \end{cases}$$

It is well-known that for every $\gamma \in \mathbf{R}$, Problem (5.5) has a unique local solution; we denote it by $y = y(t) = y(t, \gamma)$.

Let $\gamma \in (0,1)$. Then y starts below the constant solution, and $y''(t) > 0$ as long as $y < 1$. Therefore,

$$t_1 = \inf\{t : y < 1 \text{ on } (t, \infty)\} > -\infty \qquad \text{and} \qquad y(t_1) = 1.$$

There are now two possibilities: either $y' < 0$ for all $t < t_1$, or y has another critical point

$$\tau_1 = \inf\{t : y' < 0 \text{ on } (t, t_1)\} > -\infty \qquad \text{and} \qquad y'(\tau_1) = 0.$$

Since $y'' < 0$ when $y > 1$, it follows that if τ_1 exists, then y crosses the line $y = 1$ again, so that we can define the point

$$t_2 = \inf\{t : y > 1 \text{ on } (t, t_1)\} > -\infty \qquad \text{and} \qquad y(t_2) = 1.$$

Next, we name the following critical point, if it exists,

$$\tau_2 = \inf\{t : y' > 0 \text{ on } (t, t_2)\} > -\infty \qquad \text{and} \qquad y'(\tau_2) = 0,$$

and the next point of intersection

$$t_3 = \sup\{t : y < 1 \text{ on } (t, t_2)\} > -\infty \qquad \text{and} \qquad y(t_3) = 1.$$

We continue this process as long as there exist critical points. Plainly, these points all depend on γ; and by construction, they are ordered according to

$$t_1(\gamma) > \tau_1(\gamma) > t_2(\gamma) > \tau_2(\gamma) > t_3(\gamma) > \tau_3(\gamma) > \cdots.$$

Since $y = 1$ is a solution of equation (5.5a), it follows from a uniqueness argument that $y'(t_k) \neq 0$. Therefore,

$$y(\tau_{2k+1}(\gamma), \gamma) > 1 \quad \text{and} \quad y''(\tau_{2k+1}(\gamma), \gamma) < 0 \qquad \text{for } k = 1, 2, \ldots,$$

as long as they exist. Hence these critical points are all isolated. Similarly,

$$y(\tau_{2k}(\gamma), \gamma) < 1 \qquad \text{for } k = 1, 2, \ldots$$

and

$$y''(\tau_{2k}(\gamma),\gamma) > 0 \quad \text{if} \quad y(\tau_{2k}(\gamma),\gamma) > 0 \qquad \text{for } k = 1,2,\dots,$$

so that these critical points are also isolated. Since $y'' \neq 0$ at these critical points, they depend continuously on γ, and they cannot coalesce. In Figure 6, we show a graph of $y(t)$ and indicate a few of the critical points defined above.

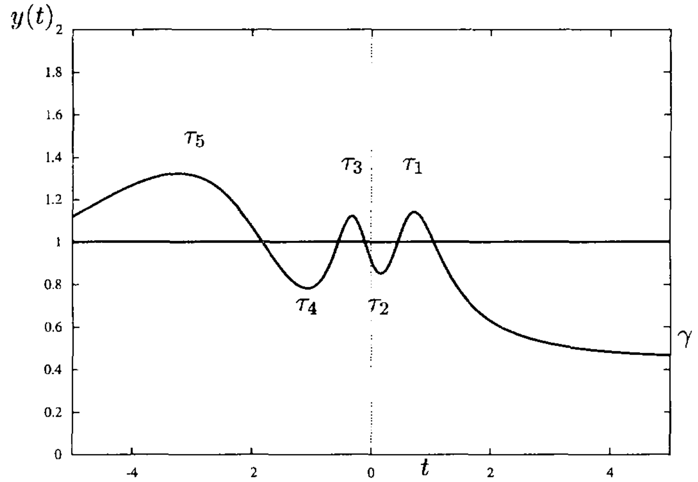

Figure 6. Solution graph $y(t)$ with critical points τ_1, τ_2, τ_3, τ_4 and τ_5.

We give an upper bound for $y(\tau_k)$ whenever $\tau_k \in [0,\infty)$. Let

$$F(y) = \int_0^y f(s)\,ds = \frac{1}{6}y^6 - \frac{1}{2}y^2.$$

Plainly, F has two zeros:

$$y = 0 \qquad \text{and} \qquad y = \sigma \stackrel{\text{def}}{=} 3^{1/4}.$$

Lemma 5.1. *We have*

$$0 < y(\tau_k) < \sigma \qquad \text{if } \tau_k \geq 0.$$

The proof of Lemma 5.1 makes use of the *energy function*

$$H(t) = \frac{1}{2|\lambda|a(t)}\,y'^2(t) + F(y(t)). \tag{5.6}$$

If $y(t)$ is a solution of Problem (5.5), then

$$H'(t) = -\frac{a'(t)}{2|\lambda|a^2(t)}\,y'^2(t). \tag{5.7}$$

Hence, $H' < 0$ on $\mathbf{R}^-$ and $H' > 0$ on $\mathbf{R}^+$. By an elementary computation,

$$(5.8)\qquad \begin{aligned} y'(t,\gamma) &= \frac{|\lambda|}{3} f(\gamma)t^{-3} + O(t^{-5}) && \text{as } t \to +\infty, \\ y(t,\gamma) &= \gamma - \frac{|\lambda|}{6} f(\gamma)t^{-2} + O(t^{-4}) && \text{as } t \to +\infty, \end{aligned}$$

and therefore

$$(5.9)\qquad H(y(t)) \to F(\gamma) \qquad \text{as } t \to +\infty.$$

Proof of Lemma 5.1. Since $H' > 0$ on $(0,\infty)$ and $\tau_k \geq 0$, it follows that

$$H(\tau_k) < H(\infty);$$

hence, because of (5.9) and the fact that $0 < \gamma < 1$,

$$F(y(\tau_k)) < F(\gamma) < 0.$$

This means that $0 < y(\tau_k) < \sigma$, as asserted. □

We now follow these critical points as γ descends from $\gamma = 1$. Thus we begin by investigating their existence and location when γ is close to $\gamma = 1$. Let us write

$$(5.10)\qquad y(t) = 1 + \varepsilon z(t),$$

in which $\varepsilon = 1 - \gamma$ is a small positive number. Then, by standard ODE theory,

$$(5.11)\qquad z(t) = \zeta(t) + O(\varepsilon) \qquad \text{as } \varepsilon \to 0,$$

uniformly on sets of the form $[T,\infty)$ for any constant $T > -\infty$, where ζ is the unique solution of the problem obtained from Problem (5.5) by linearizing around $y = 1$:

$$\begin{array}{ll} (5.12\text{a}) \\ (5.12\text{b}) \end{array} \qquad \begin{cases} \zeta'' + 4|\lambda|a(t)\zeta = 0, & -\infty < t < +\infty, \\ \zeta(t) \to -1 & \text{as } t \to +\infty. \end{cases}$$

Proceeding as for Problem (5.5), we can also define a sequence of zeros $\{t_k^0\}$ and critical points $\{\tau_k^0\}$ for the solution ζ of Problem (5.12):

$$t_1^0(\gamma) > \tau_1^0(\gamma) > t_2^0(\gamma) > \tau_2^0(\gamma) > t_3^0(\gamma) > \tau_3^0(\gamma) > \cdots$$

for as long as they exist.

Equation (5.12a) is the same as equation (4.3), but then in terms of the Emden–Fowler coordinates, and

$$(5.13)\qquad \mu = 4|\lambda|.$$

Thus, the eigenvalues and eigenfunctions are known explicitly, and since $\lambda < 0$, we define

$$\lambda_n = -\frac{1}{4}\mu_n = -\frac{1}{4}(n^2-1) \qquad \text{and} \qquad \zeta_n(t) = \varphi_n(\theta) = \frac{\sin(n\theta)}{\sin(\theta)}, \qquad n = 2, 3, \ldots.$$

In particular, we know from Section 4 that ζ_n has $n-1$ zeros and $n-2$ critical points, and that if $\lambda < \lambda_n$, then ζ has n zeros and $n-1$ critical points. We state this as a lemma.

