LIMITING EMBEDDING THEOREMS FOR $W^{s,p}$ WHEN $s \uparrow 1$ AND APPLICATIONS

By

JEAN BOURGAIN, HAÏM BREZIS AND PETRU MIRONESCU

Dedicated to the memory of T. Wolff

1 Introduction

This is a follow-up of our paper [3], where we establish that

$$\lim_{s\uparrow 1}(1-s)\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}}dxdy \sim \|\nabla f\|^p_{L^p(\Omega)}, \tag{1}$$

for any $p \in [1,\infty)$, where Ω is a smooth bounded domain in $\mathbb{R}^d$, $d \geq 1$.

On the other hand, if $0 < s < 1$, $p > 1$ and $sp < d$, the Sobolev inequality for fractional Sobolev spaces (see, e.g., [1], Theorem 7.57 or [6], Section 3.3) asserts that

$$\|f\|^p_{W^{s,p}(\Omega)} \geq C(s,p,d)\|f - \textstyle\fint f\|^p_{L^q(\Omega)}, \tag{2}$$

where

$$1/q = 1/p - s/d. \tag{3}$$

Here we use the standard semi-norm on $W^{s,p}$

$$\|f\|^p_{W^{s,p}(\Omega)} = \int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}}dxdy. \tag{4}$$

When $s = 1$, the analogue of (2) is the classical Sobolev inequality

$$\|\nabla f\|^p_{L^p(\Omega)} \geq C(p,d)\|f - \textstyle\fint f\|^p_{L^{p^*}(\Omega)}, \tag{5}$$

where

$$1/p^* = 1/p - 1/d \quad \text{and} \quad 1 \leq p < d.$$

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The behaviour of the best constant $C(p,d)$ in (5) as $p \uparrow d$ is known (see, e.g., [5], Section 7.7 and also Remark 1 below); more precisely, one has

$$\|\nabla f\|^p_{L^p(\Omega)} \geq C(d)(d-p)^{p-1}\|f - {\textstyle\fint} f\|^p_{L^{p^*}(\Omega)}. \tag{6}$$

Putting together (1), (4) and (6) suggests that (2) holds with

$$C(s,p,d) = C(d)(d-sp)^{p-1}/(1-s), \tag{7}$$

for all $s < 1$, s close to 1 and $sp < d$.

This is indeed our main result. For simplicity, we work with $\Omega =$ the unit cube Q in $\mathbb{R}^d$.

Theorem 1. *Assume $d \geq 1, p \geq 1, 1/2 \leq s < 1$ and $sp < 1$. Then*

$$\int_Q \int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}}dxdy \geq C(d)\frac{(d-sp)^{p-1}}{1-s}\|f - {\textstyle\fint} f\|^p_{L^q(Q)}, \tag{8}$$

where q is given by (3) and $C(d)$ depends only on d.

As can be seen from (8), there are two phenomena that govern the behaviour of the constant in (8). As $s \uparrow 1$, the constant gets bigger; while as $s \uparrow d/p$, the constant deteriorates. This explains why we consider several cases in the proof.

As an application of Theorem 1 with $p = 1$ and $f = \chi_A$, the characteristic function of a measurable set $A \subset Q$, we easily obtain

Corollary 1. *For all $0 < \varepsilon \leq 1/2$,*

$$|A|\,|{}^cA| \leq \left(C(d)\varepsilon \int_A \int_{{}^cA} \frac{dxdy}{|x-y|^{d+1-\varepsilon}}\right)^{d/(d-1+\varepsilon)}. \tag{9}$$

Note that in the special case $d = 1$, (9) takes the simple form

$$|A|\,|{}^cA| \leq \left(C^*\varepsilon \int_A \int_{{}^cA} \frac{dxdy}{|x-y|^{2-\varepsilon}}\right)^{1/\varepsilon} \tag{10}$$

for some absolute constant C^*. Estimate (10) is sharp, as can be easily seen when A is an interval.

The conclusion of Corollary 1 is related to a result stated in [3] (Remark 4). There is, however, an important difference. In [3], the set A was *fixed* (independent of ε); and the statement there provides a bound for $|A|\,|{}^cA|$ in terms of the limit, as $\varepsilon \to 0$, of the RHS in (9). The improved version—which requires a more delicate argument—is used in Section 7; we apply Corollary 1 (with $d = 1$) to give a proof of a result announced in [2] (Remark E.1). Namely, on $\Omega = (-1, +1)$, consider the function

$$\varphi_\varepsilon(x) = \begin{cases} 0 & \text{for } -1 < x < 0, \\ 2\pi x/\delta & \text{for } 0 < x < \delta, \\ 2\pi & \text{for } \delta < x < 1, \end{cases}$$

where $\delta = e^{-1/\varepsilon}, \varepsilon > 0$ small.

Set $u_\varepsilon = e^{i\varphi_\varepsilon}$. It is easy to check (by scaling) that

$$\|u_\varepsilon\|_{H^{1/2}} = \|u_\varepsilon - 1\|_{H^{1/2}} \le C$$

as $\varepsilon \to 0$ and consequently $\|u_\varepsilon\|_{H^{(1-\varepsilon)/2}} \le C$ as $\varepsilon \to 0$. On the other hand, a straightforward computation shows that $\|\varphi_\varepsilon\|_{H^{(1-\varepsilon)/2}} \sim \varepsilon^{-1/2}$.

The result announced in [2] asserts that *any* lifting ψ_ε of u_ε blows up in $H^{(1-\varepsilon)/2}$ (at least) in the same rate as φ_ε:

Theorem 2. *Let $\psi_\varepsilon : \Omega \to \mathbb{R}$ be any measurable function such that $u_\varepsilon = e^{i\psi_\varepsilon}$. Then*

$$\|\psi_\varepsilon\|_{H^{(1-\varepsilon)/2}} \ge c\varepsilon^{-1/2}, \quad \varepsilon \in (0, 1/2),$$

for some absolute constant $c > 0$.

Remark 1. There are various versions of the Sobolev inequality (5). All these forms hold with equivalent constants:

Form 1. $\|\nabla f\|_{L^p(Q)} \ge A_1 \|f - \fint_Q f\|_{L^q(Q)}$, $f \in W^{1,p}(Q)$.

Form 2. $\|\nabla f\|_{L^p(Q)} \ge A_2 \|f - \fint_Q f\|_{L^q(Q)}$, for all Q-periodic functions $f \in W^{1,p}_{loc}(\mathbb{R}^d)$.

Form 3. $\|\nabla f\|_{L^p(\mathbb{R}^d)} \ge A_3 \|f\|_{L^q(\mathbb{R}^d)}$, $f \in C_0^\infty(\mathbb{R}^d)$.

Form 1 $\Rightarrow$ Form 2. Obvious with $A_2 = A_1$.

Form 2 $\Rightarrow$ Form 1. Any function $f \in W^{1,p}(Q)$ can be extended by reflections to a periodic function on a larger cube $\widetilde{Q}$, so that Form 2 implies Form 1 with $A_1 \ge C A_2$, and C depends only on d.

Form 1 $\Rightarrow$ Form 3. By scale invariance, Form 1 holds with the same constant A_1 on the cube Q_R of side R. Fix a function $f \in C_0^\infty(\mathbb{R}^d)$ and let $R > \text{diam}\,(\text{Supp}\, f)$. We have

$$\|\nabla f\|_{L^p(Q_R)} \ge A_1 \|f - \fint_{Q_R} f\|_{L^q(Q_R)}.$$

As $R \to \infty$, we obtain Form 3 with $A_3 = A_1$.

