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A VARIATIONAL FORMULATION FOR THE TWO-SIDED OBSTACLE PROBLEM WITH MEASURE DATA

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We give a new variational formulation for the two-sided obstacle problem with measure data. This formulation allows to prove in particular that the solution does not depend on the part of the data which is concentrated on a set of zero newtonian capacity.

1. Introduction

We first recall a phenomenon originally observed in [1] for some variational problems of Thomas–Fermi type. Let Ω be a smooth bounded domain in $\mathbb{R}^N$ containing 0 and consider the problem

$$\begin{cases} -\Delta u + |u|^{p-1}u = \delta & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1}$$

It was proved in [1] that if $p \geq \frac{N}{N-2}$ with $N \geq 3$, then (1) has no solution in the sense of distributions with $u \in L^p(\Omega)$. However, in some sense, the unique "natural solution" is $u = 0$. This has to be interpreted in the following way: take any sequence f_n of smooth functions such that $f_n \to \delta$ in the sense of measures and $f_n \to 0$ strongly in $L^1_{loc}(\Omega\backslash\{0\})$ and let u_n be the solution of

$$\begin{cases} -\Delta u_n + |u_n|^{p-1}u_n = f_n & \text{in } \Omega\,, \\ u_n = 0 & \text{on } \partial\Omega\,, \end{cases}$$

then $u_n \to 0$ in L^1, (u_n) is bounded in L^p, $u_n \to 0$ in $L^p_{loc}(\Omega\setminus\{0\})$ and $|u_n|^{p-1}u_n \rightharpoonup \delta$ in the sense of measures (see [3]). Some variants of this result were obtained in [14], for this type of problem as well as for solutions of variational inequalities.

The obstacle problem associated to the constraint $|u| \le 1$ corresponds, roughly speaking, to the case $p = \infty$ in the previous example, i.e.

$$\begin{cases} -\Delta u + \beta(u) \ni \delta & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2}$$

where β is the graph

$$D(\beta) = [-1,1]\,, \quad \beta(u) = \begin{cases} [0,\infty) & \text{if } u = 1\,, \\ \{0\} & \text{if } -1 < u < 1\,, \\ (-\infty, 0] & \text{if } u = -1\,. \end{cases}$$

The standard formulation of problem (2) in terms of variational inequalities when δ is replaced by $f \in H^{-1}$ is

$$\inf_{u\in K} \left\{ \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega fu \right\} \tag{3}$$

where K is the convex set

$$K = \{u \in H^1_0(\Omega);\ |u| \le 1\}$$

or equivalently,

$$\begin{cases} u \in K \\ \forall\, v \in K\,, \quad \displaystyle\int_\Omega \nabla u \cdot \nabla(v-u) \ge \int_\Omega fu\,. \end{cases} \tag{4}$$

The defect of this formulation is that it makes no sense when f is a general measure, e.g. $f = \delta$, because $u \in H^1_0(\Omega)$ need not be continuous when $N \ge 2$. [When $N = 1$, problem (3) admits a unique minimizer for every measure f, and throughout the rest of the paper we will assume that $N \ge 2$.]
To overcome this difficulty, one possible approach is to consider a variant of (3):

$$\inf_{u\in K\cap C^0(\bar\Omega)} \left\{ \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega fu \right\}\,. \tag{5}$$

There are now various natural questions when f is a measure:

(1) Does every minimizing sequence for (5) converge to a limit (independent of the sequence)?

By analogy with the case of problem (1), one may ask:

(2) When $f = \delta$, does every minimizing sequence for (5) converge to 0?

The answers to both questions are indeed positive (see Theorems 1.2 and 1.3). The main new idea in this paper is to dualize problem (3). For $f \in L^2(\Omega)$, the dual problem (in the sense of convex duality, see e.g. [4], and [10] p. 108) of

$$\min_{u\in K}\left\{\frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega fu\right\} \tag{6}$$

is

$$\min_{\substack{h\in H_0^1(\Omega)\\ \Delta h\in\mathcal{M}(\bar\Omega)}}\left\{\frac{1}{2}\int_\Omega |\nabla h|^2 + \int_\Omega |\Delta h + f|\right\}. \tag{7}$$

Here $\mathcal{M}(\bar\Omega)$ denotes the space of Radon measures on $\bar\Omega$, i.e. the dual of $C^0(\bar\Omega)$. More precisely, let us recall the theorem.

Theorem 1.1 ([4]). *For any $f \in L^2(\Omega)$, there exists a unique minimizer of* (6), *say u_*, and $u_* \in H^2(\Omega)$. Moreover, u_* is also the unique minimizer of* (7) *and*

$$\frac{1}{2}\int_\Omega |\nabla u_*|^2 - \int_\Omega fu_* = -\left(\frac{1}{2}\int_\Omega |\nabla u_*|^2 + \int_\Omega |\Delta u_* + f|\right).$$

An interesting application of this theorem is for a problem arising in [15] where the analysis of the Ginzburg–Landau functional with magnetic field yields a minimization problem which is exactly of the kind (7), thus the minimizer is identified with the solution of an obstacle problem (the authors in [15] were not aware of this theorem and gave a direct proof without duality).

When $f \in L^2$ is replaced by a general measure μ, problem (6) does not make sense. However, it does make sense when μ is a special measure belonging to $L^1 + H^{-1}$. By contrast, problem (7) makes sense for a *general* measure μ and admits a unique minimizer.

Definition 1.1. The solution of the obstacle problem with data $\mu \in \mathcal{M}(\bar\Omega)$ is the unique minimizer of

$$\min_{\substack{h\in H_0^1(\Omega)\\ \Delta h\in\mathcal{M}(\bar\Omega)}}\left\{\frac{1}{2}\int_\Omega |\nabla h|^2 + \int_\Omega |\Delta h + \mu|\right\}. \tag{8}$$

Our first result relates this minimization problem with

$$\inf_{u\in K\cap C^0(\bar\Omega)}\left\{\frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega u\mu\right\}. \tag{9}$$

Theorem 1.2. *For every measure $\mu \in \mathcal{M}(\bar\Omega)$, we have*

$$\inf_{u\in K\cap C^0(\bar\Omega)}\left\{\frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega u\mu\right\}$$
$$= -\min_{\substack{h\in H_0^1(\Omega)\\ \Delta h\in\mathcal{M}(\bar\Omega)}}\left\{\frac{1}{2}\int_\Omega |\nabla h|^2 + \int_\Omega |\Delta h + \mu|\right\}. \tag{10}$$

Moreover every minimizing sequence u_n *for* (9) *converges strongly in* $H_0^1(\Omega)$ *to the solution of the obstacle problem* (*in the sense of Definition* 1.1).