Lemma 5.2. (a) *Let $n \geq 2$, and $\lambda < \lambda_n$. Then the solution $\zeta(t)$ of Problem (5.12) has zeros t_k^0 for $k = 1, 2, \ldots, n$ and critical points τ_k^0 for $k = 1, 2, \ldots, n-1$.*

(b) *Let $m \geq 1$, and $\lambda < \lambda_{2m+1} = -m(m+1)$. Then $\tau_k^0 > 0$ for $k = 1, 2, \ldots, m$.*

Uniform continuity of the solution of Problem (5.10) with respect to the initial value γ on sets of the form $[T, \infty)$ for any $T \in \mathbf{R}$, shows that the following holds.

Lemma 5.3. *Suppose that ζ has a critical point τ_k^0 for some $k \geq 1$. Then for $\gamma < 1$ sufficiently close to 1, the critical point $\tau_k(\gamma)$ also exists and*

$$\tau_k(\gamma) \to \tau_k^0 \qquad \textit{as } \gamma \nearrow 1.$$

As we lower γ, the critical points move continuously with respect to γ. In the following lemma, we show that they eventually move off to the region $t < 0$. Let

$$\gamma_k^* = \inf\{\gamma < 1 : \tau_k \text{ exists on } (\gamma, 1)\}.$$

Lemma 5.4. *Let $\tau_k(\gamma)$ be a critical point which exists for γ close to 1, so that γ_k^* is well-defined. Then there exists $\delta > 0$ such that*

$$\tau_k(\gamma) < 0 \qquad \textit{if } \gamma \in (\gamma_k^*, \gamma_k^* + \delta).$$

Proof. For simplicity, we omit the subscript k.

If $\gamma^* = 0$, then the assertion follows from the continuous dependence on initial data. Thus, we assume that $\gamma^* \in (0, 1)$.

Suppose that there exists a decreasing sequence $\{\gamma_j\}$ which converges to γ^* such that

$$\tau(\gamma_j) \geq 0 \quad \text{for all } j = 1, 2, \ldots.$$

Since the sequence $\{\tau(\gamma_j)\}$ is bounded away from $+\infty$ and the sequence $\{y(\tau_j)\}$ is bounded by Lemma 5.1, it follows that there is a subsequence $\{\gamma_{j'}\}$ such that

$$\tau(\gamma_{j'}) \to \tau^* \quad \text{and} \quad y(\tau(\gamma_{j'})) \to y^* \qquad \text{as } j' \to \infty$$

for some $\tau^* \geq 0$, and $y^* \in [0, \sigma]$. If $y^* \notin \{0, 1\}$, then it follows from the Implicit Function Theorem that $\tau(\gamma)$ is well-defined in a neighbourhood of γ^*, which contradicts the definition of γ^*. If $y^* = 0$ or $y^* = 1$, then by uniqueness, $y(t) = 0$ or $y(t) = 1$ for all $t \in \mathbf{R}$, which contradicts the initial condition.

This completes the proof of Lemma 5.4. □

We may now turn to the

Proof of Theorem 1.6. We use a continuity argument. If $\lambda < -n(n+1)$, then by Lemma 5.2, the critical points τ_k^0 for $k = 1, 2, \ldots, n$ are all positive; and hence, by Lemma 5.3, $\tau_k \in \mathbf{R}^+$ for $k = 1, 2, \ldots, n$ if $\gamma < 1$ is sufficiently close to 1. We have seen in Lemma 5.4 that as we let γ decrease, these critical points all eventually enter the region $\{t < 0\}$ and thus cross the axis $t = 0$. If $\tau_k(\gamma) = 0$ for some value γ_0, then (i) the solution $y(t, \gamma_0)$ is symmetric with respect to $t = 0$ and (ii) it has k spikes. □

In Figure 7, we show the one-, two- and three-spike ground states.

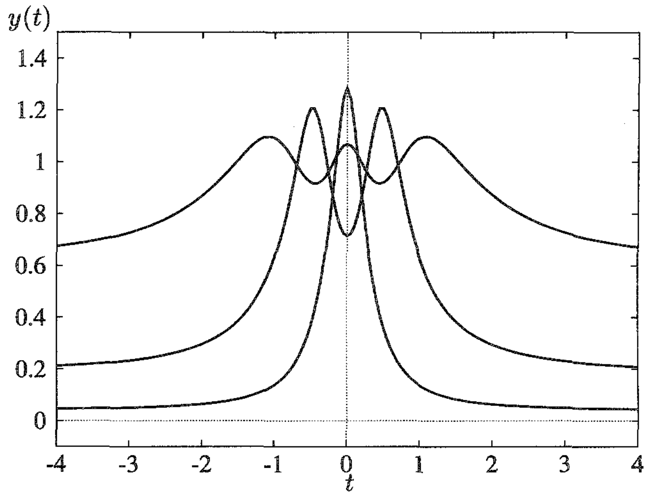

Figure 7. Three ground states, $y_1(t)$, $y_2(t)$ and $y_3(t)$ at $\lambda = -15$.

We conclude with a monotonicity property. Let y_m be a ground state with m local maxima at, say, the points $t_1 < t_2 < \cdots < t_m$. Then

$$y(t_k) > y(t_{k+1}) \qquad \text{for } k = 1, 2, \ldots, \left\lceil \frac{m-1}{2} \right\rceil, \quad m \geq 3.$$

A similar property holds for the local minima.

This follows from the observation that

$$H(t_k) = F(y(t_k))$$

and that H increases on $\mathbf{R}^+$ and decreases on $\mathbf{R}^-$. An illustration of this property is given in Figure 3 in the Introduction, where $m = 8$ and $\lambda = -100$.

6 Proof of Theorem 1.4; Part I: Single-spike solutions

As in the previous section, it is convenient to prove Theorem 1.4 by analyzing Problem (1.1) in the Emden–Fowler formulation. We scale y as in Section 5, equation (5.3a), and divide the resulting equation by $|\lambda|$ to obtain

$$\begin{cases} \varepsilon^2 y'' + a(t)f(y) = 0, & T^* < t < \infty, \qquad \varepsilon^2 = 1/|\lambda|, & \text{(6.1a)} \\ y > 0, & T^* < t < \infty, & \text{(6.1b)} \\ y(T^*) = 0 \quad \text{and} \quad \lim_{t\to\infty} y(t) \ \text{exists}, & & \text{(6.1c)} \end{cases}$$

where we recall that

$$\text{(6.2)} \quad 2T^* = \frac{1}{R^*} - R^* \ (R^* > 1), \qquad a(t) = \frac{1}{(1+t^2)^2} \qquad \text{and} \qquad f(y) = y^5 - y.$$

The main result of this section is

Theorem 6.1. *Given any $T^* < 0$, there exists a constant $\varepsilon_1 > 0$ such that if $\varepsilon < \varepsilon_1$, Problem (6.1) has at least 2 solutions for which $y(+\infty) \in (0,1)$, each solution having exactly one spike.*

The proof of Theorem 6.1 is based on a shooting method. Given $\gamma \in (0,1)$, consider the initial value problem

$$\begin{cases} \varepsilon^2 y'' + a(t)f(y) = 0, & t < \infty, & \text{(6.3a)} \\ y(t) \to \gamma & \text{as } t \to +\infty. & \text{(6.3b)} \end{cases}$$

We denote the solution by $y = y(t,\gamma)$ and define the point

$$T(\gamma) = \inf\{t \in \mathbf{R} : y(\cdot,\gamma) > 0 \text{ on } (t,\infty)\}.$$

Plainly, $T(\gamma)$ may be finite or infinite. Let $\mathcal{A}$ be the set of values of γ for which $T(\gamma)$ is finite:

$$\mathcal{A} = \{\gamma \in (0,1) : T(\gamma) > -\infty\}.$$

If $\gamma \in \mathcal{A}$, then $y(T,\gamma) = 0$ and, because $y = 0$ is a solution of (6.1a), it follows by uniqueness that $y'(T,\gamma) > 0$. Therefore, by continuous dependence, a neighborhood of γ lies in $\mathcal{A}$ as well, and $T(\gamma)$ depends continuously on γ. Thus, $\mathcal{A}$ is an open set. Note that Theorem 1.2 implies that $T(\gamma) < 0$. In general, $\mathcal{A}$ consists of several connected components (see Section 7).

