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THE FASCINATING HOMOTOPY STRUCTURE OF SOBOLEV SPACES

Abstract. — We discuss recent developments in the study of the homotopy classes for the Sobolev spaces $W^{1,p}(M; N)$. In particular, we report on the work of H. Brezis - Y. Li [5] and F.B. Hang - F.H. Lin [9].

Key words: Sobolev spaces; Homotopy theory.

0. Introduction

Let M and N be metric spaces with M compact. Standard homotopy theory deals with the question whether two given *continuous* maps, $f, g \in C^0(M; N)$ can be homotopied, *i.e.*, whether there exists a continuous deformation from f to g, that is, a path $h(t) \in C([0, 1]; C^0(M; N))$ such that $h(0) = f$ and $h(1) = g$. Here the space $C^0(M; N)$ is equipped with the standard metric

$$d(f_1, f_2) = \operatorname*{Max}_{x \in M} d_N(f_1(x), f_2(x)).$$

In other words, the objective is to determine whether $C^0(M; N)$ is path-connected (or equivalently, connected). If not, one then hopes to give a complete *classification* of the *components* of $C^0(M; N)$. This amounts to find all the *topological invariants*. If f and g belong to the same component, *i.e.*, are homotopic, we will write

$$f \sim g \text{ in } C^0.$$

For example, when $M = N = S^n$, a celebrated result of *H.* Hopf is

Theorem 1. *Let $f, g \in C^0(S^n; S^n)$, then*

$$[f \sim g \text{ in } C^0] \Leftrightarrow \deg(f) = \deg(g),$$

where deg *denotes the standard topological (Brouwer) degree.*

In what follows we assume that M and N have more structure; for example that M and N are smooth connected, compact, Riemannian manifolds with $\partial N = \emptyset$ (M with or without boundary). We have in mind simple manifolds such as spheres, products of spheres, domains in $\mathbb{R}^N$ - for example M could be a solid torus in $\mathbb{R}^3$, $M = S^1 \times B$ where B is the unit disc in $\mathbb{R}^2$, or more generally $M = S^n \times B$ where B is the unit ball in $\mathbb{R}^k$, etc.

In this case, it makes sense to consider the spaces $C^1(M; N)$, $C^2(M; N)$, ...

..., $C^j(M; N)$, ..., equipped with their own metric

$$d_j(f_1, f_2) = \sum_{i=0}^{j} \sum_{|\alpha| \leq i} \operatorname{Max}_{x \in M} d(D^\alpha f_1(x), D^\alpha f_2(x)).$$

On $C^j(M; N)$ we have a natural equivalence relation, $f \sim g$ in C^j, if there exists a path $h \in C([0, 1]; C^j(M; N))$ such that $h(0) = f$ and $h(1) = g$. In other words, $f \sim g$ in C^j if and only if f and g belong to the same component of C^j.

It turns out that such a notion has no interest because of the following two basic lemmas:

LEMMA 1. *Let* $f, g \in C^j(M; N)$; *then*

$$[f \sim g \text{ in } C^j] \Leftrightarrow [f \sim g \text{ in } C^0].$$

LEMMA 2. *Given any* $f \in C^0(M; N)$ *there exists a path* $h(t) \in C([0, 1]; C^0(M; N))$ *such that* $h(0) = f$ *and* $h(t) \in C^j(M; N)$, $\forall j$, $\forall t > 0$.

The proofs of Lemma 1 and Lemma 2 are based on regularization by convolution. Note that the approximants do not belong, in general, to the target manifold N. However, the continuity of the maps implies the *uniform* convergence of the approximants. One may then use the projection onto N which is well-defined (and smooth) in some neighborhood of N.

As a consequence of Lemmas 1 and 2 we may now assert:

COROLLARY 1. *For every* j, *any component of* $C^j(M; N)$ *contains* one and exactly one component of $C^{+1}(M; N)$.

