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Équations aux dérivées partielles/*Partial Differential Equations*

On the structure of the Sobolev space $\mathrm{H}^{1/2}$ with values into the circle

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Abstract. We are concerned with properties of $\mathrm{H}^{1/2}(\Omega;\ \mathrm{S}^1)$ where Ω is the boundary of a domain in $\mathbb{R}^3$. To every $u \in \mathrm{H}^{1/2}(\Omega;\ \mathrm{S}^1)$ we associate a distribution $T(u)$ which, in some sense, describes the location and the topological degree of singularities of u. The closure Y of $\mathrm{C}^\infty(\Omega;\ \mathrm{S}^1)$ in $\mathrm{H}^{1/2}$ coincides with the u's such that $T(u) = 0$. Moreover, every $u \in Y$ admits a unique $(\mathrm{mod.}\ 2\pi)$ lifting in $\mathrm{H}^{1/2} + \mathrm{W}^{1,1}$. We also discuss an application to the 3-d Ginzburg–Landau problem. © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Sur la structure de l'espace de Sobolev $\mathrm{H}^{1/2}$ *à valeurs dans le cercle*

Résumé. *On s'intéresse aux propriétés des fonctions de* $\mathrm{H}^{1/2}(\Omega;\ \mathrm{S}^1)$ *où* Ω *est le bord d'un domaine de* $\mathbb{R}^3$. *À tout* $u \in \mathrm{H}^{1/2}(\Omega;\ \mathrm{S}^1)$ *on associe une distribution* $T(u)$ *qui décrit l'emplacement et le degré topologique des singularités de* u. *La fermeture* Y *de* $\mathrm{C}^\infty(\Omega;\ \mathrm{S}^1)$ *dans* $\mathrm{H}^{1/2}$ *coincide avec l'ensemble des* u *tels que* $T(u) = 0$. *De plus, tout* $u \in Y$ *s'écrit de manière unique* $(\mathrm{mod.}\ 2\pi)$ *sous la forme* $u = \mathrm{e}^{i\varphi}$ *avec* $\varphi \in \mathrm{H}^{1/2} + \mathrm{W}^{1,1}$. *On présente aussi une application au problème de Ginzburg–Landau en* 3-d. © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Soit $G \subset \mathbb{R}^3$ un domaine borné régulier tel que $\Omega = \partial G$ soit simplement connexe. On s'intéresse aux propriétés des fonctions de $\mathrm{H}^{1/2}(\Omega;\ \mathrm{S}^1)$. Par analogie avec les résultats de [8] et [3], on associe à tout $u \in \mathrm{H}^{1/2}(\Omega;\ \mathrm{S}^1)$ une distribution $T(u)$ qui agit sur $\mathrm{C}^1(\Omega)$. Lorsque u admet seulement un nombre fini de singularités (a_j) de degré (d_j), on a $T(u) = 2\pi\Sigma_j d_j \delta_{a_j}$. On sait alors que $\sup\{\langle T(u), \varphi\rangle;\ \|\varphi\|_{\mathrm{Lip}} \leqslant 1\}$ est la longueur de la connexion minimale (au sens de [8]) associée aux singularités de u. On montre que $u \in Y$

Note présentée par Haïm BREZIS.

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si et seulement si $T(u) = 0$ (ceci est l'analogue $\mathrm{H}^{1/2}$ d'un résultat de Bethuel [1] concernant les fonctions de $\mathrm{H}^1(\mathrm{B}^3;\ \mathrm{S}^2)$). On prouve que toute fonction $u \in Y$ s'écrit (de manière unique $\mathrm{mod.}\ 2\pi$) sous la forme $u = \mathrm{e}^{i\varphi}$ avec $\varphi \in \mathrm{H}^{1/2}(\Omega;\ \mathbb{R}) + \mathrm{W}^{1,1}(\Omega;\ \mathbb{R})$. De plus, on a l'estimation $\|\varphi\|_{\mathrm{H}^{1/2}+\mathrm{W}^{1,1}} \leqslant C(1 + \|u\|^2_{\mathrm{H}^{1/2}})$. La preuve de cette estimée utilise la théorie des paraproduits au sens de J.-M. Bony et Y. Meyer.