Form 3 $\Rightarrow$ Form 2. Given a smooth periodic function f on $\mathbb{R}^d$, let ρ be a smooth cut-off function with $\rho = 1$ on Q and $\rho = 0$ outside $2Q$. Then

$$\|\nabla(\rho f)\|_{L^p(\mathbb{R}^d)} \geq A_3\|\rho f\|_{L^q(\mathbb{R}^d)}$$

and thus

$$A_3\|f\|_{L^q(Q)} \leq C(\|\nabla f\|_{L^p(Q)} + \|f\|_{L^p(Q)}),$$

where C depends only on d. Replacing f by $(f - \int_Q f)$ and applying Poincaré's inequality (see, e.g., [5], Section 7.8) yields

$$A_3\|f - \int f\|_{L^q(Q)} \leq C\|\nabla f\|_{L^q(Q)}.$$

The reader will check easily that the same considerations hold for the fractional Sobolev norms such as in (8). The proof of the last implication (Form 3 $\Rightarrow$ Form 2) involves a Poincaré-type inequality. What we use here is the following

Fact. Let $1 \leq p < \infty$, $1/2 \leq s < 1$; then

$$(1-s)\int_Q\int_Q \frac{|f(x) - f(y)|^p}{|x-y|^{d+sp}} \geq c(d)\|f - \int_Q f\|^p_{L^p(Q)}.$$

The proof of this fact is left to the reader. (It is an adaptation of the argument in the beginning of Section 5. In (3) of Section 5 one uses an obvious lower bound:

$$(3) \geq c\Big(\sum_r \|f_r\|_{L^p}\Big)^p \geq c\|f - \int f\|^p_{L^p}.)$$

For the convenience of the reader, we have divided the proof of Theorem 1 into several cases. The plan of the paper is the following.

1. Introduction.
2. Proof of Theorem 1 when $p = 1$ and $d = 1$.
3. Proof of Theorem 1 when $p = 1$ and $d \geq 2$.
4. Square function inequalities.
5. Proof of Theorem 1 when $1 < p < 2$.
6. Proof of Theorem 1 when $p \geq 2$.
7. Proof of Theorem 2.

Appendix: Proof of the square function inequality.

2 Proof of Theorem 1 when $p = 1$ and $d = 1$

For simplicity, we work with periodic functions of period 2π (for non-periodic functions, see Remark 1 in the Introduction). All integrals, L^p norms, etc., are understood on the interval $(0, 2\pi)$. We must prove that (with $\varepsilon = 1 - s$) for all $\varepsilon \in (0, 1/2]$,

$$C\varepsilon \iint \frac{|f(x) - f(y)|}{|x-y|^{2-\varepsilon}} dx dy \geq \|f - \textstyle\int f\|_{L^{1/\varepsilon}}. \tag{1}$$

Write the left side as

$$\varepsilon \int \frac{1}{|h|^{2-\varepsilon}} \|f - f_h\|_1 dh \sim \varepsilon \sum_{k \geq 0} 2^{k(2-\varepsilon)} \int_{|h| \sim 2^{-k}} \|f - f_h\|_1 dh. \tag{2}$$

For $|h| \sim 2^{-k}$,

$$\begin{aligned} \|f - f_h\|_1 &\geq \|(f - f_h) * F_{N_k}\|_1 \\ &= \left(N_k = 2^{k-100}, F_N(x) = \sum_{|n| \leq N} \frac{N - |n|}{N} e^{inx} = \text{Féjer kernel}\right), \end{aligned}$$

$$\left\| \sum_{|n| < N_k} \frac{N_k - |n|}{N_k} \hat{f}(n)(e^{inh} - 1) e^{inx} \right\|_1 \sim$$

$$2^{-k} \left\| \sum_{|n| < N_k} \frac{N_k - |n|}{N_k} n \hat{f}(n) e^{inx} \right\|_1 \qquad \text{(by the choice of } N_k).$$

This last equivalence is justified via a smooth truncation as in the following

Lemma 1. $\left\|\sum_{|n|<N} \hat{f}(n)(e^{inh} - 1)e^{inx}\right\|_1 \gtrsim \frac{1}{N} \left\|\sum_{|n|<N} n\hat{f}(n)e^{inx}\right\|_1$ *for* $|h| < \frac{1}{100N}$.

Proof. Write

$$\left\| \sum_{|n|<N} n\hat{f}(n)e^{inx} \right\|_1 \leq \left\| \sum_{|n|<N} \hat{f}(n)(e^{inh} - 1)e^{inx} \right\|_1 \cdot \left\| \sum \varphi\left(\frac{n}{N}\right) \frac{n}{e^{inh} - 1} e^{inx} \right\|_1,$$

where $0 \leq \varphi \leq 1$ is a smooth function with

$$\varphi(t) = \begin{cases} 1 & \text{for } |t| \leq 1, \\ 0 & \text{for } |t| \geq 2. \end{cases}$$

We have from assumption

$$\left\| \sum \varphi\left(\frac{n}{N}\right) \frac{n}{e^{inh} - 1} e^{inx} \right\|_1 \sim N \left\| \sum \varphi\left(\frac{n}{N}\right) \frac{nh}{e^{inh} - 1} e^{inx} \right\|_1$$

and the second factor remains uniformly bounded. This may be seen by expanding

$$\frac{y}{e^{iy} - 1} \sim \frac{1}{i} + O(y)$$

for $|y| < \frac{1}{50}$ and using standard multiplier bounds.

We now return to the proof of Theorem 1 ($p = 1, d = 1$).
Substitution in (2) thus gives

$$\varepsilon \sum_{k \geq 0} 2^{-\varepsilon k} \Big\| \sum_{|n| < N_k} \frac{N_k - |n|}{N_k} n \hat{f}(n) e^{inx} \Big\|_1 . \tag{3}$$

Set

$$k_0 = 10/\varepsilon.$$

For $k_0 < k < 2k_0$, minorate (using Lemma 1)

$$\Big\| \sum_{|n| < N_k} \frac{N_k - |n|}{N_k} n \hat{f}(n) e^{inx} \Big\|_1 \gtrsim \Big\| \sum_{|n| < N_{k_0}} \frac{N_{k_0} - |n|}{N_{k_0}} n \hat{f}(n) e^{inx} \Big\|_1$$

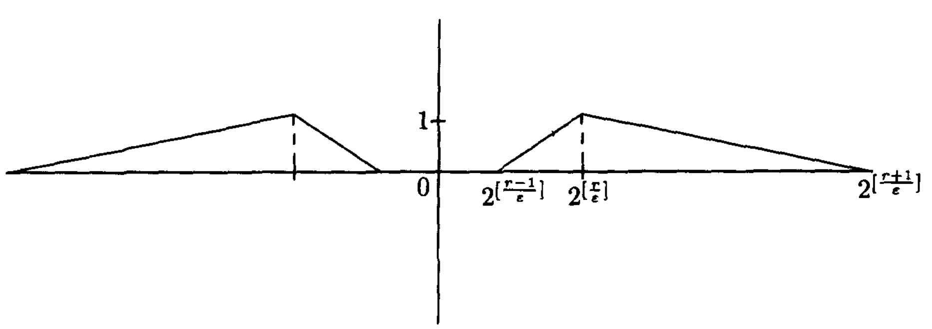
and therefore

$$\begin{aligned} (3) &\gtrsim \Big\| \sum_{|n| < N_{k_0}} \frac{N_{k_0} - |n|}{N_{k_0}} n \hat{f}(n) e^{inx} \Big\|_1 \\ &= \Big\| \sum_{|n| < N_{k_0}} \frac{N_{k_0} - |n|}{N_{k_0}} \hat{f}(n) e^{inx} \Big\|_{W^{1,1}} \\ &\geq \Big\| \sum_{0 < |n| < N_{k_0}} \frac{N_{k_0} - |n|}{N_{k_0}} \hat{f}(n) e^{inx} \Big\|_{\infty} . \end{aligned} \tag{4}$$

Next write also

$$\begin{aligned} (3) &\gtrsim \varepsilon \sum_{r \geq 1} 2^{-r} \sum_{[\frac{r+2}{\varepsilon}] \leq k < [\frac{r+3}{\varepsilon}]} \Big\| \sum_{|n| < N_k} \frac{N_k - |n|}{N_k} n \hat{f}(n) e^{inx} \Big\|_1 \\ &\gtrsim \sum_{r \geq 1} 2^{-r} \Big\| \sum_{|n| < 2^{[\frac{r+1}{\varepsilon}]}} \frac{2^{[\frac{r+1}{\varepsilon}]} - |n|}{2^{[\frac{r+1}{\varepsilon}]}} e^{inx} \Big\|_1 . \end{aligned} \tag{5}$$

Denote for each r by $\lambda_r = \{\lambda_r(n) : n \in \mathbb{Z}\}$ the following multiplier



Thus

$$\lambda_r(n) = \lambda_r(-n) \quad \text{and} \quad \left\| \sum \lambda_r(n) e^{inx} \right\|_1 < C.$$

(This multiplier may be reconstructed from Féjer kernels F_N with $N = 2^{[\frac{r+1}{\varepsilon}]}, 2^{[\frac{r}{\varepsilon}]}, 2^{[\frac{r-1}{\varepsilon}]}$.)