In other words, the components of $C^j(M; N)$ behave like the Russian «matrioshkas» dolls. We may adopt two points of view. As *j increases*, the space $C^j(M; N)$ becomes smaller. But the components of C^j shrink *without changing their shape*. A component of C^j does not «split into pieces» when passing to C^{+1}, also, it does not «disappear». As *j decreases*, the space $C^j(M; N)$ becomes larger. But two components of C^j *never coalesce* into one component of C^{-1} and no «new baby» appears. We are going to see that this simple «self-reproducing» effect is lost when we replace the scale of C^j spaces by a scale of Sobolev spaces. They have a much richer structure which is not yet fully understood.

The motivation for studying homotopy classes in Sobolev spaces is two-fold:

a) The solutions of some nonlinear PDE's admit sometimes singularities which correspond to some observations in the natural world (defects in liquid crystals, vortices in superfluids and superconductors, etc.). It is interesting to study the effect of continuous deformations under Sobolev norms, *e.g.* the energy norm.

b) The existence of *multiple solutions* in variational problems is often established via topological arguments. If the underlying function space admits several connected components one may hope to find a critical point in *each component*.

As we are going to see, the connected components of Sobolev spaces may be quite different from the classical ones. For example $H^1(S^2; S^2)$ has infinitely many components classified by a degree (as for $C^0(S^2; S^2)$); however $H^1(S^3; S^3)$ is path-connected.

Let us recall the definition of the Sobolev space $W^{1,p}(M; N)$. Let $1 \leq p < \infty$ be a real number. The definition of $W^{1,p}(M; \mathbb{R})$ is standard, as well as $W^{1,p}(M; \mathbb{R}^l)$; they are equipped with the norm

$$\|f\|_{W^{1,p}} = \|f\|_{L^p} + \|\nabla f\|_{L^p}.$$

Let $N \subset \mathbb{R}^l$ be an isometric embedding. By definition

$$W^{1,p}(M; N) = \{ f \in W^{1,p}(M; \mathbb{R}^l);\ f(x) \in N \text{ a.e.} \};$$

it is equipped with the metric

$$d(f_1, f_2) = \|f_1 - f_2\|_{W^{1,p}}.$$

When $p = 2$ we also use the notation

$$H^1(M; N) = W^{1,2}(M; N).$$

As in the previous situation of the spaces $C^j(M; N)$, we also have a *scale* of spaces, which are indexed by a real parameter $1 \leq p < \infty$, and which decrease as p increase, *i.e.*,

$$W^{1,p}(M; N) \subset W^{1,q}(M; N) \text{ if } p > q.$$

WARNING. There is an alternative definition of Sobolev maps between manifolds. Set

$$\widehat{W}^{1,p}(M; N) = \overline{C^\infty(M; N)}^{W^{1,p}}.$$

Clearly

$$\widehat{W}^{1,p}(M; N) \subset W^{1,p}(M; N),$$

but, in general, $\widehat{W}^{1,p} \neq W^{1,p}$ (see [11, 1, 9]). The study of $\widehat{W}^{1,p}(M; N)$ from the point of view of homotopy classes is widely open.

DEFINITION. We say that two maps $f, g \in W^{1,p}(M; N)$ are equivalent in $W^{1,p}(M; N)$, and we write

$$f \sim g \text{ in } W^{1,p}(M; N)$$

if there exists a homotopy connecting f to g, *i.e.*, $h \in C([0, 1]; W^{1,p}(M; N))$ with $h(0) = f$ and $h(1) = g$.