Enfin, on considère l'énergie de Ginzburg–Landau E_ε définie par (11), où g_ε est une approximation de g au sens de (10). On suppose que $g \in Y$ et on écrit $g = \mathrm{e}^{i\varphi_0}$ avec $\varphi_0 \in \mathrm{H}^{1/2}(\Omega;\ \mathbb{R}) + \mathrm{W}^{1,1}(\Omega;\ \mathbb{R})$. Alors les minimiseurs u_ε de (11) convergent vers $u_\star = \mathrm{e}^{i\varphi}$, où φ est la solution de (12).

Let $G \subset \mathbb{R}^3$ be a smooth bounded domain with $\Omega = \partial G$ simply connected. We are concerned with the properties of the space

$$\mathrm{H}^{1/2}\big(\Omega;\ \mathrm{S}^1\big) = \big\{u \in \mathrm{H}^{1/2}\big(\Omega;\ \mathbb{R}^2\big);\ |u| = 1 \text{ a.e. on } \Omega\big\}.$$

Recall (*see* [5]) that there are functions in $\mathrm{H}^{1/2}(\Omega;\ \mathrm{S}^1)$ which cannot be written in the form $u = \mathrm{e}^{i\varphi}$ with $\varphi \in \mathrm{H}^{1/2}(\Omega;\ \mathbb{R})$. For example, we may assume that locally, near a point on Ω, say 0, Ω is a disc B_1; then take

$$u(x,y) = (x,y)/\big(x^2+y^2\big)^{1/2} \quad \text{on } B_1. \tag{1}$$

Recall also (*see* [12]) that there are functions in $\mathrm{H}^{1/2}(\Omega;\ \mathrm{S}^1)$ which cannot be approximated in the $\mathrm{H}^{1/2}$-norm by functions in $\mathrm{C}^\infty(\Omega, \mathrm{S}^1)$. Consider, for example, again a function u which is the same as in (1) near 0.

It is therefore natural to introduce the classes

$$X = \big\{u \in \mathrm{H}^{1/2}\big(\Omega;\ \mathrm{S}^1\big);\ u = \mathrm{e}^{i\varphi} \text{ for some } \varphi \in \mathrm{H}^{1/2}(\Omega;\ \mathbb{R})\big\}$$

and

$$Y = \overline{\mathrm{C}^\infty\big(\Omega;\ \mathrm{S}^1\big)}^{\mathrm{H}^{1/2}}.$$

Clearly, we have

$$X \subset Y \subset \mathrm{H}^{1/2}\big(\Omega;\ \mathrm{S}^1\big).$$

Moreover, these inclusions are strict. Any function $u \in \mathrm{H}^{1/2}(\Omega;\ \mathrm{S}^1)$ which satisfies (1) does not belong to Y. On the other hand the function

$$u(x,y) = \begin{cases} \mathrm{e}^{2i\pi/r^\alpha} & \text{on } B_1, \\ 1 & \text{on } \Omega \setminus B_1, \end{cases} \tag{2}$$

with $r = (x^2+y^2)^{1/2}$ and $1/2 \leqslant \alpha < 1$, belongs to Y, but not to X (*see* [5]).

To every function $u \in \mathrm{H}^{1/2}(\Omega;\ \mathbb{R}^2)$ we associate a distribution $T = T(u) \in \mathcal{D}'(\Omega;\ \mathbb{R})$. When $u \in \mathrm{H}^{1/2}(\Omega;\ \mathrm{S}^1)$ the distribution $T(u)$ describes the location and the topological degree of its singularities.

Given $u \in \mathrm{H}^{1/2}(\Omega;\ \mathbb{R}^2)$ and $\varphi \in \mathrm{C}^1(\Omega;\ \mathbb{R})$ consider any $U \in \mathrm{H}^1(G;\ \mathbb{R}^2)$ and any $\Phi \in \mathrm{C}^1(\overline{G};\ \mathbb{R}^2)$ such that

$$U|_\Omega = u \quad \text{and} \quad \Phi|_\Omega = \varphi.$$

Set

$$H = 2(U_y \wedge U_z, U_z \wedge U_x, U_x \wedge U_y);$$

this H is independent of the choice of direct orthonormal bases in $\mathbb{R}^3$ (to compute derivatives) and in $\mathbb{R}^2$ (to compute $\wedge$-products). Next, consider

$$\int_G H \cdot \nabla \Phi. \tag{3}$$

It is not difficult to show (*see* [6]) that (3) is independent of the choice of U and Φ; it depends only on u and φ. We may thus define the distribution [1] $T(u) \in \mathcal{D}'(\Omega;\, \mathbb{R})$ by:

$$\langle T(u), \varphi \rangle = \int_G H \cdot \nabla \Phi.$$

If there is no ambiguity we will simply write T instead of $T(u)$.