The proof proceeds in a series of steps which we formulate as propositions. We first state these three propositions and then give their proofs.

Proposition 6.1. *For ε small enough there exists a $\gamma_0 \in \mathcal{A}$ such that the solution $y(t,\gamma_0)$ of Problem (6.3) with initial value γ_0 has precisely one spike.*

Let $\mathcal{A}_1 = (\gamma_1^-, \gamma_1^+)$ be the connected component of $\mathcal{A}$ which contains the value γ_0 defined in Proposition 6.1.

Proposition 6.2. *We have*

$$T(\gamma) \to -\infty \qquad \text{as } \gamma \to \gamma_1^{\pm}.$$

Thus, the branch $\Gamma_1 = \{(\gamma, T(\gamma)) : \gamma \in \mathcal{A}_1\}$ has vertical asymptotes at $\gamma_1^{\pm}$. In Figure 8, we show the branch Γ_1 of one-spike solutions in the variables r and u. It represents the graph of R^* versus $\gamma = u(0)$.

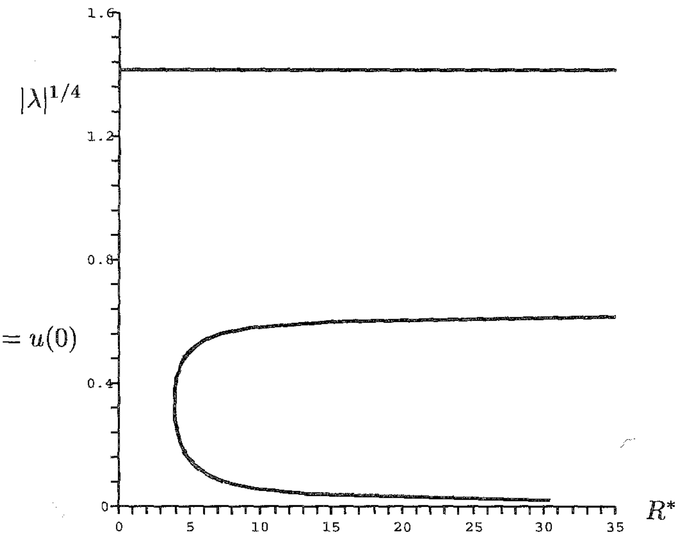

Figure 8. The branch Γ_1 of one-spike solutions $u(r)$, depicted in the $(R^*, u(0))$-plane for $\lambda = -4$.

Proposition 6.2 enables us to define

$$T^{(1)}_{\max,\varepsilon}(\gamma) \stackrel{\text{def}}{=} \max\{T(\gamma) : \gamma \in \mathcal{A}_1\}.$$

Proposition 6.3.

$$T^{(1)}_{\max,\varepsilon}(\gamma) \to 0 \qquad \text{as } \varepsilon \to 0.$$

It follows from Proposition 6.3 that, given any $T^* < 0$, it is possible to find $\varepsilon_1 > 0$ such that $T_{\max,\varepsilon} \in (T^*, 0)$ for $\varepsilon < \varepsilon_1$ and hence that Γ_1 intersects the line $T = T^*$ at least twice. This yields at least two solutions of Problem (6.1) having exactly one spike and thus completes the proof of Theorem 6.1.

Proof of Theorem 1.5. The recent result of Chen and Wei [9] implies that if $\lambda < -3/4$, then the set $\mathcal{A}$ is nonempty and thus contains an element γ^*. Let (γ^-, γ^+) be the maximal component of $\mathcal{A}$ which contains γ^*. Then it follows from Proposition 6.2 that $T(\gamma) \to \infty$ when $\gamma \to \gamma^\pm$, so that if $|T|$ (or R^*) is large enough, there exist at least two solutions of Problem (6.3) which vanish at $t = T$. This proves Theorem 1.5. □

Let us now prove Propositions 6.1–6.3.

Proof of Proposition 6.1. We fix a point $T_0 < 0$. For ε small enough, $\tau_1^0 > T_0$; and hence there exists an initial value $\gamma_0 \in (0, 1)$ such that

$$\tau_1(\gamma_0) = T_0.$$

Since γ_0 obviously depends on ε, we write $\gamma_0 = \gamma_0(\varepsilon)$.

We wish to show that for ε sufficiently small, the solution $y(t, \gamma_0)$ of Problem (6.3) has a zero $T(\gamma_0) \in (-\infty, T_0)$. We do this in two steps. First, we show that y has risen to a sufficiently high level at T_0, and then we show that the solution hits the t-axis. Henceforth, we write

$$y_0 = y(T_0, \gamma_0).$$

Since $y'' < 0$ at T_0, it is clear that $y_0 > 1$. In fact, we show that for ε small enough, $y_0 > \sigma$, where $\sigma = 3^{1/4}$ is the positive zero of $F(y) = \frac{1}{6}y^6 - \frac{1}{2}y^2$.

Lemma 6.1. *There exist constants $\alpha > 0$ and $\varepsilon_0 > 0$ such that*

$$F(y_0) > \alpha\,\varepsilon \qquad \textit{for } 0 < \varepsilon < \varepsilon_0.$$

Proof. In the proof of Lemma 6.1 and of subsequent lemmas, the *energy function*

$$H(t) = \big(\varepsilon^2/2a(t)\big)\, y'^2(t) + F(y(t)) \tag{6.4}$$

plays a central role. If $y(t)$ is a solution of equation (6.1a), then

$$H'(t) = -\frac{\varepsilon^2\, a'(t)}{2a^2(t)}\, y'^2(t). \tag{6.5}$$

Hence, $H' < 0$ on $\mathbf{R}^-$ and $H' > 0$ on $\mathbf{R}^+$. Moreover, by an elementary computation,

$$\begin{aligned} y'(t,\gamma) &= \big(1/3\varepsilon^2\big) f(\gamma) t^{-3} + O(t^{-5}) \qquad &&\text{as } t \to +\infty,\\ y(t,\gamma) &= \gamma - \big(1/6\varepsilon^2\big) f(\gamma) t^{-2} + O(t^{-4}) \qquad &&\text{as } t \to +\infty, \end{aligned} \tag{6.6}$$

and therefore

$$H(y(t)) \to F(\gamma) \qquad \text{as } t \to +\infty. \tag{6.7}$$

We integrate $H'(t)$ over (T_0, ∞) and use (6.7). This yields

$$F(\gamma_0(\varepsilon)) - F(y_0) = -\frac{\varepsilon^2}{2}\int_{T_0}^{\infty} \frac{a'}{a^2}\, y'^2(t)\, dt, \tag{6.8}$$

or

$$F(y_0) = J(\varepsilon) + F(\gamma_0(\varepsilon)),$$

where

$$J(\varepsilon) \stackrel{\text{def}}{=} \frac{\varepsilon^2}{2}\int_{T_0}^{\infty} \frac{a'}{a^2}\,\{y'(t)\}^2\, dt.$$

For convenience, we split the integration at $t = 0$, writing

$$J(\varepsilon) = J_1(\varepsilon) + J_2(\varepsilon),$$

where

$$J_1(\varepsilon) = \frac{\varepsilon^2}{2}\int_{T_0}^{0} \frac{a'}{a^2}\,\{y'(t)\}^2\, dt \qquad \text{and} \qquad J_2(\varepsilon) = \frac{\varepsilon^2}{2}\int_{0}^{\infty} \frac{a'}{a^2}\,\{y'(t)\}^2\, dt. \tag{6.9}$$

The expression for $F(y_0)$ then becomes

$$F(y_0) = J_1(\varepsilon) + J_2(\varepsilon) + F(\gamma_0(\varepsilon)). \tag{6.10}$$

In what follows, we successively estimate the three terms on the right hand side of (6.10).

