

**BLOW UP FOR  $u_t - \Delta u = g(u)$  REVISITED**

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**1. Introduction.** In this paper we are concerned with the relations between the existence of global, classical solutions of the evolution equation

$$\begin{cases} u_t - \Delta u = g(u) & \text{in } (0, \infty) \times \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \quad (1)$$

and the existence of weak solutions of the stationary problem

$$\begin{cases} -\Delta u = g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2)$$

Here, and throughout the paper,  $\Omega \subset \mathbb{R}^N$  is a smooth, bounded domain and  $g : [0, \infty) \rightarrow [0, \infty)$  is a  $C^1$  convex, nondecreasing function. For *some* results, we will also assume that there exists  $x_0 \geq 0$  such that  $g(x_0) > 0$  and

$$\int_{x_0}^{\infty} \frac{ds}{g(s)} < \infty. \quad (3)$$

Solutions  $u$  of (1) and (2) are always assumed to be nonnegative. The initial condition  $u_0$  is always assumed to be in  $L^\infty(\Omega)$  and  $u_0 \geq 0$ , so that a classical solution of (1) exists on a maximal interval  $(0, T_m)$ .

By a weak solution of (2), we mean the following.

**Definition 1.** A weak solution of (2) is a function  $u \in L^1(\Omega)$ ,  $u \geq 0$ , such that

$$g(u)\delta \in L^1(\Omega), \quad (4)$$

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where  $\delta$  denotes the function distance to the boundary,

$$\delta(x) = \text{dist}(x, \partial\Omega),$$

and

$$-\int_{\Omega} u \Delta \zeta = \int_{\Omega} g(u) \zeta,$$

for all  $\zeta \in C^2(\overline{\Omega})$  with  $\zeta = 0$  on  $\partial\Omega$ . (Note that the second integral makes sense since  $|\zeta(x)| \leq C\delta(x)$  for all  $x \in \Omega$ .)

Our first result is the following.

**Theorem 1.** *Assume (3). If there exists a global, classical solution of (1) for some  $u_0 \in L^\infty(\Omega)$ ,  $u_0 \geq 0$ , then there exists a weak solution of (2).*

**Remark 1.** Theorem 1 is quite surprising since we do not assume any bound (as  $t \rightarrow \infty$ ) for the global solution  $u$ .

**Remark 2.** The existence of a global solution of (1) does not, in general, imply the existence of a **classical** solution of (2). In many examples, the existence of a weak solution of (2) implies the existence of a classical solution of (2). However, there are situations where the stationary problem admits no classical solution, and still there exists a global, classical solution of the evolution equation. See Theorem 2 and Remark 5.

An obvious consequence of Theorem 1 is the following:

**Corollary 1.** *Assume (3). If there is no weak solution of (2), then for any initial value  $u_0 \in L^\infty(\Omega)$ ,  $u_0 \geq 0$ , the solution of (1) blows up in finite time.*

**Remark 3.** There are very sharp results concerning the existence or nonexistence of weak solutions of (2). See properties a) and d) below and Corollary 2.

There is a converse of Theorem 1, which does not require assumption (3).

**Theorem 2.** *If there exists a weak solution  $w$  of (2), then for any  $u_0 \in L^\infty(\Omega)$  with  $0 \leq u_0 \leq w$ , the solution  $u$  of (1) with  $u(0) = u_0$  is global.*

**Remark 4.** If  $w$  is a classical solution of (2), then the existence of a global solution of (1) follows immediately from the maximum principle. On the other hand, if  $w \notin L^\infty(\Omega)$ , then the conclusion is far from obvious. Indeed, suppose that the solution blows up in finite time  $T_m$ . Clearly  $u(t, x) \leq w(x)$  on  $(0, T_m) \times \Omega$ , but this estimate in itself does not prevent  $\|u(t)\|_{L^\infty}$  from blowing up in finite time. It is well known that  $u(t, x)$  can converge to a blow-up profile  $u(T_m, x)$ , which may be finite everywhere except at one point (see e.g. Weissler [16]).

A basic ingredient in the proof of Theorem 2 consists in proving that some “perturbations” of (2) have classical solutions if (2) has a weak solution. A typical result in that direction is the following:

**Theorem 3.** *If there exists a weak solution  $w$  of (2), then, for every  $\varepsilon \in (0, 1)$ , there exists a classical solution  $w_\varepsilon$  of*

$$\begin{cases} -\Delta w_\varepsilon = (1 - \varepsilon)g(w_\varepsilon) & \text{in } \Omega, \\ w_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \quad (7)$$

Theorem 3 allows us to sharpen some well-known results concerning the problem

$$\begin{cases} -\Delta u = \lambda g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (8)$$

Here we assume in addition that

$$g(0) > 0 \quad \text{and} \quad g \not\equiv g(0). \quad (9)$$

We recall that there exists  $0 < \lambda^* < \infty$  such that:

- a) For every  $0 < \lambda < \lambda^*$  equation (8) has a minimal, positive classical solution  $u(\lambda)$ , which is the unique stable solution of (8); stability means that

$$\lambda_1(-\Delta - \lambda g'(u(\lambda))) > 0.$$

(There may exist, for some values of  $\lambda \in (0, \lambda^*)$ , one or many other solutions, which are all unstable.)

- b) The map  $\lambda \mapsto u(\lambda)$  is increasing.
- c) For  $\lambda > \lambda^*$ , there is *no classical solution* of (2).
- d) For  $\lambda = \lambda^*$ , and if

$$\frac{g(u)}{u} \xrightarrow{u \rightarrow \infty} \infty, \quad (10)$$

then there is a weak solution  $u^* = \lim_{\lambda \uparrow \lambda^*} u(\lambda)$  of (8).

For all these results, we refer to I.M. Gel'fand ([7]), H.B. Keller and D.S. Cohen ([10]), H.B. Keller and J. Keener ([11]), M.G. Crandall and P.H. Rabinowitz ([3]), H. Brezis and L. Nirenberg ([2]).

Property d) is not absolutely standard; see Lemma 5.

