Calc. Var. 1, 123–148 (1993)

Asymptotics for the minimization of a Ginzburg-Landau functional

Fabrice Bethuel[1,2], **Haïm Brezis**[3,4], **and Frédéric Hélein**[2]

[1] ENPC-CERMA, La Courtine, F-93167 Noisy-le-Grand Cedex, France
[2] CMLA, ENS-Cachan, 61, avenue du Président Wilson, F-94235 Cachan Cedex, France
[3] Analyse Numérique, Université P. et M. Curie, 4, place Jussieu, F-75252 Paris Cedex 05, France
[4] Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA

Received September 11, 1992/Accepted September 14, 1992

Abstract. Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded simply connected domain. Consider the functional

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_\Omega (|u|^2-1)^2$$

on the class $H^1_g = \{u \in H^1(\Omega;\mathbb{C});\ u = g \text{ on } \partial\Omega\}$ where $g : \partial\Omega \to \mathbb{C}$ is a prescribed smooth map with $|g| = 1$ on $\partial\Omega$ and $\deg(g, \partial\Omega) = 0$. Let u_ε be a minimizer for E_ε on H^1_g. We prove that $u_\varepsilon \to u_0$ in $C^{1,\alpha}(\overline{\Omega})$ as $\varepsilon \to 0$, where u_0 is identified. Moreover $\|u_\varepsilon - u_0\|_{L^\infty} \le C\varepsilon^2$.

Mathematics Subject Classification: 35B25, 35B40, 35J55, 35J60

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded simply connected domain. Consider the functional

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_\Omega (|u|^2-1)^2 \tag{1}$$

which is defined for maps $u \in H^1(\Omega;\mathbb{C})$.

Functionals of this type are related to models introduced by Ginzburg-Landau in the study of phase transition problems occurring e.g., in superconductivity, superfluidity and XY-magnetism (see for example [5, 7, 9]). The order parameter has two degrees of freedom – so it may be described by a complex number u.

We are concerned with the minimization of the functional E_ε for a given boundary condition. More precisely let

$$H^1_g = \{u \in H^1(\Omega;\mathbb{C});\ u = g \text{ on } \partial\Omega\},$$

where $g : \partial\Omega \to \mathbb{C}$ is a prescribed smooth map with $|g(x)| = 1\ \forall x \in \partial\Omega$.

It is easy to see that

$$\operatorname*{Min}_{u\in H^1_g} E_\varepsilon(u) \tag{2}$$

is achieved by some u_ε which satisfies the Euler equation

$$\begin{cases} -\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon(1-|u_\varepsilon|^2) & \text{on} \quad \Omega\,, \\ u_\varepsilon = g & \text{on} \quad \partial\Omega\,. \end{cases} \tag{3}$$

The maximum principle implies (see Proposition 2 below) that any solution u_ε of (3) satisfies $|u_\varepsilon| \le 1$ on Ω.

Our purpose is to study the behaviour of u_ε as $\varepsilon \to 0$ and we shall always assume that $0 < \varepsilon < 1$. It turns out that the value $d = \deg(g, \partial\Omega)$ (i.e., the Brouwer degree or winding number of g considered as a map from $\partial\Omega$ into S^1) plays a crucial role in the asymptotic analysis of u_ε.

When $d = 0$ we shall prove here that $u_\varepsilon \to u_0$ in $C^1(\overline{\Omega})$ (and even in $C^k_{\text{loc}}(\Omega)\ \forall k$) where u_0 is identified.

When $d \ne 0$ the situation is much more delicate since $\int_\Omega |\nabla u_\varepsilon|^2 \to +\infty$. In this case one proves that a subsequence u_{ε_n} converges uniformly on compact sets of $\Omega\backslash S$ to some limit u_0; the singular set S consists of exactly $|d|$ points in Ω. If for instance d is positive at these points vortices of degree one appear (see [1, 2]).

In this paper we concentrate on the first case $d = 0$; the case $d \ne 0$ is studied in [2]. However, the results of the present paper – and especially the ones in Sect. 3 – are very useful for the analysis of the case $d \ne 0$ (locally) away from the singularities.