The only proof that we have for (10) is not via a simple regularization as one might expect; it is quite indirect and relies on the decomposition of the measure μ described below. Returning to the case of the Dirac mass, we may now see that the solution of the obstacle problem (in the sense of Definition 1.1) with $\mu = \delta$ is $h = 0$. Indeed, since a point has zero H^1-capacity, there is a sequence of smooth $u_n \in H_0^1(\Omega) \cap C^0(\bar{\Omega})$ such that $0 \leq u_n \leq 1$, $u_n(0) = 1$ and $u_n \to 0$ in $H_0^1(\Omega)$; hence

$$\inf_{K \cap C^0(\bar{\Omega})} \left\{ \frac{1}{2} \int_\Omega |\nabla u|^2 - u(0) \right\} \leq -1 \,.$$

Therefore, by Theorem 1.2,

$$\min_{\substack{h \in H_0^1(\Omega) \\ \Delta h \in \mathcal{M}(\bar{\Omega})}} \left\{ \frac{1}{2} \int_\Omega |\nabla h|^2 + \int_\Omega |\Delta h + \delta| \right\} \geq 1 \,;$$

hence $h = 0$ is the unique minimizer and

$$\inf_{K \cap C^0(\bar{\Omega})} \left\{ \frac{1}{2} \int_\Omega |\nabla u|^2 - u(0) \right\} = -1 \,.$$

This is a special case of a more general phenomenon, namely the solution "does not feel" the part of the measure μ which is concentrated on a set of zero capacity. To make this rigorous, we need the following lemma. The first assertion can be found in [11], the second will be proved at the beginning of Sec. 2.

Lemma 1.1. *Let* μ *be a Radon measure and* $N \geq 2$. *Then* μ *can be decomposed in a unique way as* $\tilde{\mu} + \nu$ *where*

- $\tilde{\mu}$ *is a measure that vanishes on sets of zero capacity*
- ν *is a measure concentrated on a set of zero capacity.*

Moreover,

$$\begin{cases} \mu_+ = \tilde{\mu}_+ + \nu_+ \\ \mu_- = \tilde{\mu}_- + \nu_- \,, \end{cases} \tag{11}$$

where, for any measure, μ_+, μ_- *denote the positive and negative parts of* μ.

Here, capacity refers to the standard newtonian capacity, i.e. H_0^1-capacity. Another useful lemma is

Lemma 1.2 ([7]). *A Radon measure* $\tilde{\mu}$ *vanishes on sets of zero capacity if and only if it admits a decomposition*

$$\tilde{\mu} = k + \eta$$

with $k \in L^1(\Omega)$, $\eta \in H^{-1}(\Omega)$. *Here, the decomposition is not unique.*

In particular, measures in H^{-1} vanish on sets of zero-capacity. A partial variant of Lemma 1.2, when H_0^1-capacity is replaced with $W^{2,p}$-capacities, was first proved in [12], using a lemma from [2]. In view of Lemma 1.2, we see that $\int_\Omega \tilde{\mu} u$ makes sense for every $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ provided we interpret $\int_\Omega fu$ as $\int_\Omega ku + \langle \eta, u\rangle$ where $\int_\Omega ku$ makes sense since $k \in L^1(\Omega)$ and $u \in L^\infty(\Omega)$, while $\langle \eta, u\rangle$ makes sense since $u \in H_0^1(\Omega)$. In particular, problem (6) makes sense when $f = \tilde{\mu}$, and admits a unique minimizer.

Theorem 1.3. *Let μ be any measure in $\mathcal{M}(\bar{\Omega})$ and let $\mu = \tilde{\mu} + \nu$ be its decomposition in the sense of Lemma* 1.1, *then*

$$
\begin{aligned}
&- \min_{\substack{h \in H_0^1(\Omega) \\ \Delta h \in \mathcal{M}(\bar{\Omega})}} \frac{1}{2} \int_\Omega |\nabla h|^2 + \int_\Omega |\Delta h + \mu| \\
&\quad = \min_{u \in K} \left\{ \frac{1}{2} \int_\Omega |\nabla u|^2 - \int_\Omega u \tilde{\mu} \right\} - \int_\Omega |\nu| \,. \\
&\quad = - \min_{\substack{h \in H_0^1(\Omega) \\ \Delta h \in \mathcal{M}(\bar{\Omega})}} \left\{ \frac{1}{2} \int_\Omega |\nabla h|^2 + \int_\Omega |\Delta h + \tilde{\mu}| \right\} - \int_\Omega |\nu|
\end{aligned}
$$

and minimizers coincide, *i.e. the solutions of the obstacle problem for μ and $\tilde{\mu}$ are the same.*

Another approach to the one-sided obstacle problem $u \geq -1$ with measure data was given in [8]. Their formulation is quite different from ours. It does not rely on duality but instead on methods of potential theory, i.e. the solution is the smallest function in H_0^1 satisfying

$$
\begin{cases} -\Delta v \geq \mu \,, \\ v \geq -1 \,. \end{cases} \tag{12}
$$

Their main result asserts that the smallest function exists. Moreover, it was proved in [9, Theorem 4.3] that this solution is unchanged when μ is replaced by $\tilde{\mu} + \nu_+$ (ν_+ being the positive part of ν). We believe that there is a similar formulation for the two-sided obstacle problem $-1 \leq u \leq 1$ and that it coincides with the solution we have obtained. At least when $\mu \leq 0$, the solution in the sense of (12) coincides with our notion of solution.

In Sec. 2 we present the proof of Theorems 1.2 and 1.3. In Sec. 3, we give some additional results and generalizations.