**Remark 5.** The solution  $u^*$  is sometimes a classical solution. For example when  $g(u) = e^u$  and  $N \leq 9$  or when  $g(u) = (1 + u)^p$  and  $N \leq 10$  (see F. Mignot and J.-P. Puel, [14]). However, there are important cases where there is **no classical solution** at  $\lambda = \lambda^*$ —for example when  $\Omega$  is the unit ball of  $\mathbb{R}^N$  with  $N \geq 10$  and  $g(u) = e^u$ ; in this case  $\lambda^* = 2(N - 2)$  and  $u^*(x) = \log(\frac{1}{|x|^2})$  (see D.D. Joseph and T.S. Lundgren [8]).

The main novelty is:

**Corollary 2.** *Assume (9). If  $\lambda > \lambda^*$ , then there is no weak solution of (8).*

This is an obvious consequence of Theorem 3 applied to the function  $\lambda g$ , and the characterization of  $\lambda^*$ .

**Remark 6.** A result similar to Corollary 2 was obtained by Gallouët, Mignot and Puel ([6]) in the case  $g(u) = e^u$  (and for a stronger notion of weak solution).

Putting together Theorems 1, 2 and 3, we can now state the following.

**Corollary 3.** *Assume (3) and (9), and consider the (classical) solution  $u$  of*

$$\begin{aligned} u_t - \Delta u &= \lambda g(u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \\ u(0) &= 0 && \text{in } \Omega. \end{aligned}$$

*If  $\lambda \leq \lambda^*$ , then  $u$  is global. If  $\lambda > \lambda^*$ , then  $u$  blows up in finite time.*

**Remark 7.** It is somewhat surprising that one finds the same dividing line  $\lambda^*$  in the stationary problem and in the evolution problem.

Starting with the celebrated papers of H. Fujita ([4, 5]), dealing with the case  $g(u) = e^u$ , a number of authors have investigated the question of blow up in finite time or the existence of a global solution for (11). A. Lacey ([12]) had established that the solution of (11) blows up in finite time for  $\lambda > \lambda^*$  under some additional assumption: either  $u^* \in L^\infty(\Omega)$  or  $\Omega$  is a ball. H. Bellout ([1]) had reached the same conclusion, with the additional assumption that  $(\frac{g}{g'})'' \leq 0$ . On the other hand, A. Lacey and D. Tzanetis ([13]) proved that for  $\lambda = \lambda^*$  the solution of (11) is global when  $\Omega$  is a ball and  $u_0 \leq u^*$ ,  $u_0 \in L^\infty(\Omega)$  and  $u_0$  is spherically symmetric (and also for general domains but under various restrictive conditions).

**2. Proof of Theorem 3.** We begin with a lemma concerning the linear Laplace equation.

**Lemma 1.** *Given  $f \in L^1(\Omega, \delta(x)dx)$ , there exists a unique  $v \in L^1(\Omega)$  which is a weak solution of*

$$\begin{cases} -\Delta v = f & \text{in } \Omega, \\ v|_{\partial\Omega} = 0, \end{cases}$$

*in the sense that*

$$-\int_{\Omega} v \Delta \zeta = \int_{\Omega} f \zeta,$$

*for all  $\zeta \in C^2(\overline{\Omega})$  with  $\zeta = 0$  on  $\partial\Omega$ . Moreover,*

$$\|v\|_{L^1} \leq C \|f\|_{L^1(\Omega, \delta(x)dx)},$$

*for some constant  $C$  independent of  $f$ . In addition, if  $f \geq 0$  almost everywhere in  $\Omega$ , then  $v \geq 0$  almost everywhere in  $\Omega$ .*

**Proof.** The uniqueness is clear. Indeed, let  $v_1$  and  $v_2$  be two solutions of (12). Then  $v = v_1 - v_2$  satisfies

$$\int_{\Omega} v \Delta \zeta = 0,$$

for all  $\zeta \in C^2(\overline{\Omega})$  with  $\zeta = 0$  on  $\partial\Omega$ . Given any  $\varphi \in \mathcal{D}(\Omega)$  let  $\zeta$  be the solution of

$$\begin{cases} \Delta \zeta = \varphi & \text{in } \Omega, \\ \zeta|_{\partial\Omega} = 0. \end{cases}$$

It follows that

$$\int_{\Omega} v \varphi = 0$$

Since  $\varphi$  is arbitrary, we deduce that  $v = 0$ .

For the existence, we may assume that  $f \geq 0$  (otherwise we write  $f = f_+ - f_-$ ). Given an integer  $k \geq 0$  set  $f_k(x) = \min\{f(x), k\}$ , so that  $f_k \xrightarrow[k \rightarrow \infty]{} f$  in  $L^1(\Omega, \delta(x)dx)$ .

Let  $v_k$  be the solution of

$$\begin{cases} -\Delta v_k = f_k & \text{in } \Omega, \\ v_k = 0 & \text{on } \partial\Omega. \end{cases} \quad (15)$$

The sequence  $(v_k)_{k \geq 0}$  is clearly monotone nondecreasing. It is also a Cauchy sequence in  $L^1(\Omega)$  since

$$\int_{\Omega} (v_k - v_\ell) = \int_{\Omega} (f_k - f_\ell) \zeta_0,$$

where  $\zeta_0$  is defined by

$$\begin{cases} -\Delta \zeta_0 = 1 & \text{in } \Omega, \\ \zeta_0 = 0 & \text{on } \partial\Omega \end{cases} \quad (16)$$

Hence

$$\int_{\Omega} |v_k - v_\ell| \leq C \int_{\Omega} |f_k - f_\ell| \delta(x) dx.$$

Passing to the limit in (15) (after multiplication by  $\zeta$ ), we obtain (13). Finally, taking  $\zeta = \zeta_0$  in (13), we obtain

$$\|v\|_{L^1} = \int_{\Omega} v = \int_{\Omega} f \zeta_0 \leq C \|f\|_{L^1(\Omega, \delta(x)dx)},$$

and (14) follows.  $\square$

Our next lemma is a variant of Kato's inequality (see [9]).

**Lemma 2.** *Let  $f \in L^1(\Omega, \delta(x)dx)$ , and let  $u \in L^1(\Omega)$  be the weak solution of (12). Let  $\Phi \in C^2(\mathbb{R})$  be concave, with  $\Phi'$  bounded and  $\Phi(0) = 0$ . Then*

$$-\Delta \Phi(u) \geq \Phi'(u) f,$$

in the sense that

$$-\int_{\Omega} \Phi(u) \Delta \zeta \geq \int_{\Omega} \Phi'(u) f \zeta,$$

for all  $\zeta \in C^2(\overline{\Omega})$ ,  $\zeta \geq 0$ , such that  $\zeta = 0$  on  $\partial\Omega$ .