Also

$$(6) \quad \left\| \sum_{|n|<2^{[\frac{r+1}{\varepsilon}]}} \frac{2^{[\frac{r+1}{\varepsilon}]} - |n|}{2^{[\frac{r+1}{\varepsilon}]}} n \hat{f}(n) e^{inx} \right\|_1 \gtrsim \left\| \sum_{2^{[\frac{r-1}{\varepsilon}]}<|n|<2^{[\frac{r+1}{\varepsilon}]}} \lambda_r(n)\, n \hat{f}(n) e^{inx} \right\|_1$$

and

$$(7) \quad (5) \gtrsim \sum_{r\geq 1} 2^{-r} \left\| \sum_{2^{[\frac{r-1}{\varepsilon}]}<|n|<2^{[\frac{r+1}{\varepsilon}]}} \lambda_r(n)(\text{sign } n)|n|\, \hat{f}(n) e^{inx} \right\|_1 .$$

We claim that for $q > 2$,

$$(8) \quad \left\| \sum_{N_1<|n|<N_2} \hat{g}(n) e^{inx} \right\|_q \leq C N_1^{-1/q} \left\| \sum_{N_1<|n|<N_2} |n|(\text{sign } n) \hat{g}(n) e^{inx} \right\|_1,$$

with the constant C independent of q.

Applying (8) with

$$q = 1/\varepsilon, \quad \hat{g}(n) = \lambda_r(n)\hat{f}(n), \quad N_1 = 2^{[\frac{r-1}{\varepsilon}]},\ N_2 = 2^{[\frac{r+1}{\varepsilon}]},$$

we obtain the minoration

$$(9) \quad (7) \gtrsim \sum_{r\geq 1} \left\| \sum_{2^{[\frac{r-1}{\varepsilon}]}<|n|<2^{[\frac{r+1}{\varepsilon}]}} \lambda_r(n)\hat{f}(n) e^{inx} \right\|_q .$$

By construction,

$$\sum_{r\geq 1} \lambda_r(n) = 1 \quad \text{for } |n| > 2^{[1/\varepsilon]}.$$

Using (4) together with the triangle inequality yields

$$\text{LHS in (1)} \gtrsim (3) + (8) \gtrsim \left\| \sum_{n\neq 0} \hat{f}(n) e^{inx} \right\|_q,$$

which proves the inequality.

Proof of (8). Estimate

$$\left\| \sum_{N_1<|n|<N_2} \hat{g}(n) e^{inx} \right\|_q \leq$$

$$\left\| \sum_{N_1<|n|<N_2} |n|^{-1}(\text{sign } n) e^{inx} \right\|_q \left\| \sum_{N_1<|n|<N_2} |n|(\text{sign } n)\hat{g}(n) e^{inx} \right\|_1,$$

where the first factor equals

$$\Big\| \sum_{N_1<n<N_2} \frac{1}{n} \sin nx \Big\|_q$$

$$\lesssim \Big\| \sum_{\log N_1<k<\log N_2} \Big| \sum_{n\sim 2^k} \frac{1}{n} \sin nx \Big| \Big\|_q \qquad \text{(assume } N_1, N_2 \text{ powers of 2)}$$

$$\lesssim \Big\| \sum_{\log N_1<k<\log N_2} \min(2^k|x|, 2^{-k}|x|^{-1}) \Big\|_q$$

$$\lesssim \Big\| \frac{1}{1+N_1|x|} \Big\|_q \lesssim N_1^{-1/q}. \tag{10}$$

This proves (8) and completes the proof of Theorem 1 when $p=1$ and $d=1$.

3 Proof of Theorem 1 when $p=1$ and $d \geq 2$

We have to prove that

$$\iint \frac{|f(x)-f(y)|}{|x-y|^{d+s}} dxdy \geq \frac{C(d)}{1-s} \|f - \textstyle\int f\|_q, \tag{1}$$

where $q = d/(d-s)$. We assume $d=2$. The case $d>2$ is similar. Write

$$\iint \frac{|f(x)-f(y)|}{|x-y|^{d+s}} dxdy \sim \sum_{0\leq k} 2^{k(d+s)} \int_{|h|\sim 2^{-k-10}} \|f(x+h) - f(x-h)\|_1 dh$$

$$\geq \sum 2^{k(d+s)} \underset{\substack{|h_1|\sim 2^{-k-10}\\ |h_2|\sim 2^{-k-10}}}{int} \Big\| \sum_{n\in \mathbb{Z}^d} \hat{f}(n)(\sin n.h) e^{in.x} \Big\|_1 dh_1 dh_2. \tag{2}$$

Let φ be a smooth function on $\mathbb{R}$ such that $0 \leq \varphi \leq 1$ and

$$\varphi(t) = \begin{cases} 1 & \text{for } |t| \leq 1, \\ 0 & \text{for } |t| \geq 2. \end{cases}$$

As for $d=1$, consider (radial) multipliers λ_0 and $\lambda_r, r \geq 1$,

$$\lambda_0(n) = \varphi(2^{-1/\varepsilon}|n|),$$
$$\lambda_r(n) = \varphi(2^{-(r+1)/\varepsilon}|n|) - \varphi(2^{-r/\varepsilon}|n|), \tag{3}$$

where $\varepsilon = 1-s$ and $\varepsilon \in (0, 1/2)$.

Hence

$$\sum \lambda_r(n) = 1,$$

$$\|\lambda_r\|_{M(L^1,L^1)} \leq C \qquad \text{(multiplier norm)}, \tag{4}$$

$$\text{supp } \lambda_0 \subset B(0, 2^{1/\varepsilon+1}), \tag{5}$$

$$\text{supp } \lambda_r \subset B(0, 2^{1+(r+1)/\varepsilon}) \backslash B(0, 2^{r/\varepsilon}). \tag{6}$$

Write

$$(2) = \sum_{1/\varepsilon<k<2/\varepsilon} + \sum_{r\geq 1} \sum_{(r+1)/\varepsilon<k<(r+2)/\varepsilon} . \tag{7}$$

For $2/\varepsilon > k > 1/\varepsilon$ and $|h| < 2^{-k-10}$, (4) and (5) permit us to write

$$\begin{aligned}\Big\|\sum_n \hat{f}(n)e^{in.x}\sin n.h\Big\|_1 &\gtrsim \Big\|\sum_n \lambda_0(n)\hat{f}(n)e^{in.x}\sin n.h\Big\|_1 \\ &\sim \Big\|\sum_n \lambda_0(n)(n.h)\hat{f}(n)e^{in.x}\Big\|_1 ;\end{aligned}$$

and thus

$$\begin{aligned}&2^{k(d+1-\varepsilon)} \int_{|h_1|,|h_2|\sim 2^{-k-10}} \Big\|\sum \hat{f}(n)(\sin n.h)e^{in.x}\Big\|_1 dh_1dh_2 \\ &\gtrsim 2^{k(3-\varepsilon)}8^{-k}\left(\Big\|\sum \lambda_0(n)n_1\hat{f}(n)e^{in.x}\Big\|_1 + \Big\|\sum \lambda_0(n)n_2\hat{f}(n)e^{in.x}\Big\|_1\right) \\ &= 2^{-k\varepsilon}\left(\Big\|\partial_{x_1}\Big(\sum \lambda_0(n)\hat{f}(n)e^{in.x}\Big)\Big\|_1 + \|\partial_{x_2}(\cdots)\|\right) \\ &\sim \Big\|\sum \lambda_0(n)\hat{f}(n)e^{in.x}\Big\|_{W^{1,1}} . \qquad (8)\end{aligned}$$

Similarly, for

$$(r+1)/\varepsilon < k < (r+2)/\varepsilon,$$

we have

$$2^{k(d+1-\varepsilon)} \int_{|h_1|,|h_2|\sim 2^{-k-10}} \Big\|\sum \hat{f}(n)(\sin nh)e^{in.x}\Big\|_1 \gtrsim 2^{-r}\Big\|\sum \lambda_r(n)\hat{f}(n)e^{in.x}\Big\|_{W^{1,1}} . \tag{9}$$

Since in the summation (7), each of the terms (8), (9) appear at least $1/\varepsilon$ times, we have

$$\varepsilon.(2) \gtrsim \Big\|\sum \lambda_0(n)\hat{f}(n)e^{in.x}\Big\|_{W^{1,1}} + \sum_r 2^{-r}\Big\|\sum \lambda_r(n)\hat{f}(n)e^{in.x}\Big\|_{W^{1,1}} . \tag{10}$$

Write

$$(2-s)/2 = 1 - s + s/2.$$

Then by Hölder's inequality,

$$\Big\|\sum \lambda_r(n)\hat{f}(n)e^{in.x}\Big\|_{2/(2-s)} \leq \Big\|\sum \lambda_r(n)\hat{f}(n)e^{in.x}\Big\|_2^s \Big\|\sum \lambda_r(n)\hat{f}(n)e^{in.x}\Big\|_1^{1-s} . \tag{11}$$

By the Sobolev embedding theorem ($d = 2$),

$$\Big\| \sum \lambda_r(n) \hat{f}(n) e^{in.x} \Big\|_2 \leq C \Big\| \sum \lambda_r(n) \hat{f}(n) e^{in.x} \Big\|_{W^{1,1}}. \tag{12}$$

We estimate the last factor in (11).