2. Proof of Theorems 1.2 and 1.3

As announced we start with the

Proof of the second assertion of Lemma 1.1. Let ν_+ and ν_- denote, respectively, the positive and negative parts of the measure ν. We recall that ν

is concentrated on a set of zero-capacity. We claim that there exist sequences ξ_n and ζ_n of $C_0^\infty(\Omega)$ functions that satisfy

$$\begin{cases} 0 \le \xi_n \le 1 & 0 \le \zeta_n \le 1\,, \\ \|\xi_n\|_{H^1(\Omega)} \to 0 & \|\zeta_n\|_{H^1(\Omega)} \to 0\,, \\ \xi_n \to 0 \text{ a.e.} & \zeta_n \to 0 \text{ a.e.}\,, \\ 0 \le \displaystyle\int_\Omega (1-\xi_n)\nu_+ \to 0 & 0 \le \displaystyle\int_\Omega (1-\zeta_n)\nu_- \to 0\,, \\ 0 \le \displaystyle\int_\Omega \xi_n \nu_- \to 0 & 0 \le \displaystyle\int_\Omega \zeta_n \nu_+ \to 0\,. \end{cases} \tag{13}$$

This can be found, for example, in [14, Lemma 2.1]. For the convenience of the reader, we present here a simpler proof.

Let us fix some $\delta > 0$. By definition of $\int_\Omega |\nu|$, there exists $\omega \in C_0^\infty(\Omega)$ such that $|\omega| \le 1$ and

$$\int_\Omega \omega\nu \ge \int_\Omega |\nu| - \delta\,. \tag{14}$$

Since cap(supp ν) $= 0$, there exists $\eta_n \in C_0^\infty(\Omega)$ such that $0 \le \eta_n \le 1$, $\eta_n = 1$ in a neighborhood of the support of ν, and $\|\eta_n\|_{H^1(\Omega)} \to 0$. Then, using the fact that $\eta_n = 1$ in a neighborhood of the support of ν, we have

$$\int_\Omega \eta_n\omega\nu = \int_\Omega \omega\nu \ge \int_\Omega |\nu| - \delta\,.$$

It is immediate to check that $\|\eta_n\omega\|_{H^1(\Omega)} \to 0$. Thus, for N large enough,

$$\|\eta_N\omega\|_{H^1(\Omega)} < \delta$$

and

$$\int_\Omega (\eta_N\omega)\nu \ge \int_\Omega |\nu| - \delta\,.$$

Since this is true for any $\delta > 0$, this means that we can find a sequence ω_n in $C_0^\infty(\Omega)$ such that

$$\int_\Omega \omega_n\nu \to \int_\Omega |\nu|\,, \quad \|\omega_n\|_{H^1(\Omega)} \to 0\,.$$

Then, $\xi_n = \omega_n^+$ and $\zeta_n = \omega_n^-$ (respectively positive and negative parts of ω) provide what is needed since

$$\int_\Omega (\omega_n^+ - \omega_n^-)(\nu_+ - \nu_-) \to \int_\Omega |\nu|\,.$$

Indeed this means that

$$\int_\Omega (1-\xi_n)\nu_+ + \int_\Omega \xi_n\nu_- + \int_\Omega (1-\zeta_n)\nu_- + \int_\Omega \zeta_n\nu_+ \to 0$$

but each term in this sum is nonnegative, hence each term tends to zero. This completes the proof of (13).

Consider ψ a nonnegative test-function in $C^1(\bar{\Omega})$, and $\tilde{\mu}$ given by the second assertion of the lemma. Remark that

$$\int_\Omega \psi\tilde{\mu}_+ = \sup_{\substack{0\le u\le 1\\ u\in C^\infty(\bar{\Omega})}} \int_\Omega \psi u\tilde{\mu} \quad \int_\Omega \psi\mu_+ = \sup_{\substack{0\le u\le 1\\ u\in C^\infty(\bar{\Omega})}} \int_\Omega \psi u\mu_+ \,. \tag{15}$$

Let us fix a $\delta > 0$, and consider u in $C^\infty(\bar{\Omega})$ such that $0 \le u \le 1$ and $\int_\Omega \psi u\tilde{\mu} \ge \int_\Omega \psi\tilde{\mu}_+ - \delta$. We then use the functions ξ_n and ζ_n defined in (13) and set $v_n = \xi_n(1-u) + (1-\zeta_n)u$. First, $0 \le v_n \le 1$. In addition, $(v_n - u)\psi \to 0$ in H_0^1. Indeed,

$$\nabla(v_n - u) = (1-u)\nabla\xi_n - u\nabla\zeta_n - (\xi_n + \zeta_n)\nabla u \,.$$

The first two terms tend to 0 in $L^2(\Omega)$ by strong H^1 convergence of ξ_n and ζ_n to 0. For the second term, observe that $\xi_n + \zeta_n \to 0$ a.e., and $|\xi_n + \zeta_n| \le 2$, while $|\nabla u|^2 \in L^1$, hence applying Lebesgue's dominated convergence theorem,

$$\int_\Omega |\xi_n + \zeta_n|^2 |\nabla u|^2 \to 0 \,.$$

Thus $(v_n - u)\psi \to 0$ in H_0^1, and a.e.

Next, using the result of Lemma 1.3, we can decompose $\tilde{\mu}$ as $k + \eta$ with $k \in L^1(\Omega)$, $\eta \in H^{-1}(\Omega)$. Using Lebesgue's dominated convergence theorem again, $\int_\Omega \psi k v_n \to \int_\Omega \psi k u$, and by the strong $H_0^1(\Omega)$-convergence of ψv_n to ψu, we have $\langle \eta, \psi v_n\rangle \to \langle \eta, \psi u\rangle$, thus we deduce that

$$\int_\Omega \psi v_n\tilde{\mu} = \int_\Omega \psi k v_n + \langle \eta, \psi v_n\rangle \to \int_\Omega \psi k u + \langle \eta, \psi u\rangle \,. \tag{16}$$

On the other hand,

$$\begin{aligned} \left|\int_\Omega \psi(v_n - 1)\nu_+\right| &\le \left|\int_\Omega \psi((1-\xi_n)(1-u) + \zeta_n u)\nu_+\right| \\ &\le 2\int_\Omega (1-\xi_n)\nu_+ + \zeta_n\nu_+ \to 0 \end{aligned}$$

in view of (13). Similarly,

$$\begin{aligned} \left|\int_\Omega \psi v_n\nu_-\right| &\le \left|\int_\Omega \psi(\xi_n(1-u) + (1-\zeta_n)u)\nu_-\right| \\ &\le 2\int_\Omega \xi_n\nu_- + (1-\zeta_n)\nu_- \to 0 \,. \end{aligned}$$