Lemma 6.2. *There exist positive constants A and ε_0 such that*

$$J_1(\varepsilon) \ge A\varepsilon \qquad \textit{for } 0 < \varepsilon < \varepsilon_0.$$

Proof. Write

$$t = T_0 + \varepsilon s \qquad \text{and} \qquad y(t) = z(s). \tag{6.11}$$

Then

$$\begin{cases} z'' + a(T_0 + \varepsilon s) f(z) = 0, & \text{for } s > 0, \qquad (6.12\text{a}) \\ z'(0) = 0 \quad \text{and} \quad z'(s) < 0, \quad z(s) > 0, & \text{for } s > 0. \qquad (6.12\text{b}) \end{cases}$$

Substitution into the expression (6.9) for $J_1(\varepsilon)$ yields

$$J_1(\varepsilon) = \frac{\varepsilon}{2}\int_0^{-T_0/\varepsilon} \frac{a'(T_0 + \varepsilon s)}{a^2(T_0 + \varepsilon s)}\,\{z'(s)\}^2\, ds. \tag{6.13}$$

Let $Z(s)$ be the solution of the problem obtained from (6.12) by putting $\varepsilon = 0$:

$$\begin{cases} Z'' + a(T_0) f(Z) = 0 \qquad \text{for } s > 0, & (6.14\text{a}) \\ Z'(0) = 0, & (6.14\text{b}) \end{cases}$$

endowed with the properties

$$Z'(s) < 0 \quad \text{and} \quad Z(s) > 0 \quad \text{for } s > 0.$$

These properties imply that $Z(s) \to 0$ as $s \to \infty$. It is well-known that Z is unique.

In order to continue, we need the following result.

Lemma 6.3. *We have*

$$z(s) \to Z(s) \quad \text{and} \quad z'(s) \to Z'(s) \qquad \text{as } \varepsilon \to 0, \tag{6.15}$$

uniformly on bounded intervals.

Proof. Because the family of solutions $\{z(s) : 0 < \varepsilon < \varepsilon_0\}$ is equicontinuous, it follows that

$$z(s) \to Z(s) \qquad \text{as } \varepsilon \to 0$$

along a sequence, uniformly on bounded intervals. Since Z is unique, the entire family converges to Z as $\varepsilon \to 0$. That $z'(s) \to Z'(s)$ is proved in a similar manner. □

It follows from Lemma 6.3 that for any $L > 0$,

$$\int_0^L \frac{a'(T_0 + \varepsilon s)}{a^2(T_0 + \varepsilon s)} \{z'(s)\}^2\, ds \to \frac{a'(T_0)}{a^2(T_0)} \int_0^L \{Z'(s)\}^2\, ds \qquad \text{as } \varepsilon \to 0.$$

Since

$$\frac{1}{\varepsilon} J_1(\varepsilon) = \frac{1}{2} \int_0^{-T_0/\varepsilon} \frac{a'(T_0 + \varepsilon s)}{a^2(T_0 + \varepsilon s)} \{z'(s)\}^2\, ds > \frac{1}{2} \int_0^L \frac{a'(T_0 + \varepsilon s)}{a^2(T_0 + \varepsilon s)} \{z'(s)\}^2\, ds,$$

when $-T_0/\varepsilon \geq L$, it follows that

$$\liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} J_1(\varepsilon) \geq \frac{1}{2} \frac{a'(T_0)}{a^2(T_0)} \int_0^L \{Z'(s)\}^2\, ds > 0,$$

as asserted. This completes the proof of Lemma 6.2. □

Next, we estimate the last term, $F(\gamma_0(\varepsilon))$, in (6.10). To do that, we need an estimate for $\gamma_0(\varepsilon)$, to be presented in Lemma 6.5.

Lemma 6.4. *There exist positive constants β and ε_1 such that*

$$y(0) \leq e^{-\beta/\varepsilon} \qquad \textit{for } \varepsilon \in (0, \varepsilon_1).$$

Proof. Fix $t_0 \in (T_0, 0)$. We first show that for ε small enough,

$$y(t, \gamma_0(\varepsilon)) < 1/2 \qquad \text{for } t_0 < t < \infty. \tag{6.16}$$

Choose $s_0 > 0$ such that $Z(s_0) \leq 1/4$. Because $Z(0) > 1$ and $Z(\infty) = 0$, this is always possible. Then, by Lemma 6.3,

$$y(T_0 + \varepsilon s_0) \to 1/4 \qquad \text{as } \varepsilon \to 0;$$

and hence there exists an $\varepsilon_2 > 0$ such that for $0 < \varepsilon < \varepsilon_2$, we have

$$y(T_0 + \varepsilon s_0) < 1/2 \qquad \text{as well as} \qquad T_0 + \varepsilon s_0 < t_0.$$

Since y is decreasing on (T_0, ∞), it follows that if $\varepsilon < \varepsilon_2$, then

$$y(t) < 1/2 \qquad \text{for } t_0 < t < \infty,$$

as asserted in (6.16).

Next, let $\kappa > 0$ be a constant chosen such that

$$f(s) < -\kappa\, s \qquad \text{for } 0 < s < 1/2, \tag{6.17}$$

and let φ be the solution of the problem

$$\varepsilon^2 \varphi'' - \kappa\, a(t_0)\varphi = 0 \qquad \text{for } |t| < |t_0| \qquad \text{and} \qquad \varphi(\pm t_0) = 1/2. \tag{6.18}$$

We claim that

$$y(t) < \varphi(t) \qquad \text{for } |t| < t_0. \tag{6.19}$$

Note that

$$\begin{aligned} \varepsilon^2 y'' - \kappa\, a(t_0) y &= -\kappa\, a(t_0) y - a(t)\, f(y) \\ &> -\kappa\, a(t_0) y + a(\tau)\kappa y \\ &= \kappa\{\, a(t) - a(t_0)\} y > 0 \qquad \text{for } |t| < |t_0|, \end{aligned}$$

because of (6.17) and because $a(t) > a(t_0)$ for $|t| < |t_0|$. Thus, the function $v = \varphi - y$ satisfies

$$\varepsilon^2 v'' - \kappa\, a(\tau) v < 0 \qquad \text{for } |t| < |t_0| \qquad \text{and} \qquad v(\pm t_0) > 0.$$

Hence, it follows from the Maximum Principle that $v(t) > 0$, so that $y(t) < \varphi(t)$ for $|t| \leq |t_0|$, as asserted in (6.19).

Solving Problem (6.18) explicitly, we conclude that

$$y(t) < \frac{1}{2} \frac{\cosh(\mu t/\varepsilon)}{\cosh(\mu |t_0|/\varepsilon)} \qquad \text{for } |t| \leq |t_0|,$$

where $\mu = \sqrt{\kappa\, a(t_0)}$. In particular, we find that

$$y(0) < 1/\big(2\cosh(\mu|t_0|/\varepsilon)\big) < e^{-\mu|t_0|/\varepsilon}.$$

This completes the proof of Lemma 6.4. □

Lemma 6.5. *For ε sufficiently small, we have*

$$|F(\gamma_0(\varepsilon))| \le Ce^{-2\beta/\varepsilon}$$

for some positive constant C.

Proof. In view of (6.17) we can estimate F by

$$|F(s)| \le (\kappa/2)s^2 \qquad \text{for } 0 < s < (1/2). \tag{6.20}$$

Since y is decreasing on $(0,\infty)$, it follows from Lemma 6.4 that

$$\gamma_0(\varepsilon) < y(0) < e^{-\beta/\varepsilon};$$

and it follows from (6.20) that

$$|F(\gamma_0(\varepsilon))| \le (\kappa/2)e^{-2\beta/\varepsilon}$$

for ε small enough. □

Finally, we estimate $J_2(\varepsilon)$.