**Proof.** Consider  $(f_n)_{n \geq 0} \subset \mathcal{D}(\Omega)$  such that  $f_n \xrightarrow[n \rightarrow \infty]{} f$  in  $L^1(\Omega, \delta(x)dx)$ . Let  $u_n$  be the solution of

$$\begin{cases} -\Delta u_n = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

It follows from Lemma 1 that  $u_n \xrightarrow{n \rightarrow \infty} u$  in  $L^1(\Omega)$ . On the other hand we have

$$\Delta \Phi(u_n) = \Phi'(u_n) \Delta u_n + \Phi''(u_n) |\nabla u_n|^2 \leq \Phi'(u_n) \Delta u_n = -\Phi'(u_n) f_n.$$

Therefore,

$$-\int_{\Omega} \Phi(u_n) \Delta \zeta \geq \int_{\Omega} \Phi'(u_n) f_n \zeta,$$

for all  $\zeta \in C^2(\overline{\Omega})$ ,  $\zeta \geq 0$  such that  $\zeta = 0$  on  $\partial\Omega$ ; and so the result follows easily by letting  $n \rightarrow \infty$ .  $\square$

**Lemma 3.** *Let  $\bar{w}$  be a weak super-solution of (2), in the sense that  $\bar{w} \in L^1(\Omega)$ ,  $\bar{w} \geq 0$ ,  $g(\bar{w})\delta \in L^1(\Omega)$ , where  $\delta$  is given by (5), and*

$$-\int_{\Omega} \bar{w} \Delta \zeta \geq \int_{\Omega} g(\bar{w}) \zeta,$$

*for all  $\zeta \in C^2(\overline{\Omega})$ ,  $\zeta \geq 0$  with  $\zeta = 0$  on  $\partial\Omega$ . Then there exists a weak solution  $w$  of (2) with  $0 \leq w \leq \bar{w}$ .*

**Proof.** We use a standard monotone iteration argument: define the sequence  $(w_n)_{n \geq 1}$  by

$$\begin{cases} -\Delta w_{n+1} = g(w_n) & \text{in } \Omega, \\ w_{n+1} = 0 & \text{on } \partial\Omega, \end{cases}$$

for  $n \geq 1$ , starting with  $w_1 = \bar{w}$ . It is easy to check that  $\bar{w} = w_1 \geq w_2 \geq \dots \geq 0$ . Indeed, it suffices to prove that  $w_1 \geq w_2 \geq 0$ , and the rest follows by induction, using Lemma 1. We have

$$\int_{\Omega} (w_1 - w_2)(-\Delta \zeta) \geq 0,$$

for all  $\zeta \in C^2(\overline{\Omega})$ ,  $\zeta \geq 0$  with  $\zeta = 0$  on  $\partial\Omega$ . Given  $\varphi \in \mathcal{D}(\Omega)$ ,  $\varphi \geq 0$ , let  $\zeta_{\varphi}$  be the solution of

$$\begin{cases} -\Delta \zeta_{\varphi} = \varphi & \text{in } \Omega, \\ \zeta_{\varphi} = 0 & \text{on } \partial\Omega. \end{cases}$$

Taking  $\zeta = \zeta_{\varphi}$  in (18), we obtain

$$\int_{\Omega} (w_1 - w_2) \varphi \geq 0.$$

Since  $\varphi \geq 0$  is arbitrary, we deduce that  $w_2 \leq w_1$  almost everywhere in  $\Omega$ . On the other hand, it follows from Lemma 1 that  $w_2 \geq 0$ .

Since the sequence  $(w_n)_{n \geq 1}$  is nonincreasing, it converges to a limit  $u \in L^1(\Omega)$ , which is clearly a weak solution of (2).  $\square$

An essential ingredient in the proof of Theorem 3 is the following.

**Lemma 4.** Assume  $g(0) > 0$  and set

$$h(u) = \int_0^u \frac{ds}{g(s)},$$

for all  $u \geq 0$ . Let  $\tilde{g}$  be a  $C^1$  positive function on  $[0, \infty)$  such that  $\tilde{g} \leq g$  and  $\tilde{g}' \leq g'$ . Set

$$\tilde{h}(u) = \int_0^u \frac{ds}{\tilde{g}(s)},$$

$$\Phi(u) = \tilde{h}^{-1}(h(u))$$

for all  $u \geq 0$ . Then

- (i)  $\Phi(0) = 0$  and  $0 \leq \Phi(u) \leq u$  for all  $u \geq 0$ .
- (ii)  $\Phi$  is increasing, concave and  $\Phi'(u) \leq 1$  for all  $u \geq 0$ .
- (iii) If  $h(\infty) < \infty$  and  $\tilde{g} \not\equiv g$ , then  $\Phi(\infty) < \infty$ .

**Proof.** Properties (i) and (iii) are clear. We have

$$\Phi'(u) = \frac{\tilde{g}(\Phi(u))}{g(u)} > 0,$$

$$\begin{aligned} \Phi''(u) &= \frac{g(u)\tilde{g}'(\Phi(u))\Phi'(u) - \tilde{g}(\Phi(u))g'(u)}{g(u)^2} \\ &= \frac{\tilde{g}(\Phi(u))(\tilde{g}'(\Phi(u)) - g'(u))}{g(u)^2} \end{aligned}$$

Since  $\tilde{g}'(\Phi(u)) \leq g'(\Phi(u)) \leq g'(u)$ , it follows that  $\Phi$  is concave. Hence (ii).  $\square$

**Proof of Theorem 3.** If  $g(0) = 0$ , then 0 is a weak solution of (7), so we assume  $g(0) > 0$ . We consider two cases.

**Case 1.** Suppose

$$\int_0^\infty \frac{ds}{g(s)} < \infty.$$

Let  $v = \Phi(w)$ , with the notation of Lemma 4, where  $\tilde{g} = (1 - \varepsilon)g$ . It follows from Lemmas 2 and 4 that  $v \in L^\infty(\Omega)$  is a super-solution of (7). The result follows from Lemma 3.