We can now write

$$\begin{aligned} \int_\Omega \psi v_n\mu = \int_\Omega \psi v_n(\tilde{\mu} + \nu_+ - \nu_-) &\to \int_\Omega \psi u\tilde{\mu} + \int_\Omega \psi\nu_+ \\ &\ge \int_\Omega \psi\tilde{\mu}_+ + \int_\Omega \psi\nu_+ - \delta \,. \end{aligned}$$

Therefore,

$$\sup_{\substack{0 \le v \le 1 \\ v \in C^\infty(\bar{\Omega})}} \int_\Omega \psi v \mu \ge \int_\Omega \psi \tilde{\mu}_+ + \int_\Omega \psi \nu_+ \,,$$

and using (15) again, we deduce that

$$\int_\Omega \psi \mu_+ \ge \int_\Omega \psi \tilde{\mu}_+ + \psi \nu_+ \,. \tag{17}$$

Similarly, considering $0 \le u \le 1$ such that $\int_\Omega \psi u \tilde{\mu} \ge \int_\Omega \psi \tilde{\mu}_- - \delta$ and using $v_n = -\zeta_n(1-u) - (1-\xi_n)u$, we have $-1 \le v_n \le 0$ and $\int_\Omega v_n u \psi \mu \to \int_\Omega \psi u \tilde{\mu}_- + \psi \nu_-$. Finally, we are led to

$$\int_\Omega \psi \mu_- \ge \int_\Omega \psi \tilde{\mu}_- + \psi \nu_- \,. \tag{18}$$

Adding (17) and (18), we get

$$\int_\Omega \psi |\mu| \ge \int_\Omega \psi(\tilde{\mu}_+ + \nu_+ + \tilde{\mu}_- + \nu_-) = \int_\Omega \psi |\tilde{\mu}| + \psi |\nu| \,.$$

But, by the triangle inequality, the converse inequality also holds, thus there is equality in each inequality (17) and (18), which proves that

$$\int_\Omega \psi \mu_+ = \int_\Omega \psi(\tilde{\mu}_+ + \nu_+)$$

$$\int_\Omega \psi \mu_- = \int_\Omega \psi(\tilde{\mu}_- + \nu_-) \,.$$

Since this is true for every nonnegative $\psi \in C^1(\bar{\Omega})$, we deduce the result. $\square$

Consider now the Hilbert space $H_0^1(\Omega)$ equipped with its standard scalar product $\langle u, v \rangle_{H_0^1} = \int_\Omega \nabla u \cdot \nabla v$. Given $\mu \in \mathcal{M}(\bar{\Omega})$, consider the convex function $\Phi_\mu : H_0^1 \to (-\infty, +\infty]$ defined by

$$\Phi_\mu(h) = \begin{cases} \displaystyle\int_\Omega |\Delta h + \mu| & \text{if } \Delta h \in \mathcal{M}(\bar{\Omega}) \\ +\infty & \text{otherwise}\,. \end{cases}$$

Lemma 2.1. *We have*

$$\Phi_\mu^*(f) = \begin{cases} \displaystyle\int_\Omega f \tilde{\mu} - \int_\Omega |\nu| & \text{if } |f| \le 1 \ a.e.\,, \\ +\infty & \text{otherwise}\,, \end{cases}$$

where Φ_μ^* *denotes the conjugate of the convex function* Φ_μ *i.e.*

$$\Phi_\mu^*(f) = \sup_{g \in D(\Phi_\mu)} (\langle f, g \rangle_{H_0^1} - \Phi_\mu(g)) \,.$$

Proof. Step 1. We have

$$\Phi_\mu^* = \Phi_{\tilde\mu}^* - \int_\Omega |\nu| \,. \tag{19}$$

Indeed,

$$\Phi_\mu^*(f) = \sup_{g\in D(\Phi_\mu)} \left(\int_\Omega \nabla f\cdot\nabla g - \int_\Omega |\Delta g + \mu| \right) .$$

Consider any $g \in H_0^1(\Omega) \cap D(\Phi_\mu)$; clearly $\Delta g \in H^{-1}(\Omega) \cap \mathcal{M}(\bar\Omega)$ hence $\Delta g + \tilde\mu$ is a Radon measure that vanishes on sets of zero-capacity. Thus, using the second assertion of Lemma 1.1, we can write

$$\int_\Omega |\Delta g + \mu| = \int_\Omega |\Delta g + \tilde\mu| + \int_\Omega |\nu| \,,$$

and we deduce (19). Hence, there only remains to compute $\Phi_{\tilde\mu}^*$.

Step 2. We claim that *for any* $f \in H_0^1(\Omega)$,

$$\Phi_\mu^*(f) \geq \begin{cases} \displaystyle\int_\Omega f\tilde\mu - \int_\Omega |\nu| & \text{if } |f| \leq 1 \text{ a.e.}\,, \\ +\infty & \text{otherwise.} \end{cases} \tag{20}$$

In view of Step 1, it suffices to prove that

$$\Phi_{\tilde\mu}^*(f) \geq \begin{cases} \displaystyle\int_\Omega f\tilde\mu & \text{if } |f| \leq 1 \text{ a.e.}\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

After integration by parts, we find

$$\Phi_{\tilde\mu}^*(f) = \sup_{g\in D(\Phi_{\tilde\mu})} \left(-\int_\Omega f\Delta g - \int_\Omega |\Delta g + \tilde\mu| \right) .$$

Thus,

$$\Phi_{\tilde\mu}^*(f) = \sup_{\substack{h\in H^{-1}(\Omega)\\ h\in\mathcal{M}(\bar\Omega)}} \int_\Omega fh - |h - \tilde\mu| \,.$$