In what follows we assume

$$\deg(g, \partial\Omega) = 0\,. \tag{4}$$

Since (4) holds there are smooth extensions of g from $\overline{\Omega}$ into S^1. Let

$$H^1_g(\Omega; S^1) = \{u \in H^1(\Omega; S^1);\ u = g \text{ on } \partial\Omega\}\,.$$

Consider the minimization problem

$$\operatorname*{Min}_{u\in H^1_g(\Omega;S^1)} \int_\Omega |\nabla u|^2. \tag{5}$$

Let u_0 be a minimizer for (5). By a classical result of Morrey [6] (see also [4]) one knows that u_0 is smooth and satisfies

$$\begin{cases} -\Delta u_0 = u_0|\nabla u_0|^2 & \text{on} \quad \Omega\,, \\ |u_0| = 1 & \text{on} \quad \Omega\,, \\ u_0 = g & \text{on} \quad \partial\Omega\,. \end{cases} \tag{6}$$

There is a simple relation between problem (6) and harmonic functions which implies in particular that (6) has a unique solution. First, note that since Ω is simply connected and $\deg(g, \partial\Omega) = 0$ there is a smooth function $\varphi_0 : \partial\Omega \to \mathbb{R}$ such that

$$e^{i\varphi_0} = g \quad \text{on} \quad \partial\Omega\,. \tag{7}$$

We also denote by φ_0 its harmonic extension in Ω.

Lemma 1. *We have $u_0 = e^{i\varphi_0}$ in Ω.*

Proof. Any map $u \in H^1_g(\Omega; S^1)$ may be written as

$$u = e^{i\varphi} \tag{8}$$

for some function $\varphi \in H^1(\Omega; \mathbb{R})$ [here we use again assumption (4) and the fact that Ω is simply connected]. We have

$$|\nabla u|^2 = |\nabla \varphi|^2 \tag{9}$$

and

$$\begin{cases} \Delta u = e^{i\varphi}(i\Delta\varphi - |\nabla\varphi|^2) \\ \quad\;\; = u(i\Delta\varphi - |\nabla u|^2). \end{cases}$$

Thus, the equation $-\Delta u = u|\nabla u|^2$ is equivalent to $\Delta\varphi = 0$.

Our main result is the following:

Theorem 1. *We have, as* $\varepsilon \to 0$,

$$u_\varepsilon \to u_0 \quad \text{in} \quad C^{1,\alpha}(\overline{\Omega}) \quad \forall \alpha < 1\,, \tag{10}$$

$$\|\Delta u_\varepsilon\|_{L^\infty(\Omega)} \le C\,, \tag{11}$$

$$\|u_\varepsilon - u_0\|_{L^\infty(\Omega)} \le C\varepsilon^2, \tag{12}$$

and, for every compact subset $K \subset \Omega$ *and every integer* k,

$$\|u_\varepsilon - u_0\|_{C^k(K)} \le C_{K,k}\varepsilon^2, \tag{13}$$

$$\left\| \frac{1 - |u_\varepsilon|^2}{\varepsilon^2} - |\nabla u_0|^2 \right\|_{C^k(K)} \le C_{K,k}\varepsilon^2. \tag{14}$$

Remark 1. In general u_ε does *not* converge to u_0 in $C^2(\overline{\Omega})$. Indeed, on $\partial\Omega$, $|u_\varepsilon| = |g| = 1$ and, by (2), $\Delta u_\varepsilon = 0$; on the other hand $\Delta u_0 = -u_0|\nabla u_0|^2$. Similarly (14) does not hold up to the boundary.

Remark 2. Combining (11), (12) and the interpolation inequality of Lemma A.2 (in the Appendix) we see that

$$\|\nabla(u_\varepsilon - u_0)\|_{L^\infty(\Omega)} \le C\varepsilon\,. \tag{15}$$

Section 2 is devoted to the proof of Theorem 1. In Sect. 3 we study a situation where g is not a fixed map but also depends on ε. More precisely we are given a family $g_\varepsilon : \partial\Omega \to \mathbb{C}$ such that $g_\varepsilon \to g$ uniformly on $\partial\Omega$ with $|g| = 1$ on $\partial\Omega$ and $\deg(g, \partial\Omega) = 0$. Note that here we do not assume that $|g_\varepsilon| = 1$. Let u_ε be a minimizer of

$$\operatorname*{Min}_{H^1_{g_\varepsilon}} E_\varepsilon\,.$$

Under appropriate assumptions on g_ε we prove (see Theorem 2 in Sect. 3) that $u_\varepsilon \to u_0$ in $H^1(\Omega)$ and in $C^k_{\text{loc}}(\Omega)$. As was already mentioned this result plays an important role in [2].