**Case 2.** Suppose

$$\int_0^\infty \frac{ds}{g(s)} = \infty.$$

Let  $\tilde{g} = (1 - \varepsilon)g$ , and consider the function  $\Phi$  introduced in Lemma 4. Set

$$v_1 = \Phi(w).$$

We have  $0 \leq v_1 \leq w$ . We observe that by concavity of the function  $h(u) = \int_0^u \frac{ds}{g(s)}$ ,

$$h(w) \leq h(v_1) + (w - v_1)h'(v_1) = h(v_1) + \frac{w - v_1}{g(v_1)}$$

Since  $h(v_1) = (1 - \varepsilon)h(w)$ , we deduce that

$$\varepsilon g(v_1) \leq \frac{w - v_1}{h(w)} \leq \frac{w}{h(w)} \leq C(1 + w),$$

so that in particular,  $g(v_1) \in L^1(\Omega)$ . Now, we observe that by Lemma 2,  $v_1$  is a weak super-solution of the equation

$$\begin{aligned} -\Delta u_1 &= (1 - \varepsilon)g(u_1) & \text{in } \Omega, \\ u_1 &= 0 & \text{on } \partial\Omega \end{aligned}$$

Therefore, it follows from Lemma 3 that there exists a weak solution  $u_1$  of (19) such that  $0 \leq u_1 \leq v_1$ . In particular, we have  $0 \leq g(u_1) \leq g(v_1) \in L^1(\Omega)$ , so that  $u_1 \in L^p(\Omega)$ , for all  $p \geq 1$  such that (see e.g. Stampacchia [15])

$$p < \frac{N}{N-2} \quad (p \leq \infty \text{ if } N = 1, \quad p < \infty \text{ if } N = 2).$$

By the same construction, we find a solution  $u_2$  of the equation

$$\begin{aligned} -\Delta u_2 &= (1 - \varepsilon)^2 g(u_2) & \text{in } \Omega, \\ u_2 &= 0 & \text{on } \partial\Omega, \end{aligned}$$

such that  $0 \leq u_2 \leq u_1$  and  $g(u_2) \leq C(1 + u_1)$ . In particular,  $g(u_2) \in L^p(\Omega)$ , for all  $p \geq 1$  satisfying (20). This implies that  $u_2 \in L^r(\Omega)$ , for all  $r \geq 1$  such that  $r < \frac{N}{N-4}$  ( $r \leq \infty$  if  $N = 1, 2, 3$ ,  $r < \infty$  if  $N = 4$ ). By iteration, we find that if  $k(N) = [N/2] + 1$ , then the solution  $u_k$  of the equation

$$\begin{aligned} -\Delta u_k &= (1 - \varepsilon)^k g(u_k) & \text{in } \Omega, \\ u_k &= 0 & \text{on } \partial\Omega, \end{aligned}$$

belongs to  $L^\infty(\Omega)$ . Since  $\varepsilon \in (0, 1)$  is arbitrary, this completes the proof.  $\square$

**3. Proof of Theorem 1.** We assume  $g(0) > 0$ , for otherwise  $w \equiv 0$  is a weak solution of (2). Furthermore, we may also assume that  $u_0 = 0$ , so that  $u \geq 0$  and  $u_t \geq 0$  for all  $t \geq 0$ .

Next, observe that  $g'(u) \xrightarrow{u \rightarrow \infty} +\infty$  by (3), so that there exists a constant  $M > 0$  such that

$$g(s) - \lambda_1 s \geq \frac{1}{2}g(s) \quad \text{for } s \geq M,$$

where  $\lambda_1$  is the first eigenvalue of  $-\Delta$  in  $H_0^1(\Omega)$ . Let  $\varphi \in C^2(\overline{\Omega})$  with  $\varphi|_{\partial\Omega} = 0$ . It follows from (1) that

$$\frac{d}{dt} \int_{\Omega} u(t)\varphi + \int_{\Omega} u(t)(-\Delta\varphi) = \int_{\Omega} g(u(t))\varphi.$$

We first claim that

$$\sup_{t \geq 0} \int_{\Omega} g(u)\varphi_1 \leq (1 + \lambda_1)M.$$

where  $M$  is as in (22) and  $\varphi_1$  is the first eigenfunction of  $-\Delta$  in  $H_0^1(\Omega)$  such that  $\int_{\Omega} \varphi_1 = 1$ . Indeed, taking  $\varphi = \varphi_1$  in (22), we find

$$\frac{d}{dt} \int_{\Omega} u(t)\varphi_1 + \lambda_1 \int_{\Omega} u(t)\varphi_1 = \int_{\Omega} g(u(t))\varphi_1 \geq g\left(\int_{\Omega} u(t)\varphi_1\right)$$

by Jensen's inequality. If there exists  $t_0 \geq 0$  such that  $\int_{\Omega} u(t_0)\varphi_1 > M$ , then it follows from (24) and (21) that

$$\frac{d}{dt} \int_{\Omega} u(t)\varphi_1 \geq \frac{1}{2}g\left(\int_{\Omega} u(t)\varphi_1\right)$$

for  $t \geq t_0$ , which is absurd by (3); and so

$$\int_{\Omega} u(t)\varphi_1 \leq M.$$

for all  $t \geq 0$ . Integrating (24) on  $(t, t+1)$  and since  $u_t \geq 0$ , we find

$$\begin{aligned} \int_{\Omega} g(u(t))\varphi_1 &\leq \int_t^{t+1} \int_{\Omega} g(u)\varphi_1 \leq \int_{\Omega} u(t+1)\varphi_1 + \lambda_1 \int_t^{t+1} \int_{\Omega} u\varphi_1 \\ &\leq (1 + \lambda_1)M, \end{aligned}$$

hence (23).

We next claim that there exists  $K$  such that

$$\sup_{t \geq 0} \|u(t)\|_{L^1} \leq K.$$

Indeed, let  $\zeta_0$  be the solution of (16). Taking  $\varphi = \zeta_0$  in (22) and integrating on  $(t, t+1)$ , we find

$$\int_{\Omega} u(t) \leq \int_t^{t+1} \int_{\Omega} u = \int_{\Omega} u(t)\zeta_0 - \int_{\Omega} u(t+1)\zeta_0 + \int_t^{t+1} \int_{\Omega} g(u)\zeta_0,$$

and (25) follows by applying (23).