Recalling (6), we have

$$2^{1+(r+1)/\varepsilon} > \max(|n_1|, |n_2|) > 2^{(r/\varepsilon)-1}$$

if $\lambda_r(n) \neq 0, r \geq 1$.

Hence, with φ as above,

$$\lambda_r(n) = \lambda_r(n).(1-\varphi)(2^{-(r-1)/\varepsilon} n_1) + \lambda_r(n).\varphi(2^{-(r-1)/\varepsilon} n_1).(1-\varphi)(2^{-(r-1)/\varepsilon} n_2);$$

and thus

$$\begin{aligned} &\Big\| \sum \lambda_r(n) \hat{f}(n) e^{in.x} \Big\|_1 \\ \leq &\Big\| \sum \lambda_r(n) n_1 \hat{f}(n) e^{in.x} \Big\|_1 \Big\| \sum \frac{1}{n_1} (1-\varphi)(2^{-\frac{r-1}{\varepsilon}} n_1) e^{in.x} \Big\|_1 \\ &+ \Big\| \sum \lambda_r(n) n_2 \hat{f}(n) e^{in.x} \Big\|_1 \Big\| \sum \frac{1}{n_2} \varphi(2^{-\frac{r-1}{\varepsilon}} n_1)(1-\varphi)(2^{-\frac{r-1}{\varepsilon}} n_2) e^{in.x} \Big\|_1 \\ \leq &\left(\Big\| \sum_{n_1} \frac{1}{n_1} (1-\varphi)(2^{-\frac{r-1}{\varepsilon}} n_1) e^{in_1x_1} \Big\|_{L^1_{x_1}} + \Big\| \sum_{n_2} \frac{1}{n_2} (1-\varphi)(2^{-\frac{r-1}{\varepsilon}} n_2) e^{in_2x_2} \Big\|_{L^1_{x_2}} \right) \\ &\times \Big\| \sum \lambda_r(n) \hat{f}(n) e^{in.x} \Big\|_{W^{1,1}}. \end{aligned} \tag{13}$$

Since $(1-\varphi)(2^{-(r-1)/\varepsilon} n_1) = 0$ for $|n_1| \leq 2^{(r-1)/\varepsilon}$, one easily checks that

$$\Big\| \sum_{n_1} \frac{1}{n_1} (1-\varphi)(2^{-(r-1)/\varepsilon} n_1) \varepsilon^{in_1x_1} \Big\|_{L^1_{x_1}} \lesssim \sum_{\ell \geq (r-1)/\varepsilon} 2^{-\ell} < 2^{(r-2)/\varepsilon}.$$

Similarly,

$$\Big\| \sum_{n_2} \frac{1}{n_2} (1-\varphi)(2^{-(r-1)/\varepsilon} n_2) e^{in_2x_2} \Big\|_{L^1_{x_2}} \leq 2^{-(r-2)/\varepsilon}.$$

Thus (13) implies that

$$\Big\| \sum \lambda_r(n) \hat{f}(n) e^{in.x} \Big\|_1 \leq 2^{-(r-2)/\varepsilon} \Big\| \sum \lambda_r(n) \hat{f}(n) e^{in.x} \Big\|_{W^{1,1}}. \tag{14}$$

Substitution of (12), (14) in (11) gives

$$\begin{aligned} \Big\| \sum \lambda_r(n) \hat{f}(n) e^{in.x} \Big\|_{2/(2-s)} &\lesssim 2^{-(r-2)(1-s)/\varepsilon} \Big\| \sum \lambda_r(n) \hat{f}(n) e^{in.x} \Big\|_{W^{1,1}} \\ &\sim 2^{-r} \Big\| \sum \lambda_r(n) \hat{f}(n) e^{in.x} \Big\|_{W^{1,1}}. \end{aligned} \tag{15}$$

By (12), (15),

$$\varepsilon.(2) \geq \Big\| \sum \lambda_0(n) \hat{f}(n) e^{in.x} \Big\|_2 + \sum_{r \geq 1} \Big\| \sum \lambda_r(n) \hat{f}(n) e^{in.x} \Big\|_{2/(2-s)}$$
$$\geq \| f - \textstyle\int f \|_{2/(2-s)}$$

by (3).

This proves (1) and completes the proof of Theorem 1 when $p = 1$.

4 Square function inequalities

We present here some known inequalities used in the proof of Theorem 1 when $p > 1$. Let $\{\Delta_j f\}_{j=1,2,\dots}$ be a Littlewood–Paley decomposition with $\Delta_j f$ obtained from a Fourier multiplier of the form $\varphi(2^{-j}|n|) - \varphi(2^{-j+1}|n|)$ with $0 \leq \varphi \leq 1$ a smooth function satisfying $\varphi(t) = 1$ for $|t| \leq 1$ and $\varphi(t) = 0$ for $|t| > 2$.

Recall the square-function inequality for $1 < q < \infty$:

$$\text{(1)} \qquad \frac{1}{C(q)} \Big\| \Big(\sum |\Delta_j f|^2 \Big)^{1/2} \Big\|_q \leq \|f\|_q \leq C(q) \Big\| \Big(\sum |\Delta_j f|^2 \Big)^{1/2} \Big\|_q .$$

We also consider square-functions with respect to a martingale filtration. Denote by $\{\mathbb{E}_j\}$ the expectation operators with respect to a dyadic partition of $[0,1]^d$ and

$$\text{(2)} \qquad \widetilde{\Delta}_j f = (\mathbb{E}_j - \mathbb{E}_{j-1}) f$$

the martingale differences.

We use the square-function inequality

$$\text{(3)} \qquad \|f\|_q \leq C\sqrt{q} \Big\| \Big(\sum |\widetilde{\Delta}_j f|^2 \Big)^{1/2} \Big\|_q , \qquad \infty > q \geq 2,$$

which is precise in terms of the behaviour of the constant for $q \to \infty$ (see [4] and also the Appendix for a proof of (3)).

Remark 2. One should expect (3) also to hold if $\widetilde{\Delta}_j$ is replaced by Δ_j above, but we do not need this fact.

We do require the following inequality later on.