Lemma 6.6. *There exist positive constants C and ε_0 such that*

$$J_2(\varepsilon) < C\varepsilon^{-2}e^{-2\beta/\varepsilon}, \qquad 0 < \varepsilon < \varepsilon_0. \tag{6.21}$$

Proof. Integration of equation (6.1a) over (t,∞) yields

$$-y'(t) = -\frac{1}{\varepsilon^2}\int_t^\infty a(s)f(y(s))\,ds < \frac{1}{\varepsilon^2}\int_t^\infty a(s)\,y(s)\,ds,$$

since $f(y) > -y$ for $y > 0$. Because $y(t) < y(0)$ for $t \ge 0$, we conclude that

$$|y'(t)| < \frac{1}{\varepsilon^2}y(0)\int_t^\infty a(s)\,ds < \frac{1}{\varepsilon^2}e^{-\beta/\varepsilon}A(t),$$

where

$$A(t) = \int_t^\infty a(s)\,ds.$$

Thus,

$$J_2(\varepsilon) < \frac{1}{2\varepsilon^2}e^{-2\beta/\varepsilon}\int_0^\infty \frac{|a'(t)|}{a^2(t)}\,A^2(t)\,dt.$$

Because

$$\frac{|a'(t)|}{a^2(t)} = \frac{4t}{(1+t^2)^3}(1+t^2)^4 = 4t(1+t^2) \sim 4t^3 \qquad \text{as } t \to \infty$$

and $A(t) = O(t^{-3})$ as $t \to \infty$, it follows that

$$\big(|a'(t)|/a^2(t)\big)\, A^2(t) = O(t^{-3}) \qquad \text{as } t \to \infty,$$

so that

$$\big(|a'(t)|/a^2(t)\big)\, A^2(t) \in L^1(0,\infty).$$

Therefore,

$$J_2(\varepsilon) < C\varepsilon^{-2}\, e^{-2\beta/\varepsilon}. \qquad \square$$

Summarizing, we have found that there exist constants $A > 0$, $C > 0$ and $\varepsilon_0 > 0$ such that

$$J_1(\varepsilon) > A\varepsilon, \qquad J_2(\varepsilon) < C\varepsilon^{-2}\, e^{-2\beta/\varepsilon}, \qquad |F(\gamma_0(\varepsilon))| < Ce^{-2\beta/\varepsilon}.$$

Using these bounds in (6.10), we conclude that there exists a constant $\alpha > 0$ such that

$$F(y_0) > \alpha\,\varepsilon \qquad \text{for } 0 < \varepsilon < \varepsilon_0.$$

This completes the proof of Lemma 6.1. $\square$

In the following lemma, we complete the proof of Proposition 6.1.

Lemma 6.7. *There exists a constant $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then $T(\gamma_0(\varepsilon)) > -\infty$. Set $y_\varepsilon(t) = y(t, \gamma_0(\varepsilon))$ and $T_\varepsilon = T(\gamma_0(\varepsilon))$. Then we have*

$$\begin{cases} y_\varepsilon'(t) < 0 & \text{for } T_0 < t < \infty, \\ y_\varepsilon'(t) > 0 & \text{for } T_\varepsilon \le t < T_0. \end{cases} \tag{6.22}$$

Proof. Recall that

$$y_\varepsilon(T_0) > \sigma > 1 \qquad \text{and} \qquad y_\varepsilon'(T_0) = 0.$$

Since $y_\varepsilon'' < 0$ as long as $y_\varepsilon > 1$, it follows that the graph of this solution intersects the line $y = 1$, and we can define

$$T_{1,\varepsilon} = \inf\{t < T_0 : y_\varepsilon > 1 \text{ on } (t, T_0)\} > -\infty.$$

Plainly,

$$y_\varepsilon'(T_{1,\varepsilon}) > 0. \tag{6.23}$$

Following the argument we used in the proof of Lemma 6.4, we find that

$$|T_{1,\varepsilon} - T_0| = O(\varepsilon) \qquad \text{as } \varepsilon \to 0. \tag{6.24}$$

In order to estimate $y_\varepsilon(t)$ for $t < T_{1,\varepsilon}$, we use the expression

$$G(t) = (\varepsilon^2/2)\, y_\varepsilon'^2(t) + a(t)F(y_\varepsilon(t)). \tag{6.25}$$

If $y_\varepsilon(t)$ is a solution of equation (6.1a), then

$$G'(t) = a'(t)F(y_\varepsilon(t)) < 0 \qquad \text{as long as } 0 < y_\varepsilon(t) < \sigma, \quad t < 0.$$

If we integrate $G'(t)$ over $(t, T_{1,\varepsilon})$, where $t \in (T_\varepsilon, T_{1,\varepsilon})$, we find that

$$(\varepsilon^2/2)\, y_\varepsilon'^2(t) + a(t)F(y_\varepsilon(t)) > (\varepsilon^2/2)\, y_\varepsilon'^2(T_{1,\varepsilon}) \qquad \text{for } T_\varepsilon \le t < T_{1,\varepsilon}.$$

Using once again the fact that $F < 0$, we conclude that

$$y_\varepsilon'(t) > y_\varepsilon'(T_{1,\varepsilon}) \qquad \text{for } T_\varepsilon \le t < T_{1,\varepsilon},$$

which implies that

$$T > T_{1,\varepsilon} - \sigma/\big(y_\varepsilon'(T_{1,\varepsilon})\big) > -\infty, \tag{6.26}$$

in view of (6.23). This completes the proof of Lemma 6.7 and establishes Proposition 6.1. ☐

We now turn to the

Proof of Proposition 6.2. Let (γ_-, γ_+) be any connected component of $\mathcal{A}$. Recall that $\gamma_- \ge 0$ and that $\gamma_+ \le 1$. Suppose that the assertion of Proposition 6.2 is false and that there exists a sequence $\{\gamma_n\}$ which converges to, say, γ_-, such that $T(\gamma_n)$ converges to a point $T_\infty > -\infty$. Then, by continuity, $\gamma_- \in \mathcal{A}$, which contradicts the definition of γ_-. Similarly for γ_+. This completes the proof of Proposition 6.2. ☐

We conclude with the

Proof of Proposition 6.3. In the proof of Proposition 6.1, we introduced an arbitrary point $T_0 < 0$. We may choose this point arbitrarily close to the origin $t = 0$. In the following lemma, we show that by choosing ε small enough, we can insure that T_ε is arbitrarily close to T_0. Together, these observations show that $T_{\max,\varepsilon} \to 0$ as $\varepsilon \to 0$.

Lemma 6.8. *Let $T_0 \in (-\infty, 0)$, and let $y_\varepsilon(t)$ be the solution of Problem (6.3) constructed in Lemma 6.7. Then*

$$|T_\varepsilon - T_0| = O(\sqrt{\varepsilon}) \qquad \textit{as } \varepsilon \to 0. \tag{6.27}$$

Proof. Note that

$$|T_0 - T_\varepsilon| \le |T_0 - T_{1,\varepsilon}| + |T_{1,\varepsilon} - T_\varepsilon|.$$

Hence, by (6.24) and (6.26),

$$|T_0 - T_\varepsilon| \le \sigma/\big(|y'_\varepsilon(T_{1,\varepsilon})|\big) + O(\varepsilon) \qquad \text{as } \varepsilon \to 0. \tag{6.28}$$

Thus, it remains to estimate $|y'(T_{1,\varepsilon})|$. This we do with the function $H(t)$ defined in (6.4). Since $H' < 0$ on $(T_{1,\varepsilon}, T_0)$, it follows that for some positive constants C and ε_1,

$$\frac{\varepsilon^2}{2}\, y'^2_\varepsilon(T_{1,\varepsilon}) > a(T_{1,\varepsilon})F(y_\varepsilon(T_0)) = \{a(T_0)+O(\varepsilon)\}F(y_\varepsilon(T_0)) > C\varepsilon \qquad \text{for } 0 < \varepsilon < \varepsilon_1,$$

by Lemma 6.1. Therefore,

$$y'_\varepsilon(T_1) > B\varepsilon^{-1/2} \qquad \text{for } 0 < \varepsilon < \varepsilon_1,$$

where B is another positive constant. Putting this inequality into (6.28), we obtain the estimate (6.27) we set out to prove. □

7 Proof of Theorem 1.4; Part II: Multi-spike solutions

In this section, we show that there exist branches of multi-spike solutions similar to the branch of single-spike solutions found in Section 6. The analysis of such solutions is very similar to the one presented for the one-spike solutions. The difference lies in the handling of the additional spikes. For simplicity, we first take $n = 2$. The general case is almost identical.