The equivalence classes for the equivalence relation $\sim$ are precisely the homotopy classes of $W^{1,p}$, *i.e.*, the *path-connected* components of $W^{1,p}$. As a metric space $W^{1,p}(M; N)$ also admits *connected* components. In principle, the two notions could de distinct. In fact, they coincide because of the following nontrivial result, which is implicit in [9]:

THEOREM 2. *Given $f \in W^{1,p}(M; N)$ there exists $\varepsilon > 0$ (depending on f) such*

$$[g \in W^{1,p}(M; N) \text{ and } \|g - f\|_{W^{1,p}} < \varepsilon] \Rightarrow [g \sim f \text{ in } W^{1,p}(M; N)].$$

1. The homotopy classes of $W^{1,p}(S^n, S^n)$

It is enlightening to start with the study of Sobolev classes in the simple case $M = N = S^n$. The study of homotopy classes of $W^{1,p}(S^n, S^n)$ was initiated in [4] when $p = 2$ and $n = 2$. The motivation there came from a conjecture of Giaquinta and Hildebrandt [8] concerning the existence of *multiple* solutions for an equation arising in the theory of harmonic maps.

It is natural to distinguish 3 cases:

Case 1: $p > n$,

Case 2: $p = n$,

Case 3: $p < n$.

Case 1: $p > n$. This case is the simplest case because of the Sobolev embedding

$$W^{1,p} \subset C^0.$$

The situation here is very similar to the one we encountered for the C^j-classes.
Namely we have

Lemma 1′. *Let $f, g \in W^{1,p}(S^n, S^n)$. Then*

$$[f \sim g \ \text{in} \ W^{1,p}] \Leftrightarrow [f \sim g \ \text{in} \ C^0].$$

In particular, $f \sim g$ in $W^{1,p}$ if and only if $\deg(f) = \deg(g)$.

Lemma 2′. *Given any integer $k \in \mathbb{Z}$ there exists $f \in C^\infty(S^n, S^n)$ such that* $\deg(f) = k$.

As a consequence of Lemma 1′ and 2′ we may now assert:

Corollary 1′. *For any $p > n$, every component of $C^0(S^n; S^n)$ contains one and exactly one component of $W^{1,p}(S^n; S^n)$.*

In other words, as *p increases* from n to ∞, the space $W^{1,p}(S^n, S^n)$ *decreases*. Each component «shrinks» without «changing its shape». We encounter here the same «matrioshka» effect as for C^j.

Case 2: $p = n$. This case is more delicate because $W^{1,n}(S^n; S^n)$ is *not* contained in $C^0(S^n; S^n)$. However, we were still able (in [4]) to define a degree via the *formula*;

$$\deg(f) = \frac{1}{|S^n|} \int_{S^n} \det(\nabla f). \tag{1}$$

Using a density result of Schoen and Uhlenbeck [11] one can prove that the real number defined by the integral in (1) belongs to $\mathbb{Z}$ and that it coincides with the standard topological degree when $f \in C^0 \cap W^{1,p}$.

We also have

LEMMA 1″. *Assume* $f, g \in W^{1,n}(S^n; S^n)$. *Then*

$$[f \sim g \text{ in } W^{1,n}] \Leftrightarrow [\deg(f) = \deg(g)].$$

In particular, we find that $W^{1,n}(S^n, S^n)$ has infinitely many components which are classified by this new degree.

Later on, in [7], we took a slightly different view point and defined a degree *without a formula* for the general class of VMO (=vanishing mean oscillation) maps. We observe that if $f \in \mathrm{VMO}(S^n; S^n)$ then

$$f_\varepsilon = \varrho_\varepsilon * f \in C^\infty(S^n; \mathbb{R}^{n+1})$$

and

$$|f_\varepsilon(x)| \to 1 \text{ uniformly on } S^n$$

(despite the fact that $f_\varepsilon(x)$ does *not* converge uniformly to $f(x)$, in general). Then

$$\bar{f}_\varepsilon(x) = f_\varepsilon(x) / |f_\varepsilon(x)|$$

is well-defined from S^n into S^n for ε sufficiently small, $\varepsilon < \varepsilon_0$, and $\bar{f}_\varepsilon$ is smooth. Hence $\deg(\bar{f}_\varepsilon)$ is defined and constant for $0 < \varepsilon < \varepsilon_0$ (by the standard invariance of degree under homotopy). We define

$$\deg(f) = \deg(\bar{f}_\varepsilon)(0 < \varepsilon < \varepsilon_0). \tag{2}$$

