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COMPOSITION IN FRACTIONAL SOBOLEV SPACES

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1. Introduction. A classical result about composition in Sobolev spaces asserts that if $u \in W^{k,p}(\Omega) \cap L^\infty(\Omega)$ and $\Phi \in C^k(\mathbb{R})$, then $\Phi \circ u \in W^{k,p}(\Omega)$. Here Ω denotes a smooth bounded domain in $\mathbb{R}^N$, $k \geq 1$ is an integer and $1 \leq p < \infty$. This result was first proved in [13] with the help of the Gagliardo-Nirenberg inequality [14]. In particular if $u \in W^{k,p}(\Omega)$ with $kp > N$ and $\Phi \in C^k(\mathbb{R})$ then $\Phi \circ u \in W^{k,p}$ since $W^{k,p} \subset L^\infty$ by the Sobolev embedding theorem. When $kp = N$ the situation is more delicate since $W^{k,p}$ is not contained in L^∞. However the following result still holds (see [2],[3])

Theorem 1. *Assume $u \in W^{k,p}(\Omega)$ where $k \geq 1$ is an integer, $1 \leq p < \infty$, and*

$$kp = N. \tag{1}$$

Let $\Phi \in C^k(\mathbb{R})$ with

$$D^j\Phi \in L^\infty(\mathbb{R}) \quad \forall j \leq k. \tag{2}$$

Then

$$\Phi \circ u \in W^{k,p}(\Omega)$$

The proof is based on the following

Lemma 1. *Assume $u \in W^{k,p}(\Omega) \cap W^{1,kp}(\Omega)$ where $k \geq 1$ is an integer and $1 \leq p < \infty$. Assume $\Phi \in C^k(\mathbb{R})$ satisfies (2). Then*

$$\Phi \circ u \in W^{k,p}(\Omega).$$

Proof of Theorem 1. Since $u \in W^{k,p}$ we have

$$\nabla u \in W^{k-1,p} \subset L^q$$

by the Sobolev embedding with

$$\frac{1}{q} = \frac{1}{p} - \frac{k-1}{N}.$$

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Applying assumption (1) we find $q = N = kp$ and thus $u \in W^{1,kp}$. We deduce from Lemma 1 that $\Phi \circ u \in W^{k,p}$.

Proof of Lemma 1. Note that if $u \in W^{k,p} \cap L^\infty$ with $k \geq 1$ integer and $1 \leq p < \infty$ then $u \in W^{1,kp}$ by the Gagliardo - Nirenberg inequality [14]. Thus, Lemma 1 is a generalization of the standard result about composition. In fact, it is proved exactly in the same way as in the standard case (when $u \in W^{k,p} \cap L^\infty$). When $k = 2$ the conclusion is trivial.

Assume, for example that, $k = 3$, then

$$W^{3,p} \cap W^{1,3p} \subset W^{2,3p/2}$$

by the Gagliardo - Nirenberg inequality. Then

$$D^3(\Phi \circ u) = \Phi'(u)D^3u + 3\Phi''(u)D^2uDu + \Phi'''(u)(Du)^3,$$

and thus $\Phi \circ u \in W^{3,p}$ since

$$\begin{aligned} \int |D^2u|^p|Du|^p &\leq \Big(\int |D^2u|^{3p/2}\Big)^{2/3}\Big(\int |Du|^{3p}\Big)^{1/3} \\ &\leq C\|u\|^{p/2}_{W^{3,p}}\|u\|^{3p/2}_{W^{1,3p}}. \end{aligned}$$

A simular argument holds for any $k \geq 4$.

Starting in the mid-60's a number of authors considered composition in various classes of "Sobolev spaces" $W^{s,p}$, where $s > 0$ is a real number and $1 \leq p < \infty$. The most commonly used are the Bessel potential spaces $L^{s,p}(\mathbb{R}^N) = \{f = G_s * g; g \in L^p(\mathbb{R}^N)\}$ where $\widehat{G}_s = (1+|\xi|^2)^{-s/2}$ and the Besov spaces $B^{s,p}_p(\mathbb{R}^N)$ (who's definition is recalled below when s is **not** an integer). It is well-known (see e.g. [1],[19] and [20]) that if k is an integer, $L^{k,p}$ coincides with the standard Sobolev space $W^{k,p}$; also if $p = 2$, the Bessel potential $L^{s,2}$ and the Besov spaces $B^{s,2}_2$ coincide for every s non-integer and they are usually denoted by H^s. When $p \neq 2$ the spaces $L^{s,p}$ and $B^{s,p}_p$ are distinct.

The first result about composition in fractional Sobolev spaces seems to be due to Mizohata [12] for H^s,$s > N/2$. In 1970 Peetre [15] considered $B^{s,p}_p \cap L^\infty$ using interpolation techniques; a very simple direct argument for the same class, $B^{s,p}_p \cap L^\infty$, was given by M. Escobedo [10] (see the proof of Lemma 2 below).