For convenience, we restate Theorem 1.4 with $n = 2$:

Theorem 7.1. *Given any $T^* < 0$, there exists a constant $\varepsilon_2 > 0$ such that if $\varepsilon < \varepsilon_2$, then Problem (6.1) has at least two solutions with precisely one spike and at least two solutions with two spikes.*

We start as in Section 6 and define the set $\mathcal{A}$. But instead of Proposition 6.1 we prove

Proposition 7.1. *For ε small enough, there exists $\gamma_0 \in \mathcal{A}$ such that the solution $y(t, \gamma_0)$ of Problem (6.3) with initial value γ_0 has precisely two spikes.*

Let $\mathcal{A}_2 = (\gamma_2^-, \gamma_2^+)$ be the connected component of $\mathcal{A}$ which contains the value γ_0 defined in Proposition 7.1.

Proposition 7.2. *We have*

$$T(\gamma) \to -\infty \qquad as\ \gamma \to \gamma_2^{\pm}.$$

Thus, the branch $\Gamma_2 = \{(\gamma, T(\gamma)) : \gamma \in \mathcal{A}_2\}$ has vertical asymptotes at $\gamma^{\pm}$. In Figure 9, we show the graphs of Γ_1, Γ_2, Γ_3 and Γ_4, in the variables r and u in the $(R^*, u(0))$-plane.

Define

$$T^{(2)}_{\max,\varepsilon}(\gamma) \stackrel{\text{def}}{=} \max\{T(\gamma) : \gamma \in \mathcal{A}_2\}.$$

Proposition 7.3.

$$T^{(2)}_{\max,\varepsilon}(\gamma) \to 0 \qquad as\ \varepsilon \to 0.$$

The proofs of Propositions 7.2 and 7.3 are identical to those of Propositions 6.2 and 6.3, so we omit them

It follows from Proposition 7.3 that we can choose ε so small that Γ_2 intersects the line $T = T^*$ at least twice. This yields two solutions of Problem (6.1), each with two spikes.

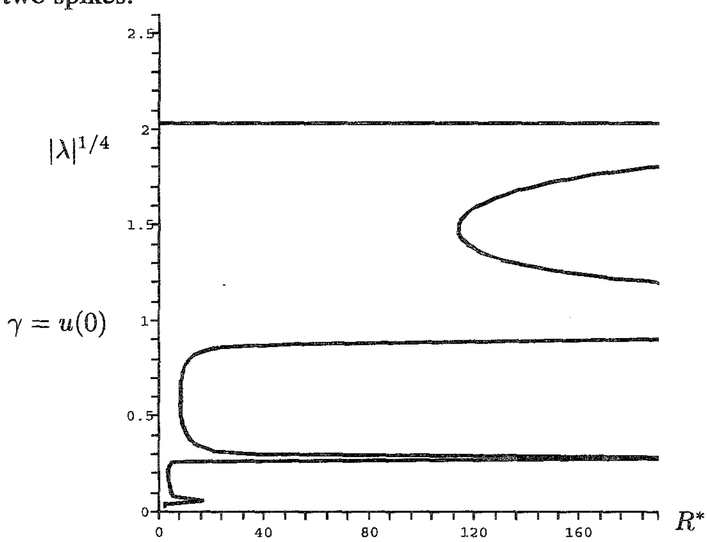

Figure 9. The branches Γ_1, Γ_2, Γ_3 and Γ_4 of one-, two-, three- and four-spike solutions $u(r)$ depicted in the $(R^*, u(0))$-plane for $\lambda = -17$.

Proof of Proposition 7.1. We fix $T_0 \in (T^*, 0)$ and choose ε so small that the second local maximum of the solution ζ of Problem (5.12)— counting from the right — lies in $\mathbf{R}^+$, i.e., $\tau_3^0 > 0$. Then, by lowering the value of γ starting from 1^-, we find a value of $\gamma_0 > 0$ such that

$$(7.1) \qquad \tau_3(\gamma_0) = T_0.$$

Plainly, γ_0 depends on ε; as before, we write $\gamma_0 = \gamma_0(\varepsilon)$ and $\tau_k = \tau_k(\varepsilon)$. By construction, the solution $y = y(t, \gamma_0(\varepsilon))$ has one local *maximum* on the interval (T_0, ∞). It is the largest critical point of y, and it is located at $\tau_1(\varepsilon)$.

Lemma 7.1. *We have*

$$\tau_1(\varepsilon) \to 0 \qquad \textit{as } \varepsilon \to 0. \tag{7.2}$$

For the moment, let us accept this result and continue with the proof of Proposition 7.1. Note that for all ε sufficiently small,

$$T_0 < \tau_2(\varepsilon) < \tau_1(\varepsilon). \tag{7.3}$$

We now integrate $H'(t)$ over $(T_0, \tau_2(\varepsilon))$ and obtain

$$F(y_0) = J(\varepsilon) + F(y(\tau_2)), \qquad J(\varepsilon) = \frac{\varepsilon^2}{2} \int_{T_0}^{\tau_2} \frac{a'}{a^2} (y')^2 \, dt. \tag{7.4}$$

By an analysis very similar to the one given in Section 6, we show that there exist positive constants C_1, C_2 and β such that $J(\varepsilon) > C_1 \varepsilon$ and $|F(y(\tau_2(\varepsilon))| \le C_2 e^{\beta/\varepsilon}$ for ε small enough. This means that there exists a constant $\alpha > 0$ such that

$$F(y_0) > \alpha\, \varepsilon \tag{7.5}$$

for small enough ε. We continue as in Section 6 and show that for ε small, the lower bound (7.5) implies that $y'(t) > 0$ for $t < T_0$ as long as $y(t) \ge 0$. This implies that y has a zero $T_\varepsilon < T_0$ for ε small enough. This completes the proof of Proposition 7.1. $\square$

It remains to prove Lemma 7.1.

Proof of Lemma 7.1. Let

$$\tau_- = \liminf_{\varepsilon \to 0} \tau_1(\varepsilon) \qquad \text{and} \qquad \tau_+ = \limsup_{\varepsilon \to 0} \tau_1(\varepsilon), \tag{7.6}$$

where τ_+ may be infinite and $\tau_- \ge T_0$.