Note that $C^0 \subset$ VMO and then, for $f \in C^0(S^n; S^n)$, the VMO-degree coincides with the standard Brouwer degree. One can also prove (see [7]) that

$$W^{1,p}(M) \subset \text{ VMO } (M) \text{ when } p = \dim M$$

(in fact, this is true in all the limiting cases of the Sobolev embedding, including the fractional Sobolev spaces). As a result $\deg(f)$ is well-defined for $f \in W^{1,n}(S^n; S^n)$ via (2) and this degree coincides with the one given by formula (2).

Case 3: $1 \leqslant p < n$. Here the situation is totally different. We have

THEOREM 3. *For any* $1 \leqslant p < n$, *the space* $W^{1,p}(S^n; S^n)$ *is path-connected.*

A weaker form of Theorem 3 was first noticed in [7]. We observed that no reasonable degree theory exists in $W^{1,p}(S^n, S^n)$ for $p < n$. Indeed

$$Id \sim \text{ const in } W^{1,p}(S^n, S^n), \qquad p < n. \tag{3}$$

To prove (3), fix any $a \in \mathbb{R}^{n+1}$ with $|a| = 2$ and set

$$h(x, t) = \frac{x - ta}{|x - ta|}, \qquad x \in S^n, \qquad t \in [0, 1].$$

Clearly

$$h \in C^\infty(S^n \times ([0, 1] \setminus \{1/2\}); S^n)$$

and an easy calculation shows that

$$h(\cdot,\, t) \in C([0,\, 1];\, W^{1,p}(S^n;\, S^n)), \quad \forall p < n\,.$$

Moreover

$$h(0) = Id, \qquad \deg(h(1)) = 0$$

(since $h(1)$ is not surjective). Hence $h(1)$ is homotopic in C^0 (and thus in C^1, by Lemma 1) to a constant. This proves (3). The same argument shows that any map $f \in C^1(S^n;\, S^n)$ is homotopic in $W^{1,p}(S^n;\, S^n)$, $\forall p < n$, to a constant. (It suffices to choose any $a \in \mathbb{R}^{n+1}$ with $|a| = 2$, $a/2$ a regular value of f and $h(x,\, t) = (f(x) - ta)/|f(x) - ta|$). The general form of Theorem 3 as stated above is due to Brezis and Li [5].

2. Topology sometimes «survives» below the Sobolev threshold $p = \dim M$

In view of the analysis of the example $W^{1,p}(S^n;\, S^n)$, one might be inclined to think that, since functions in $W^{1,p}(M;\, N)$ are *not continuous* when $p \leq \dim M$, and *not even VMO* when $p < \dim M$, there are no homotopy classes below the Sobolev threshold $p = \dim M$, *i.e.*, $W^{1,p}(M;\, N)$ is path-connected for $p < \dim M$. This is indeed true when $\dim M = 2$ (see Theorem 7 below). However, this turns out to be wrong when $\dim M \geq 3$. Such a phenomenon was first pointed out in an example by Rubinstein and Sternberg [10] which broke a «psychological barrier».

Theorem 4. *Let $M = \Omega =$ a solid torus in $\mathbb{R}^3$*, i.e., *$M = S^1 \times B$, where B is the unit disc in $\mathbb{R}^2$. Let $N = S^1$. Then, every map f in $H^1(M;\, N)$ admits a well-defined degree (in $\mathbb{Z}$). Moreover this degree is stable under H^1-convergence.*

More precisely, given $f \in H^1(M;\, N)$, write

$$f = f(x,\, \lambda) : S^1 \times B \to S^1\,.$$

Then, for a.e. $\lambda \in B$, the map

$$x \in S^1 \mapsto f(x,\, \lambda) \in S^1$$

belongs to $H^1(S^1;\, S^1)$; thus it is continuous and we set

$$\varphi(\lambda) = \deg(f(\cdot,\, \lambda)), \text{ for a.e. } \lambda \in B\,.$$

The main and nontrivial assertion is that $\varphi(\lambda)$ is a fixed integer, independent of λ a.e. in B. This integer is called the degree of f.