Using the decomposition $\tilde\mu = k + \eta$ given by Lemma 1.2, we can choose $h = \eta + \zeta$ with ζ arbitrary in $L^2(\Omega)$, and write

$$\Phi_{\tilde\mu}^*(f) \geq \sup_{\zeta\in L^2} \int_\Omega f(\eta+\zeta) - |\zeta - k| \tag{21}$$

$$\geq \sup_{\zeta\in L^2} \int_\Omega f\zeta - |\zeta| + \int_\Omega f\eta - |k| \,. \tag{22}$$

Thus, if $|f| \leq 1$ a.e. is not satisfied, we deduce from (22) that $\Phi_{\tilde\mu}^*(f) \geq +\infty$. If $|f| \leq 1$ a.e., then taking in (21) a sequence ζ_n in L^2 which converges strongly to k in L^1, we deduce that

$$\Phi_{\tilde\mu}^*(f) \geq \int_\Omega f(\eta + k) = \int_\Omega f\tilde\mu \,. \tag{23}$$

Step 3. We prove the converse inequality. If $f \in H_0^1(\Omega) \cap C^0(\bar{\Omega})$ and $|f| \le 1$ a.e., then, for all $g \in H_0^1(\Omega) \cap D(\Phi_{\tilde{\mu}})$,

$$\int_\Omega \nabla f \cdot \nabla g - f\tilde{\mu} = \int_\Omega -f\Delta g - f\tilde{\mu} \le \int_\Omega |\Delta g + \tilde{\mu}|\,,$$

hence $\Phi_{\tilde{\mu}}^*(f) \le \int_\Omega f\tilde{\mu}$. Combining this with (23), we deduce that

$$\forall\, f \in H_0^1(\Omega) \cap C^0(\bar{\Omega})\,, \quad \Phi_{\tilde{\mu}}^*(f) = \begin{cases} \displaystyle\int_\Omega f\tilde{\mu} & \text{if } |f| \le 1 \text{ a.e.}\,, \\ +\infty & \text{otherwise}\,. \end{cases} \tag{24}$$

Let now $f \in H_0^1(\Omega)$ with $|f| \le 1$ a.e. We can find a sequence $f_n \in H_0^1 \cap C^0(\bar{\Omega})$ such that

$$\begin{cases} f_n \to f \text{ in } H_0^1(\Omega)\,, \\ f_n \to f \text{ a.e.}\,, \\ |f_n| \le 1 \text{ a.e.}\,. \end{cases}$$

Then, like for (16), we have

$$\int_\Omega f_n\tilde{\mu} = \int_\Omega kf_n + \langle \eta, f_n\rangle \to \int_\Omega kf + \langle \eta, f\rangle = \int_\Omega f\tilde{\mu}\,. \tag{25}$$

Consequently, using the lower semi-continuity of $\Phi_{\tilde{\mu}}^*$ and (24),

$$\Phi_{\tilde{\mu}}^*(f) \le \liminf \Phi_{\tilde{\mu}}^*(f_n) = \liminf \int_\Omega f_n\tilde{\mu} = \int_\Omega f\tilde{\mu}\,.$$

With (20) and (19), this completes the proof of the lemma. $\square$

We are going to use the following standard lemma about duality of convex functionals.

Lemma 2.2. *Let Φ be convex lower semi-continuous from a Hilbert space H to $(-\infty, +\infty]$, then*

$$\min_{h \in H} \left(\frac{1}{2}\|h\|_H^2 + \Phi(h) \right) = -\min_{u \in H} \left(\frac{1}{2}\|u\|_H^2 + \Phi^*(-u) \right)$$

and minimizers coincide.

Combining this with Lemma 2.1, once for Φ_μ and once for $\Phi_{\tilde{\mu}}$, we deduce that

$$\begin{aligned} &-\min_{\substack{h \in H_0^1(\Omega) \\ \Delta h \in \mathcal{M}(\bar{\Omega})}} \left\{ \frac{1}{2}\int_\Omega |\nabla h|^2 + \int_\Omega |\Delta h + \mu| \right\} \\ &\quad = \min_{u \in K} \left\{ \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega u\tilde{\mu} - \int_\Omega |\nu| \right\} \\ &\quad = -\min_{\substack{h \in H_0^1(\Omega) \\ \Delta h \in \mathcal{M}(\bar{\Omega})}} \left\{ \frac{1}{2}\int_\Omega |\nabla h|^2 + \int_\Omega |\Delta h + \tilde{\mu}| \right\} - \int_\Omega |\nu|\,. \end{aligned}$$

and minimizers coincide. This concludes the proof of Theorem 1.3. In order to complete the proof of Theorem 1.2, there only remains to prove

Lemma 2.3. *We have*

$$\inf_{u\in K\cap C^0(\bar\Omega)}\left\{\frac{1}{2}\int_\Omega|\nabla u|^2-\int_\Omega u\mu\right\}=\min_{u\in K}\left\{\frac{1}{2}\int_\Omega|\nabla u|^2-\int_\Omega u\tilde\mu\right\}-\int_\Omega|\nu|\,.$$

Proof. Let u be the solution of $\min_K\{\frac{1}{2}\int_\Omega|\nabla u|^2-\int_\Omega u\tilde\mu\}-\int_\Omega|\nu|$. We can find a sequence $u_n\in H_0^1(\Omega)\cap C^0(\bar\Omega)$ such that

$$\begin{cases} u_n\to u \text{ in } H_0^1(\Omega)\,,\\ u_n\to u \text{ a.e.}\,,\\ |u_n|\le 1 \text{ a.e.}\,. \end{cases}\tag{26}$$

Using the functions ξ_n and ζ_n introduced in (13), we set

$$v_n=u_n+\xi_n(1-u_n)+\zeta_n(-1-u_n)\,.\tag{27}$$

Observe that $v_n\in C^0(\bar\Omega)\cap K$, hence it is a suitable test-function for our problem. We just need to prove that

$$\frac{1}{2}\int_\Omega|\nabla v_n|^2-\int_\Omega v_n\mu\to\frac{1}{2}\int_\Omega|\nabla u|^2-\int_\Omega u\tilde\mu-\int_\Omega|\nu|\,.\tag{28}$$

First, we show that $v_n-u\to 0$ in H_0^1 and a.e. Indeed,

$$\begin{aligned}\nabla(v_n-u)&=\nabla(u_n-u)+(1-u_n)\nabla\xi_n-\xi_n(\nabla u_n-\nabla u)-\xi_n\nabla u\\ &\quad+(-1-u_n)\nabla\zeta_n-\zeta_n(\nabla u_n-\nabla u)-\zeta_n\nabla u\,.\end{aligned}\tag{29}$$