We first prove that

$$\tau_- \ge 0. \tag{7.7}$$

Suppose to the contrary that $T_0 \le \tau_- < 0$. Then, repeating the argument of Section 6, with T_0 replaced by $\tau_1(\varepsilon)$, we find that for ε small enough, the solution $y(t, \gamma_0(\varepsilon))$ has a zero T_ε in a left neighbourhood of τ_- and is strictly increasing on $[T_\varepsilon, \tau_-)$. Since, by construction, $y(t, \gamma_0(\varepsilon))$ has a local maximum at T_0 for every $\varepsilon > 0$, which lies above the line $y = 1$, this is not possible. This completes the proof of (7.7). $\square$

Next we prove that

$$\tau_+ \le 0. \tag{7.8}$$

Suppose that $\tau_+ \in (0,\infty)$. Let $\tau_1(\varepsilon) \to \tau_+$ along a sequence $\{\varepsilon_n\}$ which tends to 0 as $n \to \infty$. Integration of H' over $(\tau_1(\varepsilon_n),\infty)$ yields

$$F(y(\tau_1(\varepsilon))) = \tilde{J}(\varepsilon) + F(\gamma_0(\varepsilon)), \qquad \tilde{J}(\varepsilon) = \frac{\varepsilon^2}{2}\int_{\tau_1}^{\infty} \frac{a'}{a^2}(y')^2\,dt, \tag{7.9}$$

where we have suppressed the subscript n. Note that

$$\tilde{J}(\varepsilon) \sim \frac{\varepsilon}{2}\frac{a'(\tau_+)}{a^2(\tau_+)}\int_0^{\infty} \{Z'(s)\}^2\,ds \qquad \text{as } \varepsilon \to 0;$$

and, as in Lemma 6.5, $F(\gamma_0(\varepsilon)) = O(e^{-2\beta/\varepsilon})$. Thus, because $a' < 0$ on $\mathbf{R}^+$, we find that for some $\tilde{\varepsilon} > 0$ small enough,

$$F(y(\tau_1(\varepsilon))) \le -\alpha\varepsilon \qquad \text{for } 0 < \varepsilon < \tilde{\varepsilon}, \tag{7.10}$$

where α is some positive constant.

We show that (7.10) implies that

$$\tau_3(\varepsilon) > 0 \tag{7.11}$$

for ε small enough. Since by construction, $\tau_3(\varepsilon) = T_0$, this contradicts the fact that $T_0 < 0$; and we may conclude that (7.8) holds.

For technical reasons, we replace $a(t)$ by the nondecreasing function

$$\overline{a}(t) = \begin{cases} a(t) & \text{for } t \ge 0, \\ a(0) & \text{for } t \le 0. \end{cases}$$

Denote the solution of Problem (6.1), with $a(t)$ replaced by $\overline{a}(t)$, $y(t)$ by $\overline{y}(t)$, and its critical points by $\overline{\tau}_k$. Plainly, $\overline{y}(t) = y(t)$ for $t \ge 0$; hence, for ε small enough, $\overline{\tau}_1(\varepsilon) = \tau_1(\varepsilon)$. The functional

$$\overline{H}(t) = \big(\varepsilon/2\overline{a}(t)\big)\{\overline{y}'(t)\}^2 + F(\overline{y}(t))$$

is now nonincreasing on the whole of $\mathbf{R}$. This means that

$$F(\overline{y}(t)) < -\alpha\varepsilon \qquad \text{for } t \le \tau_1, \quad 0 < \varepsilon < \tilde{\varepsilon}. \tag{7.12}$$

Since $\overline{y} = y$ on $\mathbf{R}^+$, to prove (7.11) it suffices to show that

$$\overline{\tau}_3(\varepsilon) > 0 \tag{7.13}$$

for ε small enough.

We recall that

$$\overline{\tau}_3 < \overline{t}_3 < \overline{\tau}_2 < \overline{t}_2 < \tau_1,$$

where $\overline{t}_3$ and $\overline{t}_2$ are the zeros of $\overline{y}$ on $(\overline{\tau}_3, \tau_1)$. We use Sturm Comparison arguments to prove the following estimates:

Lemma 7.2. *There exist constants* $c_0 > 0$ *and* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$,

$$|\tau_1(\varepsilon) - \overline{t}_2(\varepsilon)| \leq c_0\varepsilon, \tag{7.14}$$

$$|\overline{t}_3(\varepsilon) - \overline{\tau}_3(\varepsilon)| < c_0\varepsilon, \tag{7.15}$$

$$|\overline{t}_2(\varepsilon) - \overline{t}_3(\varepsilon)| < c_0\varepsilon^{3/4}. \tag{7.16}$$

Proof. On the intervals $(\overline{t}_2, \tau_1)$ and $(\tau_3, \overline{t}_3)$, we have $\overline{y} > 1$, while on the interval $(\overline{t}_3, \overline{t}_2)$, we have $\overline{y} \in (0,1)$. The proofs of (7.14) – (7.15) and of (7.16) are therefore slightly different, and we deal with them separately.

Part 1: Proof of (7.14) – (7.15). We only establish (7.14). The proof of (7.15) is similar.

We restrict ourselves to the interval $(\overline{t}_2, \tau_1)$ and recall that there $\overline{y} > 1$. Put

$$\overline{y} = 1 + \eta \qquad \text{and} \qquad f(\overline{y}) = \tilde{f}(\eta),$$

so that $\eta > 0$. For η, we then obtain the equation

$$\varepsilon^2\eta'' + \overline{a}(t)\tilde{f}(\eta) = 0. \tag{7.17}$$

Since $f'' > 0$ on $\mathbf{R}$ and $\overline{a}(t)$ is nonincreasing on $\mathbf{R}^+$, it follows that

$$\overline{a}(t)\tilde{f}(\eta) > \overline{a}(\tau_1)f'(1)\eta \overset{\text{def}}{=} k_+^2\,\eta \qquad \text{for } t \leq \tau_1, \quad \eta > 0. \tag{7.18}$$

As a comparison function, we use

$$\varphi(t) = \cos\left((k_+(t - \tau_1))/\varepsilon\right),$$

which is a solution of the problem

$$\varepsilon^2\varphi'' + k_+^2\varphi = 0, \qquad \varphi(\tau_1 - \theta) = 0, \quad \varphi'(\tau_1) = 0, \qquad \theta = \pi\varepsilon/2k_+.$$

We multiply equation (7.17) by φ and integrate over $(\tau_1 - \theta, \tau_1)$. This yields after two integrations by parts, in view of the properties of φ at τ_1 and $\tau_1 - \theta$,

$$-\varepsilon^2\varphi'(\tau_1 - \theta)\eta(\tau_1 - \theta) = \int_{\tau_1-\theta}^{\tau_1} \varphi(t)\{\overline{a}(t)\tilde{f}(\eta(t)) - k_+^2\eta(t)\}\,dt. \tag{7.19}$$

Suppose that $\eta(t) > 0$ on $(\tau_1 - \theta, \tau_1]$. Then by (7.18), the integral in (7.19) is positive. However, since $\varphi'(\tau_1 - \theta) > 0$, the left hand side of (7.19) is negative or zero. We conclude that η must have a zero on $(\tau_1 - \theta, \tau_1)$, so that

$$\tau_1(\varepsilon) - \bar{t}_2(\varepsilon) < \pi\varepsilon/2k_+.$$

In an entirely similar manner, one proves that

$$\bar{t}_3(\varepsilon) - \overline{\tau}_3(\varepsilon) < \pi\varepsilon/2k_+.$$

Part 2: Proof of (7.16). Here we restrict our attention to the interval $(\bar{t}_3, \bar{t}_2)$, where $\overline{y} < 1$. Observe that by (7.12) there exists a constant $b > 0$ such that

$$\overline{y}(t) > b\sqrt{\varepsilon} \qquad \text{for } \bar{t}_3 < t < \bar{t}_2 \qquad \text{and} \qquad 0 < \varepsilon < \tilde{\varepsilon},$$

for some small $\tilde{\varepsilon} > 0$. Since f is convex, we have

$$f(\overline{y}) - f(1) < \frac{f(1) - f(b\sqrt{\varepsilon})}{1 - b\sqrt{\varepsilon}} \cdot (\overline{y} - 1) \qquad \text{for } b\sqrt{\varepsilon} < \overline{y} < 1.$$

Because $f(1) = 0$ and $f(b\sqrt{\varepsilon}) \sim -b\sqrt{\varepsilon}$ as $\varepsilon \to 0$, it follows that for ε small enough,

$$\tilde{f}(\eta) < 2b\sqrt{\varepsilon}\eta \qquad \text{for } -1 + b\sqrt{\varepsilon} < \eta < 0.$$