2 Proof of Theorem 1

It is useful to start with some simple facts.

Proposition 1. *Let u_ε be a minimizer for* (2). *We have, as $\varepsilon \to 0$,*

$$u_\varepsilon \to u_0 \quad \text{strongly in} \quad H^1.$$

Proof. Since $u_0 \in H^1_g(\Omega; S^1)$ we have

$$\frac{1}{2}\int |\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\int (|u_\varepsilon|^2 - 1)^2 \le \frac{1}{2}\int |\nabla u_0|^2. \tag{16}$$

Hence (u_ε) is bounded in H^1 and thus

$$u_{\varepsilon_n} \rightharpoonup u \quad \text{weakly in} \quad H^1.$$

By (16) and lower semi-continuity we see that

$$\int |\nabla u|^2 \le \int |\nabla u_0|^2. \tag{17}$$

On the other hand, by (16), we also have

$$\int (|u_\varepsilon|^2 - 1)^2 \le C\varepsilon^2$$

and therefore $|u| = 1$ a.e. Hence $u \in H^1_g(\Omega; S^1)$ and in view of (17), u is a minimizer for (5), i.e., $u = u_0$. The strong H^1 convergence follows from the fact that

$$\int |\nabla u_\varepsilon|^2 \le \int |\nabla u_0|^2.$$

The convergence of the full sequence is a consequence of a uniqueness of u_0.

Proposition 2. *Let u_ε be a solution of* (3). *Then $|u_\varepsilon| \le 1$ on Ω.*

Proof. We have

$$\begin{cases} \frac{1}{2}\Delta |u_\varepsilon|^2 & = u_\varepsilon \cdot \Delta u_\varepsilon + |\nabla u_\varepsilon|^2 \\ & = \frac{1}{\varepsilon^2}|u_\varepsilon|^2(|u_\varepsilon|^2 - 1) + |\nabla u_\varepsilon|^2 \\ & \ge \frac{1}{\varepsilon^2}|u_\varepsilon|^2(|u_\varepsilon|^2 - 1). \end{cases}$$

Hence the function $v = |u_\varepsilon|^2 - 1$ satisfies

$$\begin{cases} -\Delta v + a(x)v \le 0 & \text{on} \quad \Omega \\ v = 0 & \text{on} \quad \partial\Omega \end{cases}$$

with $a(x) = \frac{2}{\varepsilon^2}|u_\varepsilon|^2 \ge 0$. By the maximum principle we conclude that $v \le 0$ on Ω.

Proposition 3. *Let u_ε be a minimizer for* (2). *Then*

$$\int_{\partial\Omega} \left|\frac{\partial u_\varepsilon}{\partial n}\right|^2 \le C \tag{18}$$

where C depends only on g and Ω.

Proof. Let $V = (V_1, V_2)$ be a smooth vector-field on Ω such that $V = n =$ outward normal on $\partial\Omega$. We multiply (3) by $V \cdot \nabla u_\varepsilon = V_1 \frac{\partial u_\varepsilon}{\partial x_1} + V_2 \frac{\partial u_\varepsilon}{\partial x_2}$. For simplicity we drop ε. Note that

$$\int_\Omega \Delta u(V \cdot \nabla u) = \int_{\partial\Omega} \left|\frac{\partial u}{\partial n}\right|^2 - \int_\Omega \sum_{i=1}^2 u_{x_i}(V \cdot \nabla u)_{x_i} \,.$$

But since $\int_\Omega |\nabla u_\varepsilon|^2$ remains bounded as $\varepsilon \to 0$ we have

$$\begin{aligned}\int_\Omega u_{x_i}(V_1 u_{x_1} + V_2 u_{x_2})_{x_i} &= \int_\Omega u_{x_i}(V_1 u_{x_1 x_i} + V_2 u_{x_2 x_i}) + O(1) \\ &= \frac{1}{2}\int_\Omega V_1(|u_{x_i}|^2)_{x_1} + V_2(|u_{x_i}|^2)_{x_2} + O(1) \\ &= \frac{1}{2}\int_{\partial\Omega} (u_{x_i})^2 + O(1)\,.\end{aligned}$$