When u has a little more regularity we may also express T in a simpler form.

LEMMA 1. – *If* $u \in \mathrm{H}^{1/2}(\Omega;\, \mathbb{R}^2) \cap \mathrm{W}^{1,1}(\Omega;\, \mathbb{R}^2) \cap \mathrm{L}^\infty(\Omega;\, \mathbb{R}^2)$, *then*

$$\langle T(u), \varphi \rangle = \int_\Omega (u \wedge u_x)\varphi_y - (u \wedge u_y)\varphi_x, \quad \forall\, \varphi \in \mathrm{C}^1(\Omega;\, \mathbb{R}),$$

for any choice of local orthonormal coordinates (x, y) *on* Ω *such that* (x, y, n) *is direct, where* n *is the outward normal to* G.

By analogy with the results of [8] and [3] we introduce, for every $u \in \mathrm{H}^{1/2}(\Omega;\, \mathbb{R}^2)$, the number

$$L(u) = \frac{1}{2\pi} \sup_{\substack{\varphi \in \mathrm{C}^1(\Omega;\, \mathbb{R} \\ \|\varphi\|_{\mathrm{Lip}} \leqslant 1}} \langle T(u), \varphi \rangle,$$

where $\|\varphi\|_{\mathrm{Lip}}$ refers to a given metric on Ω. There are three (equivalent) metrics on Ω which are of interest:

$$\begin{aligned} d_{\mathbb{R}^3}(x, y) &= |x - y|, \\ d_G(x, y) &= \text{the geodesic distance in } \overline{G}, \text{ and} \\ d_\Omega(x, y) &= \text{the geodesic distance in } \Omega. \end{aligned} \tag{4}$$

It is easy to see that

$$\big|L(u)\big| \leqslant C\|u\|^2_{\mathrm{H}^{1/2}}, \quad \forall\, u \in \mathrm{H}^{1/2}\big(\Omega;\, \mathbb{R}^2\big) \tag{5}$$

and

$$\big|L(u) - L(v)\big| \leqslant C\|u - v\|_{\mathrm{H}^{1/2}}\big(\|u\|_{\mathrm{H}^{1/2}} + \|v\|_{\mathrm{H}^{1/2}}\big), \quad \forall\, u, v \in \mathrm{H}^{1/2}\big(\Omega;\, \mathbb{R}^2\big). \tag{6}$$

When u takes its values in S^1 and has only a finite number of singularities there are very simple expressions for $T(u)$ and $L(u)$:

LEMMA 2. – *If* $u \in \mathrm{H}^{1/2}(\Omega;\, \mathrm{S}^1) \cap \mathrm{H}^1_{\mathrm{loc}}\big(\Omega \setminus \bigcup_{j=1}^k \{a_j\};\, \mathrm{S}^1\big)$, *then*

$$T(u) = 2\pi \sum_{j=1}^k d_j \delta_{a_j},$$

where $d_j = \deg(u, a_j)$ *and* $L(u)$ *is the length of the minimal connection associated to the configuration* (a_j, d_j) *and to the specific metric on* Ω (*in the sense of* [8]; see *also* [13]).

Remark 1. – Here $\deg(u, a_j)$ denotes the topological degree of u restricted to any small circle around a_j, positively oriented with respect to the outward normal. It is well defined using the degree theory for maps in $\mathrm{H}^{1/2}(\mathrm{S}^1;\ \mathrm{S}^1)$ (*see* [7] and [10]).