By monotone convergence, it follows from (25) and (23) that  $u(t)$  has a limit  $w$  in  $L^1(\Omega)$  and that  $g(u)$  converges to  $g(w)$  in  $L^1(\Omega, \delta(x)dx)$ , as  $t \rightarrow \infty$ . Let  $\varphi \in C^2(\overline{\Omega})$ ,  $\varphi|_{\partial\Omega} = 0$ . Integrating (22) on  $(t, t+1)$ , we obtain

$$\left[ \int_{\Omega} u \varphi \right]_t^{t+1} + \int_t^{t+1} \int_{\Omega} u(t)(-\Delta \varphi) = \int_t^{t+1} \int_{\Omega} g(u(t)) \varphi.$$

Letting  $t \rightarrow \infty$ , we find

$$\int_{\Omega} w(-\Delta \varphi) = \int_{\Omega} g(w) \varphi.$$

Therefore,  $w$  is a weak solution of (2).  $\square$

We now give an alternative proof of Theorem 1 in the spirit of the proof of Theorem 3. It makes use of the following lemma.

**Lemma 5.** *Assume (9), and let  $\lambda^*$  be the supremum of all  $\lambda > 0$  such that (8) has a minimal, positive, classical solution  $u(\lambda)$ . Then  $\lambda^* < \infty$ . If furthermore (10) holds, then  $\lim_{\lambda \uparrow \lambda^*} u(\lambda) = u^*$  is a weak solution of (8) with  $\lambda = \lambda^*$ .*

**Proof.** We first observe that by (9) and convexity of  $g$ , there exists  $\varepsilon > 0$  such that  $g(u) \geq \varepsilon u$ , for all  $u \geq 0$ ; and so

$$-\Delta u(\lambda) \geq \lambda \varepsilon u(\lambda).$$

Let  $\lambda_1$  be the first eigenvalue of  $-\Delta$  in  $H_0^1(\Omega)$ , and let  $\varphi_1$  be a corresponding eigenvector. Multiplying (26) by  $\varphi_1$ , we see that  $\lambda \varepsilon \leq \lambda_1$ ; and so,  $\lambda^* \leq \frac{\lambda_1}{\varepsilon}$ . If (10) holds, then there exists  $C$  such that  $g(u) \geq \frac{2\lambda_1}{\lambda^*} u - C$ , for all  $u \geq 0$ . Multiplying (8) by  $\varphi_1$ , we obtain

$$\lambda \int_{\Omega} g(u(\lambda)) \varphi_1 = \lambda \int_{\Omega} u(\lambda) \varphi_1 \leq \frac{\lambda^*}{2} \int_{\Omega} (g(u(\lambda)) + C) \varphi_1$$

Letting  $\lambda \uparrow \lambda^*$ , we deduce that

$$\lim_{\lambda \uparrow \lambda^*} \int_{\Omega} g(u(\lambda)) \varphi_1 < \infty.$$

Multiplying now (8) by the solution  $\zeta_0$  of (16), we obtain

$$\int_{\Omega} u(\lambda) = \lambda \int_{\Omega} g(u(\lambda)) \zeta_0 \leq C \lambda \int_{\Omega} g(u(\lambda)) \varphi_1,$$

so that  $u(\lambda)$  is bounded in  $L^1(\Omega)$  by (27). Since  $u(\lambda)$  is increasing in  $\lambda$ , it follows that  $u(\lambda)$  has a limit  $u^* \in L^1(\Omega)$  and that  $g(u(\lambda))$  converges to  $g(u^*)$  in  $L^1(\Omega, \delta(x)dx)$ . It follows easily that  $u^*$  is a weak solution of (8) with  $\lambda = \lambda^*$ .  $\square$

**Alternative proof of Theorem 1.** We may assume as above that  $g(0) > 0$  and  $u_0 = 0$ . Given  $0 < \varepsilon < 1$ , let  $\tilde{g} = (1 - \varepsilon)g$  and let  $\Phi$  be as in Lemma 4. Set  $v_\varepsilon(t) = \Phi(u(t))$ , for all  $t \geq 0$ . It follows from Lemma 4 that there exists  $M_\varepsilon < \infty$  such that

$$0 \leq v_\varepsilon \leq M_\varepsilon.$$

Furthermore, it follows from Lemmas 2 and 4 that

$$-\Delta v_\varepsilon \geq \Phi'(u)(-\Delta u) = \Phi'(u)(g(u) - u_t) = (1 - \varepsilon)g(v_\varepsilon) - (v_\varepsilon)_t,$$

so that  $v_\varepsilon$  is a super-solution of the equation

$$\begin{aligned} \frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon &= (1 - \varepsilon)g(u_\varepsilon), \\ u_\varepsilon &= 0 \quad \text{on} \quad \partial\Omega, \\ u_\varepsilon(0) &= 0. \end{aligned}$$

It now follows from (28) that the solution  $u_\varepsilon$  of (29) is global and bounded by  $M_\varepsilon$ . As above, we deduce that  $w_\varepsilon = \lim_{t \rightarrow \infty} u_\varepsilon(t)$  a (classical) solution of the equation

$$\begin{cases} -\Delta w_\varepsilon = (1 - \varepsilon)g(w_\varepsilon) & \text{in } \Omega, \\ w_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

It follows (see property c) in the introduction) that  $\lambda^* \geq 1$ . By Lemma 5, (2) has a weak solution.  $\square$

**4. Proof of Theorem 2.** Since the hypotheses of Theorem 2 allow  $g$  to vanish at the origin, we need a variant of Lemma 4 that applies to the case  $g(0) = 0$ .

**Lemma 6.** *Assume (3). There exist constants  $K \geq 0$  and  $\varepsilon_0 > 0$  such that for every  $0 < \varepsilon < \varepsilon_0$ , there is a function  $\Phi_\varepsilon \in C^2([0, \infty))$ , concave, increasing, with*

$$\Phi_\varepsilon(0) = 0, \tag{30}$$

$$0 < \Phi_\varepsilon(x) \leq x \quad \text{for } x > 0, \tag{31}$$

$$1 \geq \Phi'_\varepsilon(x) \geq \frac{(g(\Phi_\varepsilon(x)) - \varepsilon K)^+}{g(x)} \quad \text{for } x \geq 0.$$

Moreover,  $\sup_{x \geq 0} \Phi_\varepsilon(x) < \infty$ .