Thus

$$\int_\Omega \Delta u(V \cdot \nabla u) = \int_{\partial\Omega} \left|\frac{\partial u}{\partial n}\right|^2 - \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 + O(1)\,.$$

On the other hand

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_\Omega u(1-|u|^2)(V \cdot \nabla u) &= \frac{1}{2\varepsilon^2}\int_\Omega (1-|u|^2)\sum_{i=1}^2 V_i(|u|^2)_{x_i} \\ &= \frac{1}{4\varepsilon^2}\int_\Omega (1-|u|^2)^2 \operatorname{div} V = O(1) \quad \text{by} \quad (16)\,.\end{aligned}$$

We conclude that

$$\int_{\partial\Omega} \left|\frac{\partial u}{\partial n}\right|^2 - \frac{1}{2}|\nabla u|^2 = \frac{1}{2}\int_{\partial\Omega} \left|\frac{\partial u}{\partial n}\right|^2 - \left|\frac{\partial u}{\partial \tau}\right|^2 = O(1)$$

where $\frac{\partial}{\partial \tau}$ denotes the tangential derivative. Since $\frac{\partial u}{\partial \tau} = \frac{\partial g}{\partial \tau}$ we see that the estimate in Proposition 3 holds. It is convenient to split the proof of Theorem 1 into 2 parts:

A. Interior estimates

B. Estimates up to the boundary.

A. Interior estimates

Step A.1: $|\nabla u_\varepsilon| \leq \frac{C_K}{\varepsilon}$ on every compact subset $K \subset \Omega$.
This follows directly from Lemma A.1 in the Appendix and Proposition 2.

Step A.2: $|u_\varepsilon| \to 1$ uniformly on every compact subset $K \subset \Omega$.
In view of (16) and the fact that $u_\varepsilon \to u_0$ in H^1 we see that

$$\frac{1}{\varepsilon^2}\int_\Omega (|u_\varepsilon|^2 - 1)^2 \to 0\,. \tag{19}$$

Let $x_0 \in K$ and set $\alpha = |u_\varepsilon(x_0)|$. By Step A.1 we have

$$|u_\varepsilon(x)| \leq \alpha + \frac{C}{\varepsilon}\varrho \quad \text{if} \quad |x - x_0| < \varrho < \delta = \operatorname{dist}(K, \partial\Omega)\,.$$

Thus

$$1 - |u_\varepsilon(x)| \geq 1 - \alpha - \frac{C}{\varepsilon}\varrho \quad \text{on} \quad B(x_0, \varrho)$$

and

$$(1 - |u_\varepsilon(x)|)^2 \geq \left(1 - \alpha - \frac{C}{\varepsilon}\varrho\right)^2 \quad \text{provided} \quad \frac{C\varrho}{\varepsilon} \leq 1 - \alpha\,.$$

Since

$$(1 - |u_\varepsilon(x)|^2)^2 \geq (1 - |u_\varepsilon(x)|)^2$$

we obtain, by (19),

$$\varepsilon^2 o(1) = \int_{B(x_0,\varrho)} (1 - |u_\varepsilon|^2)^2 \geq \pi\varrho^2\left(1 - \alpha - \frac{C\varrho}{\varepsilon}\right)^2.$$

We choose

$$\varrho = \frac{\varepsilon(1-\alpha)}{2C} < \delta \quad (\text{for } \varepsilon \text{ small})\,.$$

Hence

$$\varepsilon^2 o(1) \geq \pi\,\frac{\varepsilon^2(1-\alpha)^2}{4C^2}\,\frac{(1-\alpha)^2}{4}$$

and therefore

$$(1-\alpha)^4 \leq o(1)$$

i.e., $|u_\varepsilon| \to 1$ uniformly on compact subsets of Ω.

Step A.3: Set

$$A_\varepsilon = \frac{1}{2}|\nabla u_\varepsilon|^2.$$

Then we have

$$-\Delta A_\varepsilon + \frac{1}{2}|D^2 u_\varepsilon|^2 \leq \frac{4}{|u_\varepsilon|^2}A_\varepsilon^2 \quad \text{on} \quad \Omega\,, \tag{20}$$

where $|D^2 u_\varepsilon|^2 = \sum_{i,j=1}^{2}\left(\frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j}\right)^2$.

