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Remarks on the Preceding Paper by M. Ben-Artzi "Global Solutions of Two-Dimensional Navier-Stokes and Euler Equations"

HAÏM BREZIS

In the preceding paper, M. BEN-ARTZI establishes an interesting existence and uniqueness result for the two-dimensional Navier-Stokes equation with initial vorticity in L^1. The purpose of this note is to show that one technical assumption ((1.19) in his notation) is not necessary for uniqueness. We follow the same notation as in the preceding paper.

Theorem. *Let* $\xi_0 \in L^1(\mathbb{R}^2)$ *and let* $\theta(x, t)$ *be a weak solution of* (1.14)–(1.16). *Then*

$$\theta(\cdot, t) = S(t)\xi_0 \quad \forall t > 0.$$

In other words the uniqueness part in Theorem B still holds *without assumption* (1.19). The proof relies on the following

Lemma. *Let* $K \subset L^\infty \cap L^1$ *be precompact in* L^1. *Let* $p \in (1, \infty]$ *and let* $\delta(t, K)$ *be defined by* (3.50). *Then*

(a) $$\| S(t)\xi_0 \|_p \leqq \delta(t, K) t^{-1+(1/p)} \quad \forall \xi_0 \in K.$$

Proof of Lemma. Clearly $\xi(t) = S(t)\xi_0$ satisfies the integral equation (3.51) and $M(t) = \sup_{0 \leqq \tau \leqq t} \tau^{1-(1/p)} \| \xi(\cdot, \tau) \|_p$ is continuous (since $\xi_0 \in L^\infty \cap L^1$). Hence we can repeat part of the proof of Theorem B (involving formulas (3.50)–(3.56)) to derive (a).

Proof of Theorem. By the uniqueness part in Theorem B applied to the initial condition $\theta(s)$, we know that

(b) $$S(t)\theta(s) = \theta(t + s) \quad \forall s > 0, \quad \forall t \geqq 0.$$

(Note that $\lim_{t \to 0} t \| \theta(t + s) \|_\infty = 0$ because $\theta \in C(\mathbb{R}_+; L^\infty)$.)

Applying the above Lemma with $K = \theta((0, 1])$, which is precompact in L^1 since $\theta \in C(\bar{\mathbb{R}}_+; L^1)$, we find $\delta(t, K)$ such that (combining (a) and (b)),

(c) $$\| \theta(t + s) \|_p \leqq \delta(t, K) t^{-1+(1/p)} \quad \forall s \in (0, 1].$$

As $s \to 0$ in (c) (with $t > 0$ being fixed), we obtain

$$\|\theta(t)\|_p \leqq \delta(t, K) t^{-1+(1/p)}.$$

In particular, $\lim_{t \to 0} t \|\theta(t)\|_\infty = 0$, and we are reduced to the uniqueness conditions in Theorem B.

Note added in proof. T. KATO has kindly informed me that he also had previously used conditions similar to (1.19) to establish uniqueness, beginning with his paper [1] with H. FUJITA. It seems that the observation made above, namely that assumption (1.19) is redundant for uniqueness, also applies to several other papers of T. KATO, for example, [2].

References

1. T. KATO & H. FUJITA, *On the nonstationary Navier-Stokes system*, Rend. Sem. Math. Univ. Padova **32** (1962), 243–260.
2. T. KATO, *Strong L^p-solutions of the Navier-Stokes equation in* $\mathbb{R}^n$, Math. Z. **187** (1984), 471–480.

Analyse Numérique
Université Paris VI
4 place Jussieu
75252 Paris Cedex 05

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