We will also make use of a density result of T. Rivière which is the $\mathrm{H}^{1/2}$ analogue of a result of Bethuel and Zheng [4] concerning H^1 maps from B^3 to S^2 (*see* also a related result of Bethuel [2] in fractional Sobolev spaces). Let $\mathcal{R}$ denote the class of maps in $\mathrm{H}^{1/2}(\Omega;\ \mathrm{S}^1)$ which are C^∞ on Ω except at a finite number of points.

LEMMA 3 (T. Rivière [19]). – *The class $\mathcal{R}$ is dense in* $\mathrm{H}^{1/2}(\Omega;\ \mathrm{S}^1)$.

Still some further elementary facts about T and L:

LEMMA 4. – *For every $u,\ v \in \mathrm{H}^{1/2}(\Omega;\ \mathrm{S}^1)$ we have*:

$$\begin{aligned} &T(uv) = T(u) + T(v),\\ &L\big(u\overline{v}\big) \leqslant C\|u - v\|_{\mathrm{H}^{1/2}}\big(\|u\|_{\mathrm{H}^{1/2}} + \|v\|_{\mathrm{H}^{1/2}}\big),\\ &L(uv) \leqslant L(u) + L(v). \end{aligned}$$

Here, we have identified $\mathbb{R}^2$ with $\mathbb{C}$ and uv denotes complex multiplication.
Using Lemmas 3 and 4 we may extend the representation formula of Lemma 2 to general functions in $\mathrm{H}^{1/2}(\Omega;\ \mathrm{S}^1)$:

THEOREM 1. – *Given any $u \in \mathrm{H}^{1/2}(\Omega;\ \mathrm{S}^1)$ there are two sequences of points (P_i) and (N_i) in Ω such that*

$$\sum_i |P_i - N_i| < \infty, \tag{7}$$

$$\langle T(u), \zeta\rangle = 2\pi \sum_i \big(\zeta(P_i) - \zeta(N_i)\big). \tag{8}$$

In addition, for any metric d in (4)

$$L(u) = \mathrm{Inf} \sum_i d(P_i, N_i),$$

where the infimum is taken over all possible sequences (P_i), (N_i) satisfying (7), (8). *In case the distribution T is a measure* (*of finite total mass*) *then*

$$T(u) = 2\pi \sum_{\text{finite}} d_j \delta_{a_j}$$

with $d_j \in \mathbb{Z}$ and $a_j \in \Omega$.

The last assertion in Theorem 1 is the $\mathrm{H}^{1/2}$-analogue of a result of Jerrard and Soner [15,16] (*see* also Hang and Lin [14]) concerning maps in $\mathrm{W}^{1,1}(\Omega;\ \mathrm{S}^1)$. In Theorem 1, the last assertion can be derived from the first assertion via a direct abstract argument (*see* Smets [21]).
Maps in Y can be characterized in terms of the distribution T:

THEOREM 2 (T. Rivière [19]). – *Let $u \in \mathrm{H}^{1/2}(\Omega;\ \mathrm{S}^1)$, then $T(u) = 0$ if and only if $u \in Y$.*

This result is the $\mathrm{H}^{1/2}$-counterpart of a well-known result of Bethuel [1] characterizing the closure of smooth maps in $\mathrm{H}^1(\mathrm{B}^3; \mathrm{S}^2)$ (*see* also Demengel [11]).
As was mentioned earlier, functions in Y need not belong to X, i.e., they need not have a lifting in $\mathrm{H}^{1/2}(\Omega;\ \mathbb{R})$. However we have:

THEOREM 3. – *For every $u \in Y$ there exists $\varphi \in \mathrm{H}^{1/2}(\Omega;\ \mathbb{R}) + \mathrm{W}^{1,1}(\Omega;\ \mathbb{R})$, which is unique* (modulo 2π) *such that $u = \mathrm{e}^{i\varphi}$. Conversely, if $u \in \mathrm{H}^{1/2}(\Omega;\ \mathrm{S}^1)$ can be written as $u = \mathrm{e}^{i\varphi}$ with $\varphi \in \mathrm{H}^{1/2} + \mathrm{W}^{1,1}$ then $u \in Y$.*

The existence is proved with the help of paraproducts (in the sense of J.-M. Bony and Y. Meyer) (*see* [6]). The heart of the matter is the estimate

$$\|\varphi\|_{\mathrm{H}^{1/2}+\mathrm{W}^{1,1}} \leqslant C\big(1 + \|u\|^2_{\mathrm{H}^{1/2}}\big), \tag{9}$$

which holds for smooth φ. The uniqueness (modulo 2π) of φ is established as in [9]. Theorem 3 is still valid for domains $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$ (*see* [6]).