Proof. We have, dropping ε,

$$\Delta A = |D^2 u|^2 + \sum_{i=1,2} u_{x_i} \Delta(u_{x_i}) \tag{21}$$

and then using the Euler equation (3) we find

$$\Delta u_{x_i} = u_{x_i} \frac{(|u|^2 - 1)}{\varepsilon^2} + \frac{2}{\varepsilon^2} u(u \cdot u_{x_i}) .$$

Inserting this into (21) we see that

$$\Delta A = |D^2 u|^2 + |\nabla u|^2 \frac{(|u|^2 - 1)}{\varepsilon^2} + \frac{2}{\varepsilon^2}(u \cdot \nabla u)^2 .$$

Thus

$$\Delta A \geq |D^2 u|^2 + |\nabla u|^2 \frac{|\Delta u|}{|u|} .$$

Since $|\Delta u| \leq \sqrt{2}|D^2 u|$ we have

$$-\Delta A + |D^2 u|^2 \leq 2\sqrt{2} A \frac{|D^2 u|}{|u|} \leq \frac{1}{2}|D^2 u|^2 + 4\frac{A^2}{|u|^2} .$$

Step A.4: We have

$$(u_\varepsilon) \quad \text{is bounded in} \quad H^2_{\text{loc}} \tag{22}$$

and

$$(\nabla u_\varepsilon) \quad \text{is bounded in} \quad L^\infty_{\text{loc}} . \tag{23}$$

Given $\delta > 0$ (to be determined later) we may choose R sufficiently small so that

$$\int_{B(x_0,R)} |\nabla u_\varepsilon|^2 < \delta \quad \forall x_0 \in \Omega , \quad \forall \varepsilon \tag{24}$$

[such an integral is understood on $\Omega \cap B(x_0, R)$]; this can be achieved since $u_\varepsilon \to u_0$ strongly in $H^1(\Omega)$ by Proposition 1.

Fix a point $x_0 \in \Omega$ and set $d = \text{dist}(x_0, \partial\Omega)$. Let ζ be a smooth function with support in $B(x_0, r)$ with $r = \min(d/2, R)$ such that $\zeta = 1$ on $B(x_0, r/2)$. Multiplying (20) by ζ^2 we obtain

$$\frac{1}{2}\int_\Omega \zeta^2 |D^2 u_\varepsilon|^2 \leq 4\int_\Omega \frac{\zeta^2}{|u_\varepsilon|^2} A_\varepsilon^2 + \int_\Omega (\Delta\zeta^2) A_\varepsilon . \tag{25}$$

Since $|u_\varepsilon| \to 1$ uniformly on compact subsets of Ω (Step A.2) we have, for ε sufficiently small,

$$|u_\varepsilon| \geq \frac{1}{2} \quad \text{on} \quad B(x_0, r) . \tag{26}$$

Hence we have (because u_ε is bounded in H^1)

$$\int_\Omega \zeta^2 |D^2 u_\varepsilon|^2 \leq C \int_\Omega \zeta^2 |\nabla u_\varepsilon|^4 + C . \tag{27}$$

Recall that $W^{1,1}(\Omega) \subset L^2(\Omega)$ and that (see e.g., [8])

$$\left(\int_\Omega \varphi^2\right)^{1/2} \le C \int_\Omega |\nabla\varphi| + |\varphi| \quad \forall \varphi \in W^{1,1}(\Omega). \tag{28}$$

Applying (28) with $\varphi = \zeta|\nabla u_\varepsilon|^2$ we are led to

$$\int_\Omega \zeta^2|\nabla u_\varepsilon|^4 \le C\left(\int_\Omega \zeta|\nabla u_\varepsilon|\,|D^2u_\varepsilon|\right)^2 + C$$

(we use once more the fact that u_ε is bounded in H^1). By Cauchy-Schwarz and (24) we obtain

$$\int_\Omega \zeta^2|\nabla u_\varepsilon|^4 \le C\delta \int_\Omega \zeta^2|D^2u_\varepsilon|^2 + C.$$

Hence if we choose δ sufficiently small we may absorb $\int_\Omega \zeta^2|\nabla u_\varepsilon|^4$ into the left-hand side of (27). We conclude that

$$\int_\Omega \zeta^2|D^2u_\varepsilon|^2 \le C.$$

This proves (22).