**Proof.** If  $g(0) > 0$  we apply Lemma 4 with  $\tilde{g}(u) = g(u) - \varepsilon$  and the conclusions follow with  $\varepsilon_0 = g(0)$  and  $K = 1$ .

Therefore we may assume that  $g(0) = 0$ . Let  $a > 0$  be the unique solution of  $g(a) = 1$ . Set

$$H(x) = a + \int_a^x \frac{ds}{g(s)} \quad \text{for } x \geq a$$

Since  $g(a) = 1$ , there exists  $0 < \varepsilon_0 < 1$  such that  $0 < \varepsilon < g((1 - \varepsilon)a)$ , for  $0 < \varepsilon < \varepsilon_0$ . For such an  $\varepsilon$ , let

$$H_\varepsilon(x) = a + \int_{(1-\varepsilon)a}^x \frac{ds}{g(s) - \varepsilon} \quad \text{for } x \geq (1 - \varepsilon)a.$$

Note that  $H_\varepsilon((1 - \varepsilon)a) = a = H(a)$ . Moreover,

$$\lim_{x \rightarrow \infty} H_\varepsilon(x) > a + \int_{(1-\varepsilon)a}^{\infty} \frac{ds}{g(s)} \geq \lim_{x \rightarrow \infty} H(x).$$

Thus  $\Psi_\varepsilon(x) = H_\varepsilon^{-1}(H(x))$ , is well defined for  $x \geq a$ ,  $\Psi_\varepsilon(a) = (1 - \varepsilon)a$  and  $\sup_{x \geq a} \Psi_\varepsilon(x) < \infty$ . Furthermore, for  $x \geq a$ ,

$$\Psi'_\varepsilon(x) = \frac{g(\Psi_\varepsilon(x)) - \varepsilon}{g(x)} \quad (33)$$

In addition, for  $x \geq a$  we have

$$\begin{aligned} \Psi''_\varepsilon(x) &= \frac{g(x)g'(\Psi_\varepsilon(x))\Psi'_\varepsilon(x) - (g(\Psi_\varepsilon(x)) - \varepsilon)g'(x)}{g(x)^2} \\ &\quad - \frac{(g(\Psi_\varepsilon(x)) - \varepsilon)(g'(\Psi_\varepsilon(x)) - g'(x))}{g(x)^2} \leq 0, \end{aligned}$$

since  $\Psi_\varepsilon(x) \leq x$  thus  $g'(\Psi_\varepsilon(x)) \leq g'(x)$ . We finally consider a concave function  $\Phi_\varepsilon \in C^2([0, \infty))$  such that  $\Phi_\varepsilon(x) = \Psi_\varepsilon(x)$  for  $x \geq a$ ,  $\Phi_\varepsilon(0) = 0$ , and  $\Phi'_\varepsilon(x) \leq 1$  for all  $x \geq 0$ . Such a function exists since

$$\Psi'_\varepsilon(a) \leq \frac{\Psi_\varepsilon(a)}{a} \leq$$

Clearly  $\Phi_\varepsilon$  satisfies (30) and (31). We claim that (32) holds with  $K = 1 + ag'(a)$ . Indeed, it follows from (33) that for  $x \geq a$

$$\Phi'_\varepsilon(x) \geq \frac{g(\Phi_\varepsilon(x)) - \varepsilon}{g(x)} \geq \frac{g(\Phi_\varepsilon(x)) - \varepsilon K}{g(x)}$$

so that (32) holds for  $x \geq a$  (since  $\Phi'_\varepsilon \geq 0$ ). For  $x \leq a$ , we have

$$\Phi'_\varepsilon(x) \geq \Phi'_\varepsilon(a) = g((1 - \varepsilon)a) - \varepsilon.$$

Furthermore, by convexity,

$$g((1 - \varepsilon)a) \geq g(a) - \varepsilon g'(a) = 1 - \varepsilon(K - 1);$$

and so, for  $x \leq a$ ,

$$\begin{aligned} \Phi'_\varepsilon(x) &\geq 1 - \varepsilon K = 1 - \frac{\varepsilon K}{g(a)} \geq 1 - \frac{\varepsilon K}{g(x)} \\ &= \frac{g(x) - \varepsilon K}{g(x)} \geq \frac{g(\Phi_\varepsilon(x)) - \varepsilon K}{g(x)}. \end{aligned}$$

It follows that (32) is satisfied for  $x \leq a$ , which completes the proof.  $\square$

**Lemma 7.** *Let  $\delta$  be given by (5). For every  $0 < T < \infty$ , there exists  $\varepsilon_1(T) > 0$  such that if  $0 < \varepsilon \leq \varepsilon_1$ , then the solution  $Z$  of the equation*

$$\begin{aligned} Z_t - \Delta Z &= -\varepsilon & \text{in } (0, \infty) \times \Omega, \\ Z &= 0 & \text{on } (0, \infty) \times \partial\Omega, \\ Z(0) &= \delta, \end{aligned}$$

*satisfies  $Z \geq 0$  on  $[0, T] \times \overline{\Omega}$ .*

**Proof.** Let  $(T(t))_{t \geq 0}$  be the heat semigroup with Dirichlet boundary condition, and consider the solution  $\zeta_0$  of (16). We have

$$\zeta_0 = T(t)\zeta_0 + \int_0^t T(s)1_\Omega ds,$$

for all  $t \geq 0$ . Since  $T(t)\zeta_0 \geq 0$ , it follows that

$$\int_0^t T(s)1_\Omega ds \leq \zeta_0 \leq C\delta, \quad (35)$$

for all  $t \geq 0$ . On the other hand, we have

$$Z(t) = T(t)\delta - \varepsilon \int_0^t T(s)1_\Omega ds;$$

and so,

$$Z(t) \geq T(t)\delta - \varepsilon C\delta.$$

Consider now  $c_0, c_1 > 0$  such that  $c_0\varphi_1 \leq \delta \leq c_1\varphi_1$ , where  $\varphi_1 > 0$  is the first eigenfunction of  $-\Delta$  in  $H_0^1(\Omega)$ , associated to the eigenvalue  $\lambda_1$ . We have

$$T(t)\delta \geq c_0T(t)\varphi_1 = c_0e^{-\lambda_1 t}\varphi_1 \geq \frac{c_0}{c_1}e^{-\lambda_1 t}\delta.$$