Starting in 1980 techniques of dyadic analysis and Littlewood-Paley decomposition à la Bony [5] were introduced. For example, Y. Meyer [11] considered composition in $L^{s,p}$ for $sp > N$; see also [16],[4],[9] for H^s with $s > N/2$ or for $H^s \cap L^\infty$, any $s > 0$. We refer to [17],[6],[7],[18] and their bibliographies for other directions of research concerning composition in Sobolev spaces.

In what follow we denote by $W^{s,p}(\Omega)$ the restriction of $B^{s,p}_p(\mathbb{R}^N)$ to Ω when s is not an integer. Our main result is the following

Theorem 2. *Assume $u \in W^{s,p}(\Omega)$ where $s > 1$ is a real number, $1 < p < \infty$, and*

$$sp = N. \tag{3}$$

Let $\Phi \in C^k(\mathbb{R})$, where $k = [s] + 1$, be such that

$$D^j\Phi \in L^\infty(\mathbb{R}) \quad \forall j \leq k. \tag{4}$$

Then

$$\Phi \circ u \in W^{s,p}(\Omega).$$

The proof of Theorem 2 relies on a variant of Lemma 1 for fractional Sobolev spaces.

Lemma 2. *Let $u \in W^{s,p}(\Omega)$, where $s > 1$ is a real number and $1 < p < \infty$. Assume, in addition, that $u \in W^{\sigma,q}$ for some $\sigma \in (0,1)$ with*

$$q = sp/\sigma. \tag{5}$$

Let $\Phi \in C^k(\mathbb{R})$, where $k = [s] + 1$, be such that (4) holds. Then

$$\Phi \circ u \in W^{s,p}$$

Proof of Theorem 2. By the Sobolev embedding theorem we have

$$W^{s,p} \subset W^{r,q}$$

with $r < s$ and

$$\frac{1}{q} = \frac{1}{p} - \frac{(s-r)}{N}.$$

In view of assumption (3) we find

$$q = N/r.$$

In particular,

$$u \in W^{\sigma,q}$$

for **all** $\sigma \in (0,1)$ with

$$q = \frac{N}{\sigma} = \frac{sp}{\sigma}.$$

Thus we may apply Lemma 2 and conclude that $\Phi \circ u \in W^{s,p}$.

Remark 1. Theorem 2 is known to be true when the Sobolev spaces $W^{s,p}$ are replaced by the Bessel potential spaces $L^{s,p}$ with $sp = N$; see D. Adams and M. Frazier [3]. Even though the two results are closely related it does not seem possible to deduce one from the other. Their argument relies on a variant of Lemma 2 for Bessel potential spaces:

Let $u \in L^{s,p} \cap L^{1,sp}$ where $s > 1$ is a real number and $1 < p < \infty$. Let Φ be as in Lemma 2. Then $\Phi \circ u \in L^{s,p}$.

Remark 2. The assumption in Lemma 2, $u \in W^{s,p} \cap W^{\sigma,q}$, with $q = sp/\sigma$ for some $\sigma \in (0,1)$, is **weaker** than the assumption $u \in W^{s,p} \cap L^\infty$ but it is **stronger** than the assumption $u \in W^{1,sp}$; this is a consequence of Gagliardo - Nirenberg type inequalities (see e.g. the proof of Lemma D.1 in the Appendix D of [8]). It is therefore natural to raise the following:

Open Problem. Is the conclusion of Lemma 2 valid if one assumes only $u \in W^{s,p} \cap W^{1,sp}$ where $s > 1$ is a (non-integer) real number?

Before giving the proof of Lemma 2 we recall some properties of $W^{s,p}$ when s is not an integer.

When $0 < \sigma < 1$ and $1 < p < \infty$ the standard definition of $W^{\sigma,p}$ is

$$W^{\sigma,p}(\Omega) = \{f \in L^p(\Omega);\ \int\int \frac{|f(x) - f(y)|^p}{|x-y|^{N+\sigma p}} dxdy < \infty\}.$$

If $s > 1$ is not an integer write $s = [s] + \sigma$ where $[s]$ denotes the integer part of s and $0 < \sigma < 1$. Then

$$W^{s,p}(\Omega) = \{f \in W^{[s],p}(\Omega), D^\alpha f \in W^{\sigma,p} \text{ for } |\alpha| = [s]\}.$$

There is a very useful characterization of $W^{s,p}$ in terms of finite differences (see Triebel [20], p.110). Here it is more convenient to work with functions defined on all of $\mathbb{R}^N$ and to consider their restrictions to Ω. Set

$$(\delta_h u)(x) = u(x+h) - u(x),\ h \in \mathbb{R}^N,$$

so that

$$(\delta_h^2 u)(x) = u(x+2h) - 2u(x+h) + u(x),\ \text{etc...}$$

Given $s > 1$ not integer, fix **any** integer $M > s$. Then

$$W^{s,p} = \{f \in L^p;\ \int\int \frac{|\delta_h^M f(x)|^p}{|h|^{N+sp}} dxdh < \infty\}.$$

Proof of Lemma 2. It suffices to consider the case where s is not an integer. For simplicity we treat just the case where $1 < s < 2$. The same argument extends to general $s > 2, s$ noninteger, using the same type of computations as in Escobedo [10].