From (22) and the Sobolev embedding we see that (∇u_ε) is bounded in $L^q_{\text{loc}}(\Omega)$ for every $q < \infty$. Going back to (20) we deduce that

$$-\Delta A_\varepsilon \le f_\varepsilon$$

with f_ε bounded in $L^q_{\text{loc}}(\Omega)$ for every $q < \infty$. This implies by standard elliptic theory that (A_ε) is bounded in $L^\infty_{\text{loc}}(\Omega)$.

Step A.5: We have

$$\frac{1}{\varepsilon^2}(1 - |u_\varepsilon|) \quad \text{is bounded in} \quad L^\infty_{\text{loc}} \tag{29}$$

and

$$\Delta u_\varepsilon \quad \text{is bounded in} \quad L^\infty_{\text{loc}}. \tag{30}$$

We shall often use the following

Lemma 2. *Let $\omega(r)$ be the solution of*

$$\begin{cases} -\varepsilon^2\Delta\omega + \omega = 0 & \textit{on} \quad B(0,R), \\ \omega = 1 & \textit{on} \quad \partial B(0,R). \end{cases}$$

Then, for $\varepsilon < \frac{3}{4}R$,

$$\omega(r) \le e^{\frac{1}{4\varepsilon R}(r^2 - R^2)} \quad \text{on} \quad B(0,R).$$

Proof of Lemma 2. An easy computation shows that the function $e^{\frac{1}{4\varepsilon R}(r^2-R^2)}$ is a supersolution.

Proof of Step A.5. We have

$$\frac{1}{2}\Delta|u_\varepsilon|^2 = \frac{1}{\varepsilon^2}|u_\varepsilon|^2(|u_\varepsilon|^2-1)+|\nabla u_\varepsilon|^2. \tag{31}$$

Let K be a compact subset of Ω and let $d=\operatorname{dist}(K,\partial\Omega)$. Assume $x_0=0\in K$. For ε sufficiently small we have

$$|u_\varepsilon|\geq 1/\sqrt{2} \quad \text{on} \quad B(0,d/2)\,.$$

Thus we have, by Step A.4 and (31),

$$\Delta|u_\varepsilon|^2+\frac{1}{\varepsilon^2}(1-|u_\varepsilon|^2)\leq C \quad \text{on} \quad B(0,d/2)\,.$$

Setting $\varphi = 1-|u_\varepsilon|^2$ we find

$$-\varepsilon^2\Delta\varphi+\varphi\leq\varepsilon^2 C \quad \text{on} \quad B(0,d/2)\,.$$

Applying Lemma 2 and the maximum principle we obtain that

$$\varphi\leq\varepsilon^2 C+e^{\frac{1}{2\varepsilon d}\left(|x|^2-\frac{d^2}{4}\right)}.$$

In particular

$$\frac{1}{\varepsilon^2}\,\varphi(0)\leq C+\frac{1}{\varepsilon^2}e^{-\frac{d}{8\varepsilon}}. \tag{32}$$

This proves (29) since the right-hand side in (32) remains bounded as $\varepsilon\to 0$. Finally, we use equation (3) and (29) to derive (30).

B. Estimates up to the boundary

Step B.1: Let u_ε be a solution of (3) then

$$|\nabla u_\varepsilon|\leq\frac{C}{\varepsilon} \quad \text{on} \quad \Omega\,, \tag{33}$$

where C depends only on g and Ω.

Proof. Write

$$u_\varepsilon = v_\varepsilon + w\,,$$

where v_ε is the solution of

$$\begin{cases} -\Delta v_\varepsilon = \dfrac{1}{\varepsilon^2}u_\varepsilon(1-|u_\varepsilon|^2) & \text{on} \quad \Omega\,,\\ v_\varepsilon = 0 & \text{on} \quad \partial\Omega \end{cases}$$

and w is the solution of

$$\begin{cases} -\Delta w = 0 & \text{on} \quad \Omega\,,\\ w = g & \text{on} \quad \partial\Omega\,. \end{cases}$$

It follows from Lemma A.2 in the Appendix and Proposition 2 that

$$\|\nabla v_\varepsilon\|_{L^\infty}\leq\frac{C}{\varepsilon}\|v_\varepsilon\|_{L^\infty}\leq\frac{C}{\varepsilon}(\|u_\varepsilon\|_{L^\infty}+\|w\|_{L^\infty})\leq\frac{C}{\varepsilon}\,.$$

Therefore

$$\|\nabla u_\varepsilon\|_{L^\infty} \le \|\nabla v_\varepsilon\|_{L^\infty} + \|\nabla w\|_{L^\infty} \le \frac{C}{\varepsilon} + C\,.$$

This yields (33).