But, observe that $\xi_n\to 0$ a.e. and $|\xi_n|\le 1$, while $|\nabla u|^2\in L^1(\Omega)$ hence applying Lebesgue's dominated convergence theorem,

$$\int_\Omega|\xi_n|^2|\nabla u|^2\to 0\,.$$

Similarly, $\int_\Omega|\zeta_n|^2|\nabla u|^2\to 0$, and all the other terms of (29) tend to 0 in L^2 by construction of u_n and (13), hence we deduce that $v_n\to u$ in $H_0^1(\Omega)$, and also a.e. Like for (16), we deduce that $\int_\Omega v_n\tilde\mu\to\int_\Omega u\tilde\mu$. Then,

$$\begin{aligned}\int_\Omega v_n\mu&=\int_\Omega v_n\tilde\mu+v_n\nu\\ &=\int_\Omega u\tilde\mu+\int_\Omega\nu_++\nu_-+\int_\Omega(v_n-1)\nu_+-\int_\Omega(v_n+1)\nu_-+o(1)\,.\end{aligned}$$

But,

$$\begin{aligned}\left|\int_\Omega(v_n-1)\nu_+\right|&=\left|\int_\Omega\left((1-\xi_n)(u_n-1)+\zeta_n(-1-u_n)\right)\nu_+\right|\\ &\le 2\int_\Omega(1-\xi_n)\nu_++\zeta_n\nu_+\to 0\,,\end{aligned}\tag{30}$$

in view of (13). Similarly,

$$\left|\int_\Omega (v_n+1)\nu_-\right| = \left|\int_\Omega ((1-\zeta_n)(u_n+1)+\xi_n(1-u_n))\,\nu_-\right|$$
$$\le 2\int_\Omega (1-\zeta_n)\nu_- + \xi_n\nu_- \to 0\,.$$

We deduce that

$$\int_\Omega v_n\mu \to \int_\Omega u\tilde{\mu} + \int_\Omega \nu_+ + \nu_-\,. \tag{31}$$

With the strong H_0^1-convergence of v_n to u, we deduce (28). This proves that

$$\inf_{K\cap C^0(\bar\Omega)} \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega v\mu \le \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega u\tilde{\mu} - \int_\Omega |\nu|$$
$$= \min_K \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega v\tilde{\mu} - \int_\Omega |\nu|\,.$$

On the other hand, clearly,

$$\inf_{K\cap C^0(\bar\Omega)} \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega v\mu \ge \inf_{K\cap C^0(\bar\Omega)} \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega v\tilde{\mu} - \int_\Omega |\nu|$$
$$\ge \min_K \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega v\tilde{\mu} - \int_\Omega |\nu|\,.$$

Thus, there is equality, and if v_n is a minimizing sequence for (9), it is also a minimizing sequence for $\min_K \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega v\tilde{\mu}$, thus converges strongly to the solution of the obstacle problem. □

3. Additional Results and Generalizations

3.1. *Generalizations*

All our results can be generalized to the case of general obstacles $\psi \le u \le \varphi$, where ψ and φ belong to $H^1(\Omega)\cap C^0(\bar\Omega)$ and are such that $\psi \le \varphi$ in Ω, $\psi \le 0$ on $\partial\Omega$ and $\varphi \ge 0$ on $\partial\Omega$. The convex set

$$K_{\varphi,\psi} = \{h \in H_0^1(\Omega);\ \psi \le h \le \varphi\}$$

is nonempty. Indeed, it contains for example the function

$$\omega = \max(\min(\varphi,0),\psi)\,,$$

which is also continuous. In this case, the solution of the obstacle problem with measure data μ and obstacles φ and ψ is defined as the unique minimizer of the convex functional

$$\min_{\substack{h\in H_0^1(\Omega)\\ \Delta h\in\mathcal{M}(\bar\Omega)}} \frac{1}{2}\int_\Omega |\nabla h|^2 + \int_\Omega \varphi|(\Delta h+\mu)_-| - \psi|(\Delta h+\mu)_+|\,, \tag{32}$$

where $(\Delta h + \mu)_+$ and $(\Delta h + \mu)_-$ denote the positive and negative parts of the measure $\Delta h + \mu$. Note that this functional is indeed convex because

$$\int_\Omega \varphi|(\Delta h + \mu)_-| - \psi|(\Delta h + \mu)_+| = \int_\Omega (\varphi - \omega)|(\Delta h + \mu)_-| + (\omega - \psi)|(\Delta h + \mu)_+| - \int_\Omega \omega(\Delta h + \mu)$$

which is convex in h (since $\varphi - \omega \geq 0$ and $\omega - \psi \geq 0$).

Theorem 3.1. *Let μ be any measure in $\mathcal{M}(\bar\Omega)$ and let $\mu = \tilde\mu + \nu$ be its decomposition in the sense of Lemma* 1.1, *then*

$$\begin{aligned} &- \min_{\substack{h\in H_0^1(\Omega)\\ \Delta h\in\mathcal{M}(\bar\Omega)}} \frac{1}{2}\int_\Omega |\nabla h|^2 + \int_\Omega \varphi|(\Delta h + \mu)_-| - \psi|(\Delta h + \mu)_+| \\ &= \min_{u\in K_{\varphi,\psi}} \left\{\frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega u\tilde\mu\right\} - \int_\Omega \varphi\nu_- - \psi\nu_+ \\ &= - \min_{\substack{h\in H_0^1(\Omega)\\ \Delta h\in\mathcal{M}(\bar\Omega)}} \left\{\frac{1}{2}\int_\Omega |\nabla h|^2 + \int_\Omega \varphi|(\Delta h + \tilde\mu)_-| - \psi|(\Delta h + \tilde\mu)_+|\right\} \\ &\quad - \int_\Omega \varphi\nu_- - \psi\nu_+ \,, \end{aligned} \tag{33}$$

and minimizers coincide, *i.e. the solutions of the obstacle problem for μ and $\tilde\mu$ are the same. Moreover*,

$$\begin{aligned} \inf_{u\in K_{\varphi,\psi}\cap C^0(\bar\Omega)} \left\{\frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega \mu u\right\} = &- \min_{\substack{h\in H_0^1(\Omega)\\ \Delta h\in\mathcal{M}(\bar\Omega)}} \frac{1}{2}\int_\Omega |\nabla h|^2 + \int_\Omega \varphi|(\Delta h + \mu)_-| \\ &- \psi|(\Delta h + \mu)_+| - \int_\Omega \varphi\nu_+ - \psi\nu_- \,, \end{aligned} \tag{34}$$

and minimizing sequences for this infimum converge strongly in $H_0^1(\Omega)$ to the solution of the obstacle problem.