Using the fact that $\overline{a}(t)$ is nonincreasing, we conclude that

$$\overline{a}(t)\tilde{f}(\eta) < 2b\sqrt{\varepsilon}\,\overline{a}(\bar{t}_2)\eta = \sqrt{\varepsilon}k_-^2\eta, \qquad \text{for } \bar{t}_3 < t < \bar{t}_2, \quad \eta < 0, \tag{7.20}$$

where we have put $k_-^2 = 2b\overline{a}(\bar{t}_2)$. We now use the comparison function

$$\psi(t) = \sin\left(\frac{k_-(t - \bar{t}_2)}{\varepsilon^{3/4}}\right),$$

which is the solution of the problem

$$\varepsilon^{3/2}\psi'' + k_-^2\psi = 0, \qquad \psi(\bar{t}_2 - \vartheta) = 0, \quad \psi(\bar{t}_2) = 0, \qquad \text{where } \vartheta = \pi\varepsilon^{3/4}/k_-.$$

We multiply the equation for η by ψ and integrate over the interval $(\bar{t}_2 - \vartheta, \bar{t}_2)$. Proceeding as in Part 1, we find that η must have a zero on this interval, i.e.,

$$0 < \bar{t}_2(\varepsilon) - \bar{t}_3(\varepsilon) < \pi\varepsilon^{3/4}/k_-.$$

This completes Lemma 7.2. □

Proof of Lemma 7.1, continued. Combining the three bounds of Lemma 7.2, we conclude that

$$\begin{aligned}\overline{\tau}_3 &= \{\overline{\tau}_3(\varepsilon) - \bar{t}_3(\varepsilon)\} + \{\bar{t}_3(\varepsilon) - \bar{t}_2(\varepsilon)\} + \{\bar{t}_2(\varepsilon) - \tau_1(\varepsilon)\} + \tau_1(\varepsilon) \\ &> \tau_1(\varepsilon) - c_0(2\varepsilon + \varepsilon^{3/4}) \qquad \text{for } 0 < \varepsilon < \varepsilon_0.\end{aligned}$$

Therefore, if $\tau_+ > 0$, then for ε small enough, $\overline{\tau}_3 > 0$ as well. This proves (7.13), and the proof of Lemma 7.1 is complete. □

The n-spike solution. Finally, we turn to solutions with n spikes. They are located at the points $\{\tau_{2k-1} : k = 1, 2, \ldots, n\}$. In the construction, we fix $\tau_{2n+1} = T_0$; and we show that the remaining spikes all converge to the origin as $\varepsilon \to 0$. Specifically, we show that

$$\limsup_{\varepsilon \to 0} \tau_1(\varepsilon) \le 0 \qquad \text{and} \qquad \liminf_{\varepsilon \to 0} \tau_{2(n-1)-1}(\varepsilon) \ge 0.$$

This can be done with the methods developed in this section.

This completes the proof of Theorem 1.4. □

8 Further open problems

Let M be a 3-dimensional compact Riemannian manifold without boundary. Consider first the problem

$$\begin{cases} -\Delta_M u = \lambda u + u^5 & \text{in } M, & (8.1a) \\ u > 0 & \text{in } M, & (8.1b) \end{cases}$$

where Δ_M denotes the Laplace–Beltrami operator on M.

Let $\nu_2, \nu_3, \ldots, \nu_k, \ldots$ denote the sequence of positive eigenvalues of $-\Delta_M$ (the first eigenvalue is $\nu_1 = 0$). When $M = \mathbf{S}^3$, we keep the notation $0 = \mu_1 < \mu_2 < \cdots$ for the radial eigenvalues (see Section 4).

Open Problem 8.1. *Is it true that for $-3/4 < \lambda < 0$, the only solution of the problem*

$$\begin{cases} -\Delta_{\mathbf{S}^3} u = \lambda u + u^5 & \text{in } \mathbf{S}^3 & (8.2a) \\ u > 0 & \text{in } \mathbf{S}^3 & (8.2b) \end{cases}$$

is the constant solution $u = |\lambda|^{1/4}$? Same question for the general manifold M when $-\nu_2/4 < \lambda < 0$.

Remark 8.1. After completing this paper, we were informed by Qinian Jin and Y. Y. Li that for the sphere $\mathbf{S}^3$, the answer to Open Problem 8.1 is "Yes" and that the result is due to Gidas and Spruck [13] (see Theorem B.2 in Appendix B). Their argument is quite involved and relies on some remarkable identities. A different proof was recently given by Brezis and Li [5] which relies on the method of moving planes. Concerning the second part of Open Problem 8.1, which deals with general manifolds, it has been established in [5] that for $\lambda < 0$ and $|\lambda|$ sufficiently small, the only solution of (8.2) is constant.

The bifurcation analysis in [5] and [8] suggests that for a generic manifold M, nonconstant solutions of (8.2) exist when $|\lambda+\nu_2/4|$ is sufficiently small, $\lambda \neq -\nu_2/4$

Open Problem 8.2. *Assume that $\lambda < -\nu_2/4$. Is there a non-constant solution of Problem (8.1)? Are there more solutions as $\lambda \to -\infty$?*

Even for the case $M = S^3$, it is not known whether non-constant solutions of (8.2) exist for all $\lambda \in [-2, -3/4)$. By Theorem 1.6, they do exist for $\lambda < -2$. Bifurcation analysis (as in [5] and [8]) shows that nonconstant solutions also exist for $\lambda > -2$, provided $(\lambda + 2)$ is sufficiently small. Recall that when $\lambda = -3/4$, there is a whole family of non-constant solutions (see Remark 4.1).

By analogy with Problem (1.1), let $\omega \subset M$ be an open set with smooth boundary and consider the problem

$$
\begin{cases}
-\Delta_M u = \lambda u + u^5 & \text{in } M \setminus \omega, \quad (8.3a)\\
u > 0 & \text{in } M \setminus \omega, \quad (8.3b)\\
u = 0 & \text{on } \partial(M \setminus \omega). \quad (8.3c)
\end{cases}
$$

Open Problem 8.3. *Which domains ω have the property that a solution of (8.3) exists for all λ sufficiently large negative? For such domains, does the number of solutions increase as $\lambda \to -\infty$?*

In another direction, we have

Open Problem 8.4. *Assume that $\lambda < -\nu_2/4$. Is it true that, for ω "sufficiently small" in some appropriate sense (capacity ?), there are at least two solutions of Problem (8.3)? More generally, assume that $\lambda < -\nu_{2k}/4$ and ω is sufficiently small. Are there at least $2k$ solutions ?*

The techniques introduced by Coron [10] for "small holes" might possibly be useful here.

Similar questions can be asked if M is a d-dimensional manifold and the power 5 is replaced by $p \geq (d+2)/(d-2)$.

Next, we consider the problem

$$
\begin{cases}
-\Delta v + \mu V(x) v = v^5 & \text{in } B_1, \quad (8.4a)\\
v > 0 & \text{in } B_1, \quad (8.4b)\\
v = 0 & \text{on } \partial B_1, \quad (8.4c)
\end{cases}
$$

where $\mu > 0$ and $V(x) > 0$ for $x \in B_1$.

When V is a radial function, the result of [1], [2], combined with the Pohozaev inequality, establishes

Theorem 8.1. *There exists a solution of Problem (8.4) for μ large enough if and only if*

$$r^2 V(r) \text{ has at least 1 critical point on } (0,1). \tag{8.5}$$

Open Problem 8.5. *Assume (8.5). Does the number of solutions of Problem (8.4) increase as μ increases to infinity?*

If $r^2 V(r)$ has a strict maximum, the answer should be "Yes". This is suggested by the results in [16] concerning the corresponding Neumann problem; see also the earlier paper [14].

Some conjectures along these lines have been formulated in [1] and [2] concerning the same problem on all of $\mathbf{R}^N$.

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