As a consequence, we see that $H^1(M;\, N)$ admits infinitely many components even though, here, $p = 2 < \dim M = 3$.

Subsequently, we generalized Theorem 4 and gave a more direct proof (the argument of Rubinstein and Sternberg relied on earlier results of Bethuel [1] and White [12]):

Theorem 5 [6]. *Let $M = \Omega = S^n \times \Lambda$ where $n \geq 1$ and Λ is any open connected set in $\mathbb{R}^k$, $k \geq 1$. Let $N = S^n$. Then every map $f \in W^{1,n+1}(M;\, N)$ has a well-defined degree. Moreover this degree is stable under $W^{1,n+1}$ convergence.*

More precisely for a.e. $\lambda \in \Lambda$, the map

$$x \in S^n \mapsto f(x, \lambda) \in S^n$$

belongs to $W^{1, n+1}(S^n; S^n)$; thus it is continuous by the Sobolev embedding. The main point is that $\varphi(\lambda)$ is a constant a.e. on Λ.

Note that $p = n + 1$ can be much smaller than the Sobolev threshold, $\dim M = n + k$.

The assertion that the function $\varphi : \Lambda \to \mathbb{Z}$ defined only a.e. (and not necessarily continuous) is constant is quite striking. This fact can also be related to the following:

LEMMA 3 [2, 3]. *Let Λ be any open connected set in $\mathbb{R}^k$, $k \geq 1$. Let $\varphi : \Lambda \to \mathbb{Z}$ be any measurable function satisfying*

$$\int_\Lambda \int_\Lambda \frac{|\varphi(\lambda) - \varphi(\mu)|^p}{|\lambda - \mu|^{k+1}} d\lambda\, d\mu < \infty, \quad \textit{with } p \geq 1 .$$

Then φ is constant a.e. on Λ.

REMARK. Surprizingly, the exponent $p = n + 1$ in Theorem 5 is *optimal.* If $n < p < n + 1$ (resp. $p = n$), the map

$$x \in S^n \mapsto f(x, \lambda) \in S^n$$

belongs to $W^{1,p}(S^n; S^n)$ for a.e. $\lambda \in \Lambda$. Thus, it is continuous (resp. VMO) and we may still define

$$\varphi(\lambda) = \deg(f(\cdot, \lambda)) \quad \text{for a.e. } \lambda \in \Lambda .$$

However φ need *not* be constant. It is not difficult to construct examples, even with $\Lambda = (0, 1)$, where φ takes two different values.

The precise homotopy structure of $W^{1,p}(S^n \times \Lambda; S^n)$ might be quite complicated when Λ has a rich topological structure. However if Λ is simple, *e.g.* a ball, we have a complete description of the homotopy classes of $W^{1,p}$ for all values of p, $1 \leq p < \infty$:

THEOREM 6 [5]. *Let $M = \Omega = S^n \times B$, where $n \geq 1$ and B is the unit ball in $\mathbb{R}^k$, $k \geq 1$. Let $N = S^n$. Let $f, g \in W^{1,p}(M; N)$.*

If $p \geq n + 1$, then

$$[f \sim g \textit{ in } W^{1,p}] \Leftrightarrow [\deg(f) = \deg(g)],$$

where the degree is defined in Theorem 5.

If $p < n + 1$, then

$$f \sim g \textit{ in } W^{1,p} \quad \forall f, g ,$$

i.e., $W^{1,p}(M; N)$ *is path-connected* $\forall p < n + 1$.

In other words we have a threshold at $p = n + 1$. As p increases from 1 to ∞, the space $W^{1,p}(M; N)$ decreases. When $1 < p < (n + 1)$, $W^{1,p}$ is path-connected. At $p =$

$= n+1$ the space $W^{1,n+1}$ «suddenly» splits into infinity many components. As p increases from $n+1$ to ∞, each components «shrinks without changing its shape» (no further splitting occurs).