Let

$$p < q \quad \text{and} \quad s = d(1/p - 1/q) \geq \tfrac{1}{2}.$$

Then, for $q \geq 2$,

$$\text{(4)} \qquad \|f\|_q \leq C\sqrt{q} \Big[\sum_k (2^{ks} \|\Delta_k f\|_p)^2 \Big]^{1/2} .$$

Proof of (4). It follows from (3) that since $q \geq 2$,

$$\|f\|_q \leq C\sqrt{q}\Big(\sum_j \|\widetilde{\Delta}_j f\|_q^2\Big)^{1/2}. \tag{5}$$

Write

$$\widetilde{\Delta}_j f = \sum_{k\leq j} \widetilde{\Delta}_j \Delta_k f + \sum_{k>j} \widetilde{\Delta}_j \Delta_k f,$$

$$\|\widetilde{\Delta}_j f\|_q \lesssim \sum_{k\leq j} 2^{k-j}\|\Delta_k f\|_q + \sum_{k>j} 2^{js}\|\Delta_k f\|_p$$

$$\lesssim \sum_{k\leq j} 2^{k-j}(2^{ks}\|\Delta_k f\|_p) + \sum_{k>j} 2^{(j-k)s}(2^{ks}\|\Delta_k f\|_p). \tag{6}$$

Substitution of (6) in (5) gives

$$\|f\|_q \leq C\sqrt{q}\Big\{\Big(\sum_{k\leq j}(j-k)^2 4^{k-j}(2^{ks}\|\Delta_k f\|_p)^2\Big)^{1/2} + \Big(\sum_{k>j}(k-j)^2 4^{(j-k)s}(2^{ks}\|\Delta_k f\|_p)^2\Big)^{1/2}\Big\}$$

$$\leq C\sqrt{q}\Big(\sum_k (2^{ks}\|\Delta_k f\|_p)^2\Big)^{1/2}. \tag{7}$$

5 Proof of Theorem 1 when $1 < p < 2$

Write

$$\iint \frac{|f(x)-f(y)|^p}{|x-y|^{d+ps}}dxdy \sim \sum_{k\geq 0} 2^{k(d+ps)} \int_{|h|\sim 2^{-k-10}} \|f(x+h)-f(x-h)\|_p^p dh$$

$$\geq \sum_{k\geq 0} 2^{k(d+ps)} \int_{|h|\sim 2^{-k-10}} \Big\|\sum \hat{f}(n)(\sin n.h)e^{in.x}\Big\|_p^p dh. \tag{1}$$

Following the argument in Section 3 (formula (10)), we get again for

$$s = d(1/p - 1/q), \quad 1 - s = \varepsilon \tag{2}$$

$$\varepsilon.(1) \gtrsim \sum_r \Big(2^{-r}\Big\|\sum_n \lambda_r(n)\hat{f}(n)e^{in.x}\Big\|_{W^{1,p}}\Big)^p, \tag{3}$$

where the multipliers λ_r are defined as before.

Case $\underline{d=1}$

Define

$$f_r = \sum_n \lambda_r(n) \hat{f}(n) e^{in.x}.$$

We make two estimates.

First write

$$f_r = \Big(\sum n\lambda_r(n)\hat{f}(n)e^{in.x} \Big) * \Big(\sum_{2^{r/\varepsilon}<|n|<2^{(r+1)/\varepsilon}} \frac{1}{n} e^{in.x} \Big),$$

which implies

$$\|f_r\|_q \leq \|f_r\|_{W^{1,p}} \Big\| \sum_{2^{r/\varepsilon}<n<2^{(r+1)/\varepsilon}} \frac{1}{n} \sin nx \Big\|_{(1/p'+1/q)^{-1}} \tag{4}$$

and, by estimate (10) in Section 2,

$$\|f_r\|_q \lesssim 2^{-\frac{r}{\varepsilon}(\frac{1}{p'}+\frac{1}{q})} \|f_r\|_{W^{1,p}} = 2^{-\frac{r}{\varepsilon}(1-s)} \|f_r\|_{W^{1,p}} = 2^{-r} \|f_r\|_{W^{1,p}}. \tag{5}$$

Estimate then

$$\|f\|_q \leq \sum_r \|f_r\|_q \leq C \sum_r (2^{-r} \|f_r\|_{W^{1,p}}). \tag{6}$$

Next apply inequality (4) of Section 4. Observe that

$$|\Delta_k f| \leq \sum_r |\Delta_k f_r|,$$

where, by construction, there are, for fixed k, at most two nonvanishing terms.

Thus

$$\|\Delta_k f\|_p^2 \lesssim \sum_r \|\Delta_k f_r\|_p^2. \tag{7}$$

Also, for fixed r,

$$\sum_k (2^{ks} \|\Delta_k f_r\|_p)^2 = \sum_r 4^{-k\varepsilon} \|\Delta_k f_r\|^2_{W^{1,p}} \lesssim \frac{1}{\varepsilon} 4^{-r} \|f_r\|^2_{W^{1,p}}. \tag{8}$$

Substituting (7), (8) in (4) of Section 4 gives

$$\|f\|_q \lesssim C\sqrt{q} \Big[\sum_k \sum_r (2^{ks} \|\Delta_k f_r\|_p)^2 \Big]^{1/2} \leq C\sqrt{q/\varepsilon} \Big[\sum_r (2^{-r} \|f_r\|_{W^{1,p}})^2 \Big]^{1/2}, \tag{9}$$

which is the second estimate.

Interpolation between (6) and (9) thus implies

$$\|f\|_q \le C(\sqrt{q/\varepsilon})^{2(1-1/p)} \Big[\sum_r (2^{-r}\|f_r\|_{W^{1,p}})^p\Big]^{1/p}. \tag{10}$$

Recalling (3) and also (2) (which implies that $1-\varepsilon = 1/p - 1/q < 1/p$, hence $\varepsilon > 1 - 1/p$) we obtain

$$\varepsilon.(1) \gtrsim (1/q)^{p-1}\|f\|_q^p, \tag{11}$$

which gives the required inequality.

Case $\underline{d>1}$

We distinguish two cases.

Case A: $0 < 1/p - 1/d$ is not near 0.

Case B: $1/p - 1/d$ is near 0.

Observe that case B may only happen for $d=2$ and p near 2 (we assumed $1<p<2$).

Case A.

Define q_1 by

$$1 = d(1/p - 1/q_1), \tag{12}$$

so that $q < q_1$ and q_1 is bounded from above by assumption.

Thus we have the Sobolev inequality

$$\|g\|_{q_1} \le C\|g\|_{W^{1,p}}. \tag{13}$$

Next, we make the obvious adjustment of the argument in Section 3, (11)–(15).

Thus Hölder's inequality gives

$$\|f_r\|_q \le \|f_r\|_{q_1}^{1-\theta}\|f_r\|_q^{\theta}, \tag{14}$$

with

$$1/q = (1-\theta)/q_1 + \theta/p, \qquad \text{hence } \theta = 1 - s = \varepsilon \text{ by (2), (12).}$$

Hence, by (13),

$$\|f_r\|_q \le C\|f_r\|_{W^{1,p}}^{1-\varepsilon}\|f_r\|_p^{\varepsilon}. \tag{15}$$

To estimate $\|f_r\|_p$, proceed as in (13) of Section 3. Thus

$$\begin{aligned}\|f_r\|_p &\lesssim \Big\|\sum \frac{1}{n}(1-\varphi)(2^{-(r-1)/\varepsilon}n)e^{inx}\Big\|_{L^1_x(\mathbb{T})} \|f_r\|_{W^{1,p}} \\ &\lesssim 2^{-(r-1)/\varepsilon}\|f_r\|_{W^{1,p}}.\end{aligned} \tag{16}$$

Substituting (16) in (15), we get

(17) $$\|f_r\|_q \lesssim 2^{-r}\|f_r\|_{W^{1,p}}.$$

Substituting (17) in (3) gives (since q is bounded by case A hypothesis)

$$\varepsilon.(1) \gtrsim \sum_r \|f_r\|_q^p \sim \sum_r \left\| \left(\sum_j |\Delta_j f_r|^2 \right)^{1/2} \right\|_q^p$$

$$\gtrsim \left\| \left(\sum_{r,j} |\Delta_j f_r|^2 \right)^{1/2} \right\|_q^p$$

(18) $$\gtrsim \left\| \left(\sum_j |\Delta_j f|^2 \right)^{1/2} \right\|_q^p \sim \|f\|_q^p$$

(the second inequality requires distinction of the cases $q \geq 2$ and $p < q \leq 2$).

Now (18) gives the required inequality.

Case B.

Thus $d = 2$ and p is near 2.

Going back to (3) and applying (1), (4) of Section 4, we obtain

$$\varepsilon.(1) \gtrsim \sum_r (2^{-r}\|f_r\|_{W^{1,p}})^p$$

$$\gtrsim \left(\sum_r 4^{-r} \sum_j \|\Delta_j f_r\|_p^2 4^j \right)^{p/2}$$

$$\gtrsim \left(\sum_j (2^{sj}\|\Delta_j f\|_p)^2 \right)^{p/2}$$

(19) $$\gtrsim q^{-p/2}\|f\|_q^p,$$

where

(20) $$q^{-p/2} = (1/p - s/2)^{p/2} \sim (2 - ps)^{p-1},$$

which again gives the required inequality.