Therefore,

$$Z(t) \geq \left(\frac{c_0}{c_1}e^{-\lambda_1 t} - \varepsilon C\right)\delta.$$

It follows that  $Z(t) \geq 0$  on  $[0, T]$ , provided  $\varepsilon \leq \frac{c_0}{c_1 C}e^{-\lambda_1 T}$   $\square$

**Proof of Theorem 2.** If (3) fails, then the solution of

$$\theta' = g(\theta), \quad \theta(0) = \|u_0\|_{L^\infty},$$

is global. Since  $\theta(t)$  is a super-solution of (1) and 0 is a sub-solution, it follows that all the solutions of (1) are global.

We now assume that (3) holds. Furthermore, we may assume

$$w \notin L^\infty(\Omega), \quad (36)$$

since otherwise  $u(t) \leq w$  by the maximum principle, and so  $u$  is global. We denote by  $[0, T_m)$  the maximal interval of existence of  $u$ , and we now proceed in five steps.

**Step 1.** We have  $u(t) \leq w$  for all  $t \in [0, T_m)$ . (Note that if  $w$  were a smooth solution of (2), this would follow from the maximum principle.) Fix  $T < T_m$ . Let  $h(t, x) \in \mathcal{D}((0, T) \times \Omega)$ ,  $h \geq 0$ , and let  $\zeta$  be the solution of

$$-\zeta_t - \Delta \zeta = h, \quad \zeta|_{\partial\Omega} = 0, \quad \zeta(T) = 0.$$

We have in particular  $\zeta \in C([0, T], C^2(\overline{\Omega}) \cap C_0(\Omega))$ . Multiplying (1) by  $\zeta$  and integrating on  $(0, T) \times \Omega$ , we find

$$-\int_{\Omega} u_0 \zeta(0) + \int_0^T \int_{\Omega} u h = \int_0^T \int_{\Omega} g(u) \zeta.$$

On the other hand,

$$\int_0^T \int_{\Omega} w \zeta_t - \int_{\Omega} w \zeta(0) = 0,$$

and

$$-\int_0^T \int_{\Omega} w \Delta \zeta = \int_0^T \int_{\Omega} g(w) \zeta.$$

Therefore,

$$\int_{\Omega} (u_0 - w) \zeta(0) + \int_0^T \int_{\Omega} (u - w) h = \int_0^T \int_{\Omega} (g(u) - g(w)) \zeta.$$

Since  $\zeta \geq 0$  and  $u_0 - w \leq 0$ , this yields

$$\int_0^T \int_{\Omega} (u - w) h \leq \int_0^T \int_{\{u \geq w\}} (g(u) - g(w)) \zeta \leq C \int_0^T \int_{\Omega} (u - w)^+ \zeta.$$

(Note that  $\|u\|_{L^\infty((0, T) \times \Omega)} < \infty$ , so that  $g$  is Lipschitz on  $[0, \|u\|_{L^\infty((0, T) \times \Omega)}]$ .) Therefore,

$$\int_0^T \int_{\Omega} (u - w) h \leq C \left( \int_0^T \int_{\Omega} [(u - w)^+]^2 \right)^{\frac{1}{2}} \left( \int_0^T \int_{\Omega} \zeta^2 \right)^{\frac{1}{2}}$$

On the other hand,

$$\zeta(t) = \int_t^T T(s - t) h(s) ds,$$

where  $(T(t))_{t \geq 0}$  is the heat semigroup with Dirichlet boundary condition, thus

$$\|\zeta(t)\|_{L^2}^2 \leq \left( \int_t^T \|h(s)\|_{L^2} ds \right)^2 \leq (T-t) \int_0^T \int_{\Omega} h^2.$$

Therefore,

$$\int_0^T \int_{\Omega} \zeta^2 \leq \frac{T^2}{2} \int_0^T \int_{\Omega} h^2;$$

and so,

$$\int_0^T \int_{\Omega} (u-w)h \leq \frac{CT}{\sqrt{2}} \left( \int_0^T \int_{\Omega} [(u-w)^+]^2 \right)^{\frac{1}{2}} \left( \int_0^T \int_{\Omega} h^2 \right)^{\frac{1}{2}}$$

Now we observe that  $(u-w)^+ \in L^\infty((0, T) \times \Omega)$ , and we let  $h$  converge to  $(u-w)^+$  in  $L^2((0, T) \times \Omega)$  and be bounded in  $L^\infty((0, T) \times \Omega)$ . Since  $u-w \in L^1(\Omega)$ , we obtain

$$\int_0^T \int_{\Omega} [(u-w)^+]^2 \leq \frac{CT}{\sqrt{2}} \int_0^T \int_{\Omega} [(u-w)^+]^2.$$

It follows that  $u \leq w$  provided  $C^2 T^2 < 2$ . The result follows by iteration.

**Step 2.** There exist  $0 < \tau < T_m$  and  $C_0, c_0 > 0$  such that

$$u(\tau) \leq C_0 \delta,$$

and

$$u(\tau) \leq w - c_0 \delta.$$

Set  $v_0 = \min\{w, 1 + u_0\}$ . We have  $v_0 \geq u_0$  and  $v_0 \not\equiv u_0$  by (36). In particular, there exists a function  $\gamma : [0, \infty) \rightarrow \mathbb{R}$  such that  $\gamma(t) > 0$  for  $t > 0$  and

$$T(t)(v_0 - u_0) \geq \gamma(t)\delta,$$

where  $\delta$  is defined by (5) and  $(T(t))_{t \geq 0}$  is the heat semigroup with Dirichlet boundary condition. Let  $v$  be the solution of (1) with the initial value  $v(0) = v_0$ , and let  $[0, \bar{T})$  be the maximal interval of existence of  $v$ . We have  $v \geq 0$ , and by Step 1,  $v \leq w$ . Let  $z(t) = u(t) + T(t)(v_0 - u_0)$  for  $0 \leq t < \bar{T}$ . We have