3. The homotopy structure of $W^{1,p}(M; N)$ for general M and N

At this stage, we decided with Yanyan Li, in 1999, to investigate the homotopy structure of $W^{1,p}$ for general manifolds M and N and general $p \in [1, \infty)$. We obtained several general results. We also analyzed completely several examples (see *e.g.* Theorem 6 and Theorem 8 below). On the basis of these observations we made several general conjectures which were beautifully solved by Hang and Lin [9].

Here is a first universal result:

Theorem 7 [5]. *Assume* $\dim M \geq 2$. *Then* $W^{1,p}(M; N)$ *is path-connected for any* p, $1 \leq p < 2$ *and any* N.

Our original proof of Theorem 7 is rather sophisticated. Theorem 7 may also be deduced form a general result of Hang-Lin (see Theorem 10 below). In view of the extreme simplicity of the statement of Theorem 7, it would be desirable to have a simpler proof.

Another example we studied carefully is $N = S^1$ with M arbitrary and $p \geq 2$:

Theorem 8 [5]. *Assume* $\dim M \geq 2$ *and* $N = S^1$. *Let* $f, g \in W^{1,p}(M; N)$ *with* $p \geq 2$.

Then

$$[f \sim g \text{ in } W^{1,p}] \Leftrightarrow [f \sim g \text{ in } H^1].$$

Moreover any map f *in* $H^1(M; N)$ *is homotopic in* H^1 *to a smooth map.*

As a consequence we see that if $\dim M \geq 2$ and if $N = S^1$:

– for $1 \leq p < 2$, $W^{1,p}(M; N)$ is path-connected,

– for $p = 2$, $H^1(M; S^1)$ splits into infinitely many components (provided $\pi_1(M) \neq 0$),

– as p increases from 2 to ∞, $W^{1,p}$ shrinks without changing its shape.

Let us summarize some of the examples discussed above:

Ex. 1: $M = N = S^n$,

Ex. 2: $M = S^n \times B$, $N = S^n$,

Ex. 3: M arbitrary with $\dim M \geq 2$, $N = S^1$,

In all these example we see that there is exactly *one* distinguished value of p, say p^*, such that:

– when $p < p^*$, $W^{1,p}$ is path-connected

– when $p \geq p^*$, $W^{1,p}$ shrinks without changing its shape.

We discovered with YanYan Li, that more interesting scenarios might occur with a *cascade* of numbers p^* where a splitting (resp. collapse) of components occurs as p increases (resp. decreases) and p crosses the distinguished values p^*. Here is such an example:

Let $M = N = S^1 \times S^2$. For any map $f\colon M \to N$ write

$$f = (f_1, f_2) = (f_1(x_1, x_2), f_2(x_1, x_2)).$$

In view of Theorem 5 we may consider

$$d_1 = \deg(f_1(\cdot, x_2))$$

which is well-defined (and independent of x_2 a.e.) when $f \in W^{1,p}$ and $p \geqslant 2$.

We may also consider

$$d_2 = \deg(f_2(\cdot, x_1))$$

when $f \in W^{1,p}$ and $p \geqslant 3$.

If $1 \leqslant p < 2$, $W^{1,p}$ is path-connected. For p in the interval $[2, 3)$, $W^{1,p}$ admits infinitely many components classified by d_1. When p reaches the value $p = 3$ a *further* splitting of $W^{1,p}$ occurs (due, in particular, to the appearance of d_2). As p increases in the interval $[3, \infty)$, no new «catastrophe» occurs: the components of $W^{1,p}$ shrink without changing their shapes.

In this example we see that there are *two distinguished values* of p, $p = 2$ and $p = 3$, where the homotopy type of $W^{1,p}$ changes. This motivated us to introduce a new concept which expresses rigorously that a change of homotopy type occurs at p.