The key observation is that $\delta_h^2(\Phi \circ u)$ can be expressed in terms of $\delta_h^2 u$ and $\delta_h u$. This is the purpose of our next computation.

Set

$$X = u(x+2h)$$
$$Y = u(x+h)$$
$$Z = u(x).$$

Since $\Phi'' \in L^\infty(\mathbb{R})$ we have

$$\Phi(X) - \Phi(Y) = \Phi'(Y)(X-Y) + 0(|X-Y|^2) \tag{6}$$

and since $\Phi' \in L^\infty(\mathbb{R})$ we also have

$$\Phi(X) - \Phi(Y) = \Phi'(Y)(X-Y) + 0(|X-Y|). \tag{7}$$

Combining (6) and (7) we find

$$\Phi(X) - \Phi(Y) = \Phi'(Y)(X-Y) + 0(|X-Y|^a)$$

for any $1 \leq a \leq 2$ (we will choose a specific value of a later) Similarly

$$\Phi(Z) - \Phi(Y) = \Phi'(Y)(Z-Y) + 0(|Z-Y|^a)$$

Since
$$\delta_h^2(\Phi \circ u)(x) = (\Phi(X) - \Phi(Y)) + (\Phi(Z) - \Phi(Y)),$$
one finds
$$|\delta_h^2(\Phi \circ u)(x)| \leq C(|\delta_h^2 u(x)| + |\delta_h u(x+h)|^a + |\delta_h u(x)|^a). \tag{8}$$
This yields
$$\int\int \frac{|\delta_h^2(\Phi \circ u)(x)|^p}{|h|^{N+sp}} dxdh \leq C \int\int \frac{|\delta_h^2 u(x)|^p}{|h|^{N+sp}} dxdh + C \int\int \frac{|\delta_h u(x)|^{ap}}{|h|^{N+sp}} dxdh. \tag{9}$$
The first integral on the right-hand side of (9) is finite since $u \in W^{s,p}$. To handle the second integral we argue as follows. From the assumption $u \in W^{s,p} \cap W^{\sigma,q}$ with $\sigma \in (0,1)$ and q given by (5) we know that
$$\int\int \frac{|\delta_h^2 u(x)|^p}{|h|^{N+sp}} dxdh < \infty \text{ and } \int\int \frac{|\delta_h^2 u(x)|^q}{|h|^{N+sp}} dxdh < \infty. \tag{10}$$
From (10) and Hölder's inequality we derive that
$$\int\int \frac{|\delta_h^2 u(x)|^r}{|h|^{N+sp}} dxdh < \infty \tag{11}$$
for all $r \in [p,q]$, i.e., $u \in W^{\tau,r}$ with $\tau = sp/r$. We now choose
$$a = min\{2, s/\sigma\}, \text{ so that } a \in [1,2]$$
and $r = ap \in [p,q]$. It follows that
$$\int\int \frac{|\delta_h u(x)|^{ap}}{|h|^{N+sp}} dxdh < \infty,$$
which is the desired in equality.

Remark 3. There could be another natural proof of Theorem 2 by induction on [s]. One might attempt to prove that
$$D(\Phi \circ u) = \Phi'(u)Du \in W^{s-1,p}.$$
Note that $u \in W^{(s-1),N/(s-1)}$ and thus (by induction) we would have $\Phi'(u) \in W^{(s-1),N/(s-1)}$. On the other hand $Du \in W^{s-1,p}$. In order to conclude we need a lemma about products, but we are not aware of any such tool.

Remark 4. When s (or equivalently p) is a **rational** number, and $\Phi \in C^\infty$ with $D^j\Phi \in L^\infty \, \forall j$, there is a simple proof of Theorem 2 based on trace theory and Theorem 1. Assume for simplicity that $\Omega = \mathbb{R}^N$. Suppose that s is not an integer, but that $s_1 = s + 1/p$ is an integer. Then u is the trace of some function $u_1 \in W^{s_1,p}(\mathbb{R}^{N+1})$. Then $s_1 p = N + 1$ and by Theorem 1 we deduce that $\Phi \circ u_1 \in W^{s_1,p}(\mathbb{R}^{N+1})$. Taking traces we find $\Phi \circ u \in W^{s,p}(\mathbb{R}^N)$. If s_1 is not an integer we keep extending u_1 to higher dimensions and stop at the first integer k such that $s_k = s + k/p$ is an integer (this is possible since p is rational and $s + k/p = (N+k)/p$ becomes an integer for some integer k). We have an extension

$u_k \in W^{s_k,p}(\mathbb{R}^{N+k})$ of u. Then $\Phi \circ u_k \in W^{s_k,p}(\mathbb{R}^{N+k})$ by Theorem 1. Taking back traces yields $u \in W^{s,p}$.

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