$$\begin{aligned} z_t - \Delta z &= g(u) \leq g(z) && \text{in } (0, \bar{T}) \times \Omega \\ z &= 0 && \text{on } \partial\Omega, \\ z(0) &= v_0 && \text{in } \Omega, \end{aligned}$$

so that  $z \leq v$  by the maximum principle. Therefore,

$$u(t) < v(t) - T(t)(v_0 - u_0) \leq w - T(t)(v_0 - u_0) \leq w - \gamma(t)\delta.$$

for  $0 \leq t < \bar{T}$  by (39). Fix  $0 < T < \min\{\bar{T}, T_m\}$ .  $u$  is bounded by some constant  $M$  on  $[0, T] \times \bar{\Omega}$ , so that

$$u(t) \leq MT(t)1_\Omega + g(M) \int_0^t T(s)1_\Omega ds.$$

There exists a function  $\bar{C} : (0, \infty) \rightarrow \mathbb{R}$  such that  $T(t)1_\Omega \leq C(t)\delta$  for  $t > 0$ , so that we deduce from (35) that

$$u(t) \leq MC(t)\delta + g(M)C\delta,$$

for  $0 < t \leq T$ . (37) and (38) now follow from (40) and (41).

**Step 3.** We may assume without loss of generality that

$$u_0 \leq C_0\delta,$$

and

$$u_0 \leq w - c_0\delta,$$

where  $C_0, c_0$  are as in Step 2. Indeed, we need only consider  $u(\cdot + \tau)$  instead of  $u(\cdot)$ .

**Step 4.** Let  $\varepsilon_0$  and  $\Phi_\varepsilon$  be as in Lemma 6, and set  $w_\varepsilon = \Phi_\varepsilon(w)$  for  $0 < \varepsilon < \varepsilon_0$ . Then

$$w_\varepsilon \in L^\infty(\Omega),$$

and

$$\int_\Omega \zeta(-\Delta w_\varepsilon) \geq \int_\Omega (g(w_\varepsilon) - \varepsilon K)\zeta$$

for all  $\zeta \in C^2(\bar{\Omega})$ ,  $\zeta \geq 0$  on  $\Omega$  and  $\zeta|_{\partial\Omega} = 0$ . Moreover, there exists  $0 < \varepsilon_1 \leq \varepsilon_0$  such that

$$u_0 \leq w_\varepsilon - \frac{c_0}{2}\delta, \tag{46}$$

for  $0 < \varepsilon < \varepsilon_1$ , where  $c_0$  is as in (43). Indeed, (44) and (45) follow from Lemmas 2 and 6. In order to prove (46), set

$$\eta = \min\{w, (C_0 + c_0)\delta\}, \quad \text{and} \quad \eta_\varepsilon = \Phi_\varepsilon(\eta).$$

Here,  $\delta$  is given by (5) and  $C_0$  is as in (42). It follows from (42) and (43) that

$$u_0 \leq \eta - c_0\delta.$$

We claim that

$$\eta \leq \eta_\varepsilon + \frac{c_0}{2}\delta, \tag{48}$$

for  $\varepsilon > 0$  small enough. Note that it follows from (47) and (48) that  $u_0 \leq \eta_\varepsilon - \frac{c_0}{2}\delta$ , and (46) follows since  $\eta_\varepsilon \leq w_\varepsilon$  (since  $\Phi_\varepsilon$  is nondecreasing). Thus we need only

prove (48). Note that  $\eta_\varepsilon \leq \eta \leq M$ , where  $M = (C_0 + c_0)\|\delta\|_{L^\infty}$ , and that  $\Phi'_\varepsilon(x) \xrightarrow{\varepsilon \downarrow 0} 1$ , uniformly on  $[0, M]$  by Lemma 6. Therefore,

$$\eta - \eta_\varepsilon \leq \eta \sup_{0 \leq x \leq M} (1 - \Phi'_\varepsilon(x)) \leq (C_0 + c_0)\delta \sup_{0 \leq x \leq M} (1 - \Phi'_\varepsilon(x)) \leq \frac{c_0}{2}\delta,$$

for  $\varepsilon$  small enough, and (48) follows.

**Step 5. Conclusion.** Assume for the sake of contradiction that  $T_m < \infty$ . Let  $\varepsilon > 0$  be small enough so that

$$u_0 \leq w_\varepsilon - \frac{c_0}{2}\delta$$

(see Step 4), and so that the solution  $Z$  of the equation

$$\begin{aligned} Z_t - \Delta Z &= -\varepsilon K && \text{in } (0, T_m) \times \Omega, \\ Z &= 0 && \text{on } \partial\Omega, \\ Z(0) &= \frac{c_0}{2}\delta && \text{in } \Omega, \end{aligned}$$

is nonnegative on  $[0, T_m] \times \overline{\Omega}$  (see Lemma 7; here,  $K$  is given by Lemma 6). Let  $v$  be the solution of

$$\begin{aligned} v_t - \Delta v &= g(|v|) - \varepsilon K && \text{in } (0, T) \times \Omega, \\ v &= 0 && \text{on } \partial\Omega, \\ v(0) &= w_\varepsilon && \text{in } \Omega. \end{aligned}$$

Let  $[0, S_m]$  be the maximal interval of existence of  $v$ . Set  $z(t) = Z(t) + u(t)$  for  $0 \leq t < T_m$ . We have  $z \geq u \geq 0$  and

$$\begin{aligned} z_t - \Delta z &= g(u) - \varepsilon K \leq g(z) - \varepsilon K && \text{on } (0, T_m) \times \Omega, \\ z|_{\partial\Omega} &= 0, \\ z(0) &= u_0 + \frac{c_0}{2}\delta \leq w_\varepsilon && \text{in } \Omega \end{aligned}$$

By the maximum principle, we have  $z \leq v$  on  $[0, \min\{T_m, S_m\})$ . In particular,  $v \geq 0$  on  $[0, \min\{T_m, S_m\})$ ; by the maximum principle and (45),  $v \leq w_\varepsilon$ . Since  $w_\varepsilon \in L^\infty(\Omega)$ , this implies that  $T_m < S_m = +\infty$ . Therefore,  $u \leq z \leq v \leq w_\varepsilon$  on  $[0, T_m)$ , which is absurd.  $\square$

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