Let p be a real number $p \geqslant 1$ and $f \in W^{1,p}(M; N)$. We denote by $[f]_p$ the equivalence class of f, *i.e.*,

$$[f]_p = \{g \in W^{1,p}(M; N);\ g \sim f \text{ in } W^{1,p}\}.$$

The quotient space of $W^{1,p}$ by this equivalence relation is denoted $\widetilde{W}^{1,p}$.

Assume $p > 1$ and $0 < \varepsilon < p - 1$. The map

$$i_{p,\varepsilon}\colon [f]_{p+\varepsilon} \to [f]_{p-\varepsilon}$$

is well-defined from $\widetilde{W}^{1,p+\varepsilon}$ in $\widetilde{W}^{1,p-\varepsilon}$.

Set

$$CT(M, N) = \{p \in (1, \infty);\ i_{p,\varepsilon} \text{ is notbijective for all } 0 < \varepsilon < p - 1\}$$

Let me explain, roughly speaking, the meaning of this concept. The fact that $i_{p,\varepsilon}$ is *not injective* says that we can find two maps $f, g \in W^{1,p+\varepsilon}$ such that:

a) $f \sim g$ in $W^{1,p-\varepsilon}$

b) f is not homotopic to g in $W^{1,p+\varepsilon}$

In other words, 2 components of $W^{1,q}$ have *coalesced* as q decreases from $p + \varepsilon$ to $p - \varepsilon$.

The fact that $i_{p,\varepsilon}$ is *not surjective* says that some $f \in W^{1,p-\varepsilon}$ is not homotopic to any $g \in W^{1,p+\varepsilon}$. In other words, a *new component* of $W^{1,q}$ is «born» as q decreases from $p + \varepsilon$ to $p - \varepsilon$.

We made in [5] two conjectures:

Conjecture 1. $CT(M, N)$ consists only of integers, *i.e.*, a change of topology for $W^{1,p}$ occurs only when p is an integer.

CONJECTURE 2. Any map $f \in W^{1,p}(M, N)$ can always be connected in $W^{1,p}$ to a smooth map (any p, any M, any N).

They were both solved in a beautiful piece of work by Hang and Lin [9].

The answer to Conjecture 1 is positive:

THEOREM 9 [9]. *$CT(M, N)$ consists only of integers.*

Theorem 9 is an immediate consequence of the following remarkable result which provides a «reduction» of the study of homotopy classes for $W^{1,p}$ to more classical concepts in Topology. Given $p \geq 2$, let $M^{[p]-1}$ be the $([p]-1)$-skeleton of M.

THEOREM 10 [9]. *Let $f, g \in W^{1,p}(M, N)$. Then $f \sim g$ in $W^{1,p}$ if and only if $f_{|\Sigma} \sim g_{|\Sigma}$ in C^0 for a generic Σ in $M^{[p]-1}$.*

Note that $\dim \Sigma \leq [p] - 1$, and for a generic Σ, $f_{|\Sigma} \in W^{1,p}(\Sigma)$, while $p > \dim \Sigma$ (since $[p] \geq \dim \Sigma + 1$). Thus for a generic Σ, $f_{|\Sigma} \in C^0$, by the Sobolev embedding.

Concerning Conjecture 2, we had presented in [5] several cases where Conjecture 2 is true: for example when $\dim M = 2$ (any N), or $\dim M = 3$ and $\partial M \neq \phi$ (any N), or $N = S^1$ (any M). However Hang and Lin [9] found some situations where Conjecture 2 fails.

Here is such an example:

THEOREM 11 [9]. *Let $M = \mathbb{RP}^3$ and $N = \mathbb{RP}^2$. Then $CT(M, N) = \{2, 3\}$. Moreover, there are maps $f \in W^{1,p}(M, N)$, with $p \in [2, 3)$, such that f cannot be path-connected in $W^{1,p}$ to any smooth map.*

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