6 Proof of Theorem 1 when $p \geq 2$

From (3) in Section 5, we now get the minoration

(1) $$\varepsilon.(1) \gtrsim \sum_j (2^{sj}\|\Delta_j f\|_p)^p,$$

which we use to majorize $\|f\|_q$.

We already have inequality (4) of Section 5; thus

$$\|f\|_q \le C\sqrt{q}\Big(\sum_j (2^{sj}\|\Delta_j f\|_p)^2\Big)^{1/2}. \tag{2}$$

Our aim is to prove that

$$\|f\|_q \le Cq^{1-1/p}\Big(\sum_j (2^{sj}\|\Delta_j f\|_p)^p\Big)^{1/p}, \tag{3}$$

which will give the required inequality together with (1).

Using interpolation for $2 \le p < d/s$, it clearly suffices to establish (3) for large values of q. To prove (3), we assume $2 \le p \le 4$ (other cases may be treated by adaption of the argument presented below). Assume further (taking the previous comment into account) that

$$q \ge 2p. \tag{4}$$

Again by interpolation, (3) will follow from (2) and the inequality

$$\|f\|_q \le Cq^{\frac{3}{4}}\Big(\sum_j (2^{sj}\|\Delta_j f\|_p)^4\Big)^{1/4}. \tag{5}$$

We use the notation from Section 4 and start from the martingale square function inequality (3) in Section 4; thus

$$\|f\|_q \le C\sqrt{q}\Big\|\Big(\sum |\widetilde{\Delta}_j f|^2\Big)^{1/2}\Big\|_q. \tag{6}$$

Write

$$|\widetilde{\Delta}_j f| \le \sum_k |\widetilde{\Delta}_j \Delta_k f| = \sum_{m\in\mathbb{Z}} |\widetilde{\Delta}_j \Delta_{j+m} f|$$

(putting $\Delta_k = 0$ for $k < 0$).

Writing

$$\Big\|\Big(\sum_j |\widetilde{\Delta}_j f|^2)^{1/2}\Big\|_q \le \sum_{m\in\mathbb{Z}} \Big\|\Big(\sum_j |\widetilde{\Delta}_j \Delta_{j+m} f|^2\Big)^{1/2}\Big\|_q, \tag{7}$$

we estimate each summand.

Fix m. Write

$$\begin{aligned}\Big\|\Big(\sum_j |\widetilde{\Delta}_j \Delta_{j+m} f|^2\Big)^{1/2}\Big\|_q^4 &= \Big\|\Big(\sum_j |\widetilde{\Delta}_j \Delta_{j+m} f|^2\Big)^2\Big\|_{q/4}\\ &\le 2\sum_{j_1\le j_2} \big\| |\widetilde{\Delta}_{j_1} \Delta_{j_1+m} f|^2\, |\widetilde{\Delta}_{j_2} \Delta_{j_2+m} f|^2\big\|_{q/4}\end{aligned} \tag{8}$$

and

$$\| |\widetilde{\Delta}_{j_1}\Delta_{j_1+m}f|^2 \, |\widetilde{\Delta}_{j_2}\Delta_{j_2+m}f|^2\|_{q/4}$$

$$= \left[\int |\widetilde{\Delta}_{j_1}\Delta_{j_1+m}f|^{q/2}.\mathbb{E}_{j_1}\left[|\widetilde{\Delta}_{j_2}\Delta_{j_2+m}f|^{q/2}]\right]\right]^{4/q}$$

$$\leq \|\widetilde{\Delta}_{j_1}\Delta_{j_1+m}f\|_q^2 \, \left\|\left(\mathbb{E}_{j_1}[|\widetilde{\Delta}_{j_2}\Delta_{j_2+m}f|^{q/2}]\right)^{2/q}\right\|_q^2$$

$$\leq 4^{d(j_2-j_1)(1/p-2/q)}\|\widetilde{\Delta}_{j_1}\Delta_{j_1+m}f\|_q^2 \, \left\|\left(\mathbb{E}_{j_1}[|\widetilde{\Delta}_{j_2}\Delta_{j_2+m}f|^{p}]\right)^{1/p}\right\|_q^2$$

$$\leq 4^{d(j_2-j_1)(1/p-2/q)}\, 4^{dj_1(1/p-1/q)}\|\widetilde{\Delta}_{j_1}\Delta_{j_1+m}f\|_q^2 \, \|\widetilde{\Delta}_{j_2}\Delta_{j_2+m}f\|_p^2. \tag{9}$$

Assume $\underline{m \leq 0}$,

Estimate

$$\|\widetilde{\Delta}_{j_1}\Delta_{j_1+m}f\|_q \lesssim 2^m\|\Delta_{j_1+m}f\|_q \leq 2^m 2^{d(j_1+m)(1/p-1/q)}\|\Delta_{j_1+m}f\|_p, \tag{10}$$

$$\|\widetilde{\Delta}_{j_2}\Delta_{j_2+m}f\|_p \lesssim 2^m\|\Delta_{j_2+m}f\|_p. \tag{11}$$

Substitution of (10), (11) in (9) gives

(12)

$$4^{(1-d(\frac{1}{p}-\frac{1}{q}))m+m}\, 4^{-\frac{d}{q}(j_2-j_1)}[2^{d(\frac{1}{p}-\frac{1}{q})(j_1+m)}\|\Delta_{j_1+m}f\|_p]^2\, [2^{d(\frac{1}{p}-\frac{1}{q})(j_2+m)}\|\Delta_{j_2+m}f\|_p]^2$$

where

$$d(1/p-1/q) = s.$$

Summing (12) for $j_1 < j_2$ and applying Cauchy-Schwarz implies for $m < 0$

$$(8) < 4^{(2-s)m}\left(\sum_{\ell\geq 0} 4^{-\frac{d}{q}\ell}\right)\left[\sum_j (2^{sj}\|\Delta_j f\|_p)^4\right]$$

$$\lesssim 4^{(2-s)m} q\left[\sum_j (2^{sj}\|\Delta_j f\|_p)^4\right]. \tag{13}$$

Assume next $\underline{m > 0}$.

Estimate

$$\|\widetilde{\Delta}_{j_1}\Delta_{j_1+m}f\|_q \lesssim 2^{dj_1(1/p-1/q)}\|\Delta_{j_1+m}f\|_p$$

and

$$(9) \leq 4^{d(j_2-j_1)(\frac{1}{p}-\frac{2}{q})}\, 16^{dj_1(\frac{1}{p}-\frac{1}{q})}\|\Delta_{j_1+m}f\|_p^2\, \|\Delta_{j_2+m}f\|_p^2$$

$$\leq 16^{-ms}4^{-(j_2-j_1)\frac{d}{q}}\|2^{s(j_1+m)}\Delta_{j_1+m}f\|_p^2\, \|2^{s(j_2+m)}\Delta_{j_2+m}f\|_p^2. \tag{14}$$

Summing over $j_1 < j_2$ implies that for $m > 0$,

$$(8) \lesssim 16^{-ms} q\left[\sum_j (2^{sj}\|\Delta_j f\|_p)^4\right]. \tag{15}$$

Summing (13), (15) in m implies that

$$(7) \leq \Big(\sum_{m\leq 0} 2^{(1-s/2)m} + \sum_{m>0} 2^{-sm}\Big) q^{1/4}\Big[\sum_j (2^{sj}\|\Delta_j f\|_p)^4\Big]^{1/4}$$

$$\leq q^{1/4}\Big[\sum_j (2^{sj}\|\Delta_j f\|_p)^4\Big]^{1/4}. \tag{16}$$

To bound $\|f\|_q$, apply (6), which introduces an additional $q^{1/2}$-factor. This establishes (5) and completes the argument and the proof of Theorem 1.

7 Proof of Theorem 2

We make use of the following two lemmas

Lemma 2. *Let $I \subset \mathbb{R}$ be an interval and let $\psi : I \to \mathbb{Z}$ be any measurable function. Then there exists $k \in \mathbb{Z}$ such that*

$$|\{x \in I; \psi(x) \neq k\}| \leq 2\Big(C^*\varepsilon \int_I \int_I \frac{|\psi(x) - \psi(y)|^2}{|x-y|^{2-\varepsilon}} dxdy\Big)^{1/\varepsilon}$$

for all $\varepsilon \in (0, 1/2]$, where C^* is the absolute constant in Corollary 1 (inequality (10) in Section 1).