Proof. The proof is basically the same as for the case of obstacles 1 and -1. One defines

$$\Phi_\mu(h) = \begin{cases} \displaystyle\int_\Omega \varphi|(\Delta h + \tilde\mu)_-| - \psi|(\Delta h + \tilde\mu)_+| & \text{if } \Delta h \in \mathcal{M}(\bar\Omega)\,, \\ +\infty & \text{otherwise}\,. \end{cases} \tag{35}$$

Using Lemma 1.1 and arguing as in Lemma 2.1,

$$\Phi_\mu^*(f) = \sup_{g\in D(\Phi_\mu)} \int_\Omega \nabla f\cdot\nabla g - \Phi_\mu(g) = \Phi_{\tilde\mu}^*(f) - \int_\Omega (\varphi\nu_- - \psi\nu_+)\,. \tag{36}$$

Then, we claim that

$$\Phi^*_{\tilde\mu}(f) = \begin{cases} \displaystyle\int_\Omega f\tilde\mu & \text{if } \psi \le f \le \varphi\,, \\ +\infty & \text{othewise}\,. \end{cases} \tag{37}$$

Indeed,

$$\begin{aligned}\Phi^*_{\tilde\mu}(f) &= \sup_{g\in D(\Phi_{\tilde\mu})} -\int_\Omega f\Delta g - \int_\Omega \varphi(\Delta g+\tilde\mu)_- - \psi(\Delta g+\tilde\mu)_+ \\ &= \sup_{\substack{h\in H^{-1}(\Omega)\\ h\in\mathcal{M}(\bar\Omega)}} \int_\Omega -fh - \varphi(h+\tilde\mu)_- + \psi(h+\tilde\mu)_+\,.\end{aligned}$$

Using $\tilde\mu = k+\eta$ and taking $h = -\eta+\zeta$ with $\zeta \in L^2(\Omega)$, we have

$$\Phi^*_{\tilde\mu}(f) \ge \sup_{\zeta\in L^2} \int_\Omega f(\eta-\zeta) - \varphi(k+\zeta)_- + \psi(k+\zeta)_+\,.$$

If $\psi \le f \le \varphi$ is not satisfied, then $\Phi^*_{\tilde\mu}(f) \ge +\infty$, and if it is, taking a sequence ζ_n which converges strongly to $-k$ in L^1, one gets $\Phi^*_{\tilde\mu}(f) \ge \int_\Omega f\tilde\mu$ which proves one inequality in (37). Conversely, if $f \in H^1_0(\Omega)\cap C^0(\bar\Omega)$, and $\psi \le f \le \varphi$ then

$$\int_\Omega \nabla f\cdot\nabla g - \int_\Omega f\tilde\mu = \int_\Omega -f\Delta g - f\tilde\mu \le \int_\Omega \varphi(\Delta g+\tilde\mu)_- - \psi(\Delta g+\tilde\mu)_+\,,$$

hence $\Phi^*_{\tilde\mu} \le \int_\Omega f\tilde\mu$. The rest of the proof is as in Lemma 2.1, and one gets (37). Therefore, applying Lemma 2.2, we obtain the first part of the theorem.
For the second part, we adjust the proof of Lemma 2.3 as follows. Consider u, the minimizer of

$$\min_{K_{\varphi,\psi}} \frac12\int_\Omega |\nabla u|^2 - \int_\Omega u\tilde\mu\,,$$

and approximate it in H^1_0 and a.e. by a sequence $u_n \in K_{\varphi,\psi}\cap C^0(\bar\Omega)$. Define

$$v_n = u_n + \xi_n(\varphi-u_n) + \zeta_n(\psi-u_n)\,. \tag{38}$$

First $v_n \to u$ in $H^1_0(\Omega)$ and a.e., indeed,

$$\begin{aligned}\nabla(v_n-u) &= \nabla(u_n-u) + (\varphi-u_n)\nabla\xi_n + (\psi-u_n)\nabla\zeta_n + \xi_n(\nabla\varphi-\nabla u) \\ &\quad + \zeta_n(\nabla\psi-\nabla u) + (\xi_n+\zeta_n)(\nabla u-\nabla u_n)\,, \end{aligned}\tag{39}$$

and all the terms tend to 0 strongly in $L^2(\Omega)$. Like for (16) and (25), one has $\int_\Omega v_n\tilde\mu \to \int_\Omega u\tilde\mu$. In addition,

$$\int_\Omega v_n\nu = \int_\Omega \varphi\nu_+ - \psi\nu_- + \int_\Omega (v_n-\varphi)\nu_+ - (v_n-\psi)\nu_-\,.$$

But, like for (30),

$$\left|\int_\Omega (v_n - \varphi)\nu_+\right| \to 0\,.$$

$$\left|\int_\Omega (v_n - \psi)\nu_-\right| \to 0\,.$$

We deduce that

$$\int_\Omega v_n \mu \to \int_\Omega u\tilde{\mu} + \int_\Omega \varphi\nu_+ - \psi\nu_-$$

and

$$\begin{aligned}\inf_{K_{\varphi,\psi}\cap C^0(\bar\Omega)} \left(\frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega v\mu\right) &\le \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega u\tilde{\mu} - \int_\Omega \varphi\nu_+ - \psi\nu_- \\ &= \min_{K_{\varphi,\psi}} \left(\frac{1}{2}|\nabla u|^2 - \int_\Omega u\tilde{\mu}\right) - \int_\Omega \varphi\nu_+ - \psi\nu_-\,,\end{aligned} \tag{40}$$

while the converse inequality is straightforward. Thus

$$\begin{aligned}&\inf_{K_{\varphi,\psi}\cap C^0(\bar\Omega)} \left(\frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega v\mu\right) \\ &\quad= \min_{K_{\varphi,\psi}} \left(\frac{1}{2}|\nabla u|^2 - \int_\Omega u\tilde{\mu}\right) - \int_\Omega \varphi\nu_+ - \psi\nu_- \\ &\quad= -\min_{\substack{h\in H_0^1(\Omega)\\ \Delta h\in\mathcal{M}(\bar\Omega)}} \left(\frac{1}{2}\int_\Omega |\nabla h|^2 + \int_\Omega \varphi|(\Delta h+\tilde{\mu})_-| - \psi|(\Delta h+\tilde{\mu})_+|\right) \\ &\qquad - \int_\Omega \varphi\nu_+ - \psi\nu_-\,.\end{aligned}$$