Remark 2. – Using Theorem 3 and the basic estimate (9) one may prove that for every $u \in \mathrm{H}^{1/2}(\Omega;\ \mathrm{S}^1)$ there exists $\varphi \in \mathrm{H}^{1/2}(\Omega;\ \mathbb{R}) + \mathrm{BV}(\Omega;\ \mathbb{R})$ such that $u = \mathrm{e}^{i\varphi}$. Of course this φ is not unique, but there are some "distinguished" φ's (*see* [6]).

The link between the Ginzburg–Landau energy and minimal connections has been first pointed out in the important work of T. Rivière [18] (*see* also [17] and [20]) in the case of boundary data with a finite number of singularities. We are concerned here with a general boundary condition g in $\mathrm{H}^{1/2}$.

Given $g \in \mathrm{H}^{1/2}(\Omega;\ \mathrm{S}^1)$ we may always approximate it by a sequence $g_\varepsilon \in \mathrm{C}^\infty(\Omega;\ \mathbb{R}^2)$ such that

$$\begin{cases} \|g_\varepsilon - g\|_{\mathrm{L}^2} \leqslant C\sqrt{\varepsilon}, \quad \|\nabla g_\varepsilon\|_{\mathrm{L}^\infty} \leqslant C/\varepsilon, \quad \|g_\varepsilon\|_{\mathrm{L}^\infty} \leqslant 1, \text{ and} \\ g_\varepsilon \to g \quad \text{in } \mathrm{H}^{1/2}. \end{cases} \tag{10}$$

Set

$$E_\varepsilon = \mathrm{Min}\left\{\frac{1}{2}\int_G |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_G \big(|u|^2 - 1\big)^2;\ u \in \mathrm{H}^1\big(G;\ \mathbb{R}^2\big) \text{ and } u = g_\varepsilon \text{ on } \Omega\right\}. \tag{11}$$

THEOREM 4. – *We have, as $\varepsilon \to 0$,*

$$E_\varepsilon = \pi L(g)\log(1/\varepsilon) + \mathrm{o}\big(\log(1/\varepsilon)\big),$$

where $L(g)$ corresponds to the metric d_G on Ω.

Finally, we study the convergence of minimizers (u_ε) of (11). If $g \in X$ we may write $g = \mathrm{e}^{i\varphi_0}$ with $\varphi_0 \in \mathrm{H}^{1/2}(\Omega;\ \mathbb{R})$. A natural choice for g_ε is $g_\varepsilon = \mathrm{e}^{i\varphi_\varepsilon}$ where φ_ε is an ε-regularization of φ as in (10). In this case it is easy to prove that

$$u_\varepsilon \to u_* = \mathrm{e}^{i\varphi} \quad \text{in } \mathrm{H}^1(G),$$

where φ is the solution of

$$\Delta\varphi = 0 \quad \text{in } G, \quad \varphi = \varphi_0 \quad \text{on } \Omega. \tag{12}$$

When $g \in Y$ we prove (*see* [6]):

THEOREM 5. – *For every $g \in Y$ write* (*as in Theorem* 3) $g = \mathrm{e}^{i\varphi_0}$ *with $\varphi_0 \in \mathrm{H}^{1/2} + \mathrm{W}^{1,1}$. Then* (*for any choice of g_ε satisfying* (10)) *we have*

$$u_\varepsilon \to u_* = \mathrm{e}^{i\varphi} \quad \textit{in } \mathrm{W}^{1,p}(G) \cap \mathrm{C}^\infty(G),\ \forall p < 3/2,$$

where φ is defined in (12).

Remark 3. – We shall also present in [6] results concerning the convergence of u_ε when $g \in \mathrm{H}^{1/2}(\Omega;\ \mathrm{S}^1)$ does not belong to Y.

[1] The distribution $T(u)$ and the corresponding number $L(u)$ were originally introduced, for a general $u \in \mathrm{H}^{1/2}$, by the authors in 1996 and these concepts were presented in various lectures.

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