Proof of Lemma 2. After scaling and shifting, we may assume that $I = (-1, +1)$. For each $k \in \mathbb{Z}$, set

$$A_k = \{x \in I : \psi(x) < k\}.$$

Note that A_k is nondecreasing, $\lim_{k\to-\infty} |A_k| = 0$, and $\lim_{k\to+\infty} |A_k| = 2$. Thus, there exists $k \in \mathbb{Z}$ such that

$$|A_k| \leq 1 \quad \text{and} \quad |A_{k+1}| > 1. \tag{1}$$

Applying Corollary 1 with $A = A_k$ and with $A = A_{k+1}$, we find (using (1))

$$|A_k| \leq |A_k|\,|{}^c\!A_k| \leq \Big(C^*\varepsilon \int_{A_k}\int_{{}^cA_k} \frac{dxdy}{|x-y|^{2-\varepsilon}}\Big)^{1/\varepsilon} \tag{2}$$

and

$$|{}^cA_{k+1}| \leq |A_{k+1}|\,|{}^cA_{k+1}| \leq \Big(C^*\varepsilon \int_{A_{k+1}}\int_{{}^cA_{k+1}} \frac{dxdy}{|x-y|^{2-\varepsilon}}\Big)^{1/\varepsilon}. \tag{3}$$

On the other hand,

$$|\psi(x) - \psi(y)| \geq 1 \quad \text{for a.e. } x \in A_k,\ \ y \in {}^cA_k$$

and

$$|\psi(x)-\psi(y)| \geq 1 \quad \text{for a.e. } x \in A_{k+1},\ y \in {}^cA_{k+1}.$$

Therefore,

$$\begin{aligned}|\{x \in I; \psi(x) \neq k\}| &= |A_k| + |{}^cA_{k+1}| \\ &\leq 2\left(C^*\varepsilon \int_I \int_I \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{2-\varepsilon}} dxdy\right)^{1/\varepsilon}.\end{aligned}$$

Lemma 3. *If $\alpha > 0, a < b < x$, and $A \subset (a,b)$ is measurable, then*

$$\int_{(a,b)\setminus A} \frac{dy}{(x-y)^\alpha} \geq \int_a^{b-|A|} \frac{dy}{(x-y)^\alpha};$$

similarly, if $x < a < b$, then

$$\int_{(a,b)\setminus A} \frac{dy}{(y-x)^\alpha} \geq \int_{a+|A|}^{b} \frac{dy}{(y-x)^\alpha}.$$

The proof of Lemma 3 is elementary and left to the reader.

Proof of Theorem 2. Let $\psi_\varepsilon : \Omega = (-1,+1) \to \mathbb{R}$ be any measurable function such that $u_\varepsilon = e^{i\psi_\varepsilon}$. We have to prove that for all $\varepsilon < 1/2$,

$$\|\psi_\varepsilon\|_{H^{(1-\varepsilon)/2}(\Omega)} \geq c\varepsilon^{-1/2} \tag{4}$$

for some absolute constant c to be determined.

We argue by contradiction and assume that for some $\varepsilon < 1/2$,

$$\|\psi_\varepsilon\|_{H^{(1-\varepsilon)/2}(\Omega)} < \eta\varepsilon^{-1/2}. \tag{5}$$

We reach a contradiction if η is less than some absolute constant. Set

$$\psi = \frac{1}{2\pi}(\psi_\varepsilon - \varphi_\varepsilon),$$

so that $\psi : \Omega \to \mathbb{Z}$; recall that $u_\varepsilon = e^{i\varphi_\varepsilon}$ and the function φ_ε is defined by

$$\varphi_\varepsilon(x) = \begin{cases} 0 & \text{for } -1 < x < 0, \\ 2\pi x/\delta & \text{for } 0 < x < \delta, \\ 2\pi & \text{for } \delta < x < 1, \end{cases}$$

where $\delta = e^{-1/\varepsilon}$.

A straightforward computation (using the fact that ψ takes values in $\mathbb{Z}$) shows that

$$|\psi(x)-\psi(y)| \leq |\psi_\varepsilon(x)-\psi_\varepsilon(y)| \quad \text{for a.e. } x, y \in (-1, 2\delta/3) \tag{6}$$

and

(7) $$|\psi(x) - \psi(y)| \leq |\psi_\varepsilon(x) - \psi_\varepsilon(y)| \quad \text{for a.e. } x, y \in (\delta/3, 1).$$

Applying Lemma 2 with $I = (-1, 2\delta/3)$ and $I = (\delta/3, 1)$, together with (5), (7) and (8) yields the existence of $\ell, m \in \mathbb{Z}$ such that

$$|\{x \in (-1, 2\delta/3); \psi(x) \neq \ell\}| \leq 2(C^*\eta^2)^{1/\varepsilon}$$

and

$$|\{x \in (x \in \delta/3, 1); \psi(x) \neq m\}| \leq 2(C^*\eta^2)^{1/\varepsilon}.$$

We choose η in such a way that

$$4(C^*\eta^2)^{1/\varepsilon} < \delta/3 \quad \text{for } \varepsilon < 1/2,$$

for example,

(8) $$\eta^2 < 1/4eC^*.$$

It follows that $\ell = m$. Without loss of generality (after adding a constant to ψ_ε), we may assume that

(9) $$\ell = m = 0.$$

Therefore,

(10) $$\psi_\varepsilon(x) = \varphi_\varepsilon(x) \quad \text{for } x \in [(-1, 0)\backslash A] \cup [(\delta, 1)\backslash B],$$

where

$$A = \{x \in (-1, 0); \psi(x) \neq 0\}$$

and

$$B = \{x \in (\delta, 1); \psi(x) \neq 0\},$$

with

(11) $$|A| < \delta/6, |B| < \delta/6.$$

From (11) and the definition of φ_ε, we have

$$\begin{aligned}
\varepsilon \int_\Omega \int_\Omega \frac{|\psi_\varepsilon(x) - \psi_\varepsilon(y)|^2}{|x - y|^{2-\varepsilon}} dx dy &\geq \varepsilon \int_{-1}^0 dx \int_0^1 \frac{|\psi_\varepsilon(x) - \psi_\varepsilon(y)|^2}{|x - y|^{2-\varepsilon}} dy \\
&\geq \varepsilon \int_{(-1,0)\backslash A} dx \int_{(\delta,1)\backslash B} \frac{|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)|^2}{|x - y|^{2-\varepsilon}} dy \\
&\geq \varepsilon \int_{(-1,0)\backslash A} dx \int_{(\delta,1)\backslash B} \frac{4\pi^2 dy}{|x - y|^{2-\varepsilon}}.
\end{aligned}$$

Applying Lemma 3 and (5), we find

$$\eta^2 > \varepsilon \int_\Omega \int_\Omega \frac{|\psi_\varepsilon(x) - \psi_\varepsilon(y)|^2}{|x-y|^{2-\varepsilon}} dxdy \geq \varepsilon \int_{-1}^{-|A|} dx \int_{\delta+|B|}^{1} \frac{4\pi^2 dy}{|x-y|^{2-\varepsilon}}$$
$$\geq \varepsilon \int_{-1}^{-\delta/6} dx \int_{\delta+\delta/6}^{1} \frac{4\pi^2 dy}{|x-y|^{2-\varepsilon}} = 4\pi^2(1-e^{-1}) + o(1)$$

as $\varepsilon \to 0$. We obtain a contradiction for an appropriate choice of η.

Appendix

Proof of the square function inequality

Let $\{\mathcal{F}_n\}_{n=0,1,2,\dots}$ be refining finite partitions such that

$$\#\mathcal{F}_n = K^n,$$
$$|Q| = K^{-n} \quad \text{if } Q \text{ is an } \mathcal{F}_n\text{-atom.}$$

(If $\Omega = [0,1]^d, K = 2^d$.)

Denote $\mathbb{E}_n$ the $\mathcal{F}_n$-expectation

$$\Delta_n f = \mathbb{E}_n f - \mathbb{E}_{n-1} f \qquad \text{(we used the notation } \widetilde{\Delta}_n f \text{ in Section 4),}$$
$$Sf = \left(\sum |\Delta_n f|^2\right)^{1/2} \qquad \text{(the square function),}$$
$$f \underset{a.e.}{\leq} f^* = \sup |\mathbb{E}_n f| \qquad \text{(the maximal function).}$$

Proposition 1.