This completes the proof of Theorem 3.1. □

Remark 3.1. The approach followed here immediately generalizes to data in $\mathcal{M}(\bar\Omega) + H^{-1}(\Omega)$. Indeed, if the data is $\mu + g$ with $\mu \in \mathcal{M}(\bar\Omega)$, g in H^{-1}, the decomposition $\mu + g = (\tilde{\mu} + g) + \nu$ holds, the proofs of Theorems 1.2, 1.3 and 3.1 still apply, and the solution of the obstacle problem coincides with the solution for the data $\tilde{\mu} + g$ (and is the solution of the variational problem $\min_K \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega u(\tilde{\mu} + g)$ which makes sense). The class of testing functions for the dual problem becomes $\{h \in H_0^1(\Omega); \Delta h + \mu \in \mathcal{M}(\bar\Omega)\}$.

More general scalar products on H_0^1 such as $\int_\Omega \nabla f \cdot \nabla g + fg$ can be considered as well.

3.2. Additional results

The solution u of the obstacle problem with data μ, a general measure, satisfies $\Delta u \in H^{-1} \cap \mathcal{M}(\bar{\Omega})$. A natural question is to ask whether the measure Δu is in L^1. Here is a partial result in that direction. (We don't know any necessary and sufficient condition on μ which guarantees that $\Delta u \in L^1$.)

Theorem 3.2. *Let $\mu \in \mathcal{M}(\bar{\Omega})$ and $\mu = \tilde{\mu} + \nu$ its decomposition given by Lemma* 1.1. *Let u be the solution of the obstacle problem with obstacles $+1, -1$ and data μ. If $\tilde{\mu} \in L^1$, then $\Delta u \in L^1$.*

This relies on the following theorem by Brezis and Strauss [5]: let β be a maximal monotone graph, then $\forall\, f \in L^1(\Omega)$, there exists a unique $u \in W_0^{1,1}(\Omega)$, with $\Delta u \in L^1(\Omega)$ satisfying

$$-\Delta u + \beta(u) \ni f\,.$$

In the case of the obstacle problem, this solution coincides with our solution. This can be proved by approximating f by a sequence of smooth functions f_n converging to f in L^1. Let u_n be the solution of the obstacle problem for f_n. By the regularity theory [13, 6], one knows that $\Delta u_n \in L^\infty$. One can also check that $u_n \to u$ in H_0^1, where u is the solution of the obstacle problem for f. By the method of Brezis–Strauss [5] (Cauchy sequence argument), we also know that Δu_n is a Cauchy sequence in L^1, thus $\Delta u \in L^1$.

The conclusion of Theorem 3.2 is not true for a general measure μ, as can be shown by the following counter-example. Take $\Omega = B(0,1)$ and the radially symmetric function f defined by

$$\begin{cases} f(r) = 1 & \text{for } 0 \le r \le \dfrac{1}{2} \\[2ex] f(r) = 2 - 2r & \text{for } \dfrac{1}{2} \le r \le 1\,. \end{cases} \tag{41}$$

One can easily check that $f \in H_0^1$ and Δf is a measure which is not in L^1. Then, f satisfies the obstacle constraint $|f| \le 1$ and is the solution of the obstacle problem with data $\mu = -\Delta f$, but Δf is not in L^1. One could think that a result such as $\Delta u + \mu \in L^1$ or $\Delta u + \tilde{\mu} \in L^1$ for u solution of the obstacle problem with data μ, could hold. But again, this is wrong for general measures and obstacles. To see this, consider again the function f defined in (41). Then, let u be the solution of the obstacle problem with obstacles -1 and 1 and with data $\mu = -\Delta f|_{\{\frac{1}{2} < r < 1\}}$, ($\mu \in L^1$), i.e. solution of

$$\min_{u \in H_0^1 \cap K_{1,-1}} \frac{1}{2} \int_\Omega |\nabla u|^2 + \int_{\frac{1}{2} \le r \le 1} u \Delta f\,. \tag{42}$$

We claim that $u = f$. From the duality argument, u minimizes

$$\min_{\substack{h \in H_0^1(\Omega) \\ \Delta h \in \mathcal{M}(\bar{\Omega})}} \frac{1}{2} \int_\Omega |\nabla h|^2 + \int_\Omega |\Delta h + \mu|\,. \tag{43}$$

Taking f as test-function in (42), one has

$$\min_{u\in H_0^1\cap K_{1,-1}} \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_{\frac{1}{2}\le r\le 1} u\Delta f \le -\frac{1}{2}\int_\Omega |\nabla f|^2 + \pi f'_d\left(\frac{1}{2}\right),$$

where $f'_d(\frac{1}{2})$ is the right-derivative of f at $\frac{1}{2}$ i.e. -2. On the other hand, taking f as a test-function in (43), we get

$$\min_{\substack{h\in H_0^1(\Omega)\\ \Delta h\in\mathcal{M}(\bar\Omega)}} \frac{1}{2}\int_\Omega |\nabla h|^2 + \int_\Omega |\Delta h+\mu| \le \frac{1}{2}\int_\Omega |\nabla f|^2 + \pi\left|f'_d\left(\frac{1}{2}\right)\right|.$$

But in view of the duality result (Theorem 1.3), the two minima are the same up to a minus sign, hence there was equality, and $u = f$. Now it is easy to check that $\Delta f + \mu$ is not in L^1.

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