(1) $$\operatorname{mes}(\{|f| > \lambda \|Sf\|_\infty\}) < e^{-c\lambda^2} \qquad (\lambda \geq 1),$$

where $c = c(K) > 0$ is a constant.

Proposition 2 (*good-λ inequality*).

(2) $$\operatorname{mes}(\{f^* > 2\lambda, Sf < \varepsilon\lambda, \sup \mathbb{E}_{n-1}[|\Delta_n f|] < \varepsilon\lambda\}) < e^{-c/\varepsilon^2} \operatorname{mes}(\{f^* > \lambda\})$$
$$(0 < \varepsilon < 1).$$

Proposition 3.

(3) $$\|f^*\|_q \leq C\sqrt{q}\|Sf\|_q \quad \textit{for } q \geq 2.$$

We follow essentially [4].

Proof of Proposition 1. One verifies that there is a constant $A = A(K)$ such that if φ is $\mathcal{F}_n$-measurable and $\mathbb{E}_{n-1}\varphi = 0$, then

$$\mathbb{E}_{n-1}[e^{\varphi - A\varphi^2}] \leq 1. \tag{4}$$

Hence

$$\mathbb{E}_{n-1}[e^{\Delta_n f - A(\Delta_n f)^2}] \leq 1 \tag{5}$$

and, writing $S_n f = \left(\sum_{m \leq n} |\Delta_m f|^2\right)^{1/2}$, we have

$$\begin{aligned}\int e^{\mathbb{E}_n f - A(S_n f)^2} &= \int e^{\mathbb{E}_{n-1} f - A(S_{n-1} f)^2}\mathbb{E}_{n-1}[e^{\Delta_n f - A(\Delta_n f)^2}]\\ &\leq \int e^{\mathbb{E}_{n-1} f - A(S_{n-1} f)^2} \qquad \text{(by (5))}\\ &\leq 1.\end{aligned}$$

Thus

$$\int e^{f - A(Sf)^2} \leq 1. \tag{6}$$

Assume $\|Sf\|_\infty \leq 1$. Applying (6) to tf ($t > 0$ a parameter), we get

$$\int e^{tf} \leq e^{At^2}, \qquad \mathrm{mes}(\{f > \lambda\}) \leq e^{At^2 - t\lambda},$$

so for appropriate choice of t

$$\mathrm{mes}(\{f > \lambda\}) < e^{-\lambda^2/4A}.$$

This proves (1).

Proof of Proposition 2. This is a standard stopping time argument.

Consider a collection of maximal atoms $\{Q_\alpha\} \subset \bigcup \mathcal{F}_n$ such that if Q_α is an $\mathcal{F}_n$-atom, then $|\mathbb{E}_n f| > \lambda$ on Q_α. Thus $Q_\alpha \cap Q_\beta = \emptyset$ for $\alpha \neq \beta$. Fix α. From the maximality,

$$|\mathbb{E}_{n-1} f| \leq \lambda \quad \text{on } Q_\alpha. \tag{7}$$

Therefore,

$$\{f^* > 2\lambda, Sf < \varepsilon\lambda, \sup \mathbb{E}_{m-1}[|\Delta_m f|] < (1/K)\varepsilon\lambda\} \cap Q_\alpha \subset$$
$$\{(f - \mathbb{E}_n f)^* > (1-\varepsilon)\lambda, Sf < \varepsilon\lambda, \sup \mathbb{E}_{m-1}[|\Delta_m f|] < (1/K)\varepsilon\lambda\} \cap Q_\alpha = (8).$$

For $m > n$, denote χ_m the indicator function of the set

$$Q_\alpha \cap \left\{ \left(\sum_{\ell=n+1}^{m-1} |\Delta_\ell f|^2 \right)^{1/2} < \varepsilon\lambda \right\} \cap \left\{ \mathbb{E}_{m-1}[|\Delta_m f|] < \frac{1}{K}\varepsilon\lambda \right\}$$
$$\cap \bigcap_{n \le \ell < m} \{|\mathbb{E}_\ell f - \mathbb{E}_n f| \le (1-\varepsilon)\lambda\} = (9).$$

Thus

$$\chi_m = \mathbb{E}_{m-1}\, \chi_m$$

and

$$g = \sum_{m>n} \chi_m \Delta_m f$$

is an $\{\mathcal{F}_m : m \ge n\}$-martingale on Q_α.

From the definition of χ_m, we have clearly

(10) $$S(g) = \left(\sum_{m>n} \chi_m |\Delta_m f|^2 \right)^{1/2} < \varepsilon\lambda + \varepsilon\lambda \lesssim \varepsilon\lambda$$

and

$$|g| > (1-\varepsilon)\lambda \quad \text{on the set (8).}$$

From Proposition 1 and (10),

(11) $$\operatorname{mes}(\{x\, \varepsilon Q_\alpha;\ |g| > (1-\varepsilon)\lambda\}) < e^{-c/\varepsilon^2}|Q_\alpha|$$

hence

(12) $$\operatorname{mes}(8) \lesssim e^{-c/\varepsilon^2}|Q_\alpha|.$$

Summing (12) over α implies

$$\operatorname{mes}(\{f^* > 2\lambda, Sf < \varepsilon\lambda, \sup \mathbb{E}_{m-1}[|\Delta_m f|] < (1/K)\varepsilon\lambda\})$$
$$< e^{-c/\varepsilon^2} \sum |Q_\alpha| \le e^{-c/\varepsilon^2} \operatorname{mes}[f^* > \lambda],$$

which is (2).

Proof of Proposition 3.

$$\|f^*\|_q^q = q \int \lambda^{q-1} \operatorname{mes}(\{f^* > \lambda\}) d\lambda$$
$$= 2^q q \int \lambda^{q-1} \operatorname{mes}(\{f^* > 2\lambda\}) d\lambda$$

(13) $$\le 2^q q \int \lambda^{q-1} [\operatorname{mes}(\{Sf \ge \varepsilon\lambda\}) + \operatorname{mes}(\{\sup \mathbb{E}_{n-1}[|\Delta_n f|] \ge (\varepsilon/K)\lambda\})$$
$$+ e^{-c/\varepsilon^2} \operatorname{mes}(\{f^* > \lambda\})]$$
$$< (2/\varepsilon)^q (\|Sf\|_q^q + K^q \|\sup \mathbb{E}_{n-1}[|\Delta_n f|]\|_q^q) + 2^q e^{-c/\varepsilon^2} \|f^*\|_q^q.$$

Take $1/\varepsilon \sim \sqrt{q}$, so that the last term in (13) is at most $\frac{1}{2}\|f^*\|_q^q$. Thus

$$\|f^*\|_q < C\sqrt{q}(\|Sf\|_q + \|\sup \mathbb{E}_{n-1}[|\Delta_n f|]\|_q). \tag{14}$$

Also,

$$\begin{aligned}\|\sup \mathbb{E}_{n-1}[|\Delta_n f|]\|_q &\le \Big(\sum_n \|\mathbb{E}_{n-1}[|\Delta_n f|]\|_q^q\Big)^{1/q} \\ &\le \Big(\sum_n \|\Delta_n f\|_q^q\Big)^{1/q} \\ &\le \|Sf\|_q. \qquad (15)\end{aligned}$$

Proposition 3 follows from (14) and (15).

Added in proof

An alternative, more elementary, proof of Theorem 1 was given by V. Mazya and T. Shaposhnikova, *On the Bourgain, Brezis and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces*, J. Funct. Anal. (to appear).

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Jean Bourgain
SCHOOL OF MATHEMATICS
INSTITUTE FOR ADVANCED STUDY
PRINCETON, NJ 08540, USA
email: bourgain@math.ias.edu

Haim Brezis
ANALYSE NUMÉRIQUE
UNIVERSITÉ P. ET M. CURIE, B.C. 187
4 PL. JUSSIEU
75252 PARIS CEDEX 05, FRANCE
email: brezis@ccr.jussieu.fr

DEPARTMENT OF MATHEMATICS
RUTGERS UNIVERSITY
HILL CENTER, BUSCH CAMPUS
110 FRELINGHUYSEN RD
PISCATAWAY, NJ 08854, USA
email: brezis@math.rutgers.edu

Petru Mironescu
DÉPARTEMENT DE MATHÉMATIQUES
UNIVERSITÉ PARIS-SUD
91405 ORSAY, FRANCE
email: Petru.Mironescu@math.u-psud.fr

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