Step B.2: $|u_\varepsilon| \to 1$ uniformly on $\overline{\Omega}$.
The proof is exactly the same as the proof of Step A.2. We allow here x_0 to be in $\overline{\Omega}$ and we use the fact that

$$\text{meas}(\Omega \cap B(x_0, \varrho)) \ge C\varrho^2,$$

where C depends only on the smoothness of $\partial\Omega$.

Step B.3: u_ε remains bounded in $H^2(\Omega)$.
We already know (Step A.4) that (u_ε) is bounded in $H^2_{\text{loc}}(\Omega)$; thus we have only to establish H^2 estimates near the boundary. Let $x_0 \in \partial\Omega$. We drop the subcript ε.

For simplicity we first describe the proof when $\partial\Omega$ is flat near x_0, i.e.,

$$\Omega \cap B(x_0, d) = \{(x_1, x_2);\ x_2 > 0\} \cap B(x_0, d)$$

for some positive d. Let ζ be a smooth function with support in $B(x_0, r)$ with $r = \min(d, R)$ such that $\zeta = 1$ on $B(x_0, r/2)$. Multiplying (20) by ζ^2 we obtain

$$\frac{1}{2}\int_\Omega \zeta^2 |D^2 u|^2 \le 4\int_\Omega \frac{\zeta^2}{|u|^2} A^2 + \int_\Omega \zeta^2 \Delta A\,. \tag{34}$$

By Step B.2 $|u_\varepsilon| \to 1$ uniformly on $\overline{\Omega}$ and therefore we have, for ε sufficiently small,

$$|u_\varepsilon| \ge \frac{1}{2} \quad \text{on} \quad \Omega\,. \tag{35}$$

From (34) and (35) we obtain

$$\frac{1}{2}\int_\Omega \zeta^2 |D^2 u|^2 \le 4\int_\Omega \zeta^2 |\nabla u|^4 - \int_{[x_2=0]} \zeta^2 \frac{\partial A}{\partial x_2} + \int_{[x_2=0]} \frac{\partial \zeta^2}{\partial x_2} A + \int_\Omega (\Delta\zeta^2) A\,. \tag{36}$$

The last two integrals on the right-hand side of (36) are bounded by Proposition 3 and by (16).

We claim that

$$\int_{[x_2=0]} \zeta^2 \frac{\partial A}{\partial x_2} \quad \text{remains bounded}\,. \tag{37}$$

Proof of (37). We have

$$\begin{aligned}\frac{\partial}{\partial x_2} A &= \frac{1}{2}\frac{\partial}{\partial x_2}|\nabla u|^2 = (u_{x_1} u_{x_1 x_2} + u_{x_2} u_{x_2 x_2}) \\ &= (u_{x_1} u_{x_1 x_2} - u_{x_2} u_{x_1 x_1})\end{aligned}$$

since $\Delta u = \frac{1}{\varepsilon^2} u(|u|^2 - 1) = 0$ on $\partial\Omega$. Hence

$$\int\limits_{[x_2=0]} \zeta^2 \frac{\partial A}{\partial x_2} = \int\limits_{[x_2=0]} \zeta^2 (g_{x_1} u_{x_1 x_2} - g_{x_1 x_1} u_{x_2}) = -2 \int\limits_{[x_2=0]} u_{x_2} (\zeta^2 g_{x_1 x_1} + \zeta \zeta_{x_1} g_{x_1}) .$$

This remains bounded by Proposition 3.

Thus we have shown that

$$\frac{1}{2} \int\limits_{\Omega} \zeta^2 |D^2 u|^2 \leq 4 \int\limits_{\Omega} \zeta^2 |\nabla u|^4 + C .$$

Using the same argument as in Step A.4 we conclude that

$$\int\limits_{\Omega} \zeta^2 |D^2 u|^2 \leq C .$$

In the general case where Ω is not flat near $x_0 = 0$ we introduce local coordinates which straighten the boundary. In the new coordinates the function u becomes $\tilde{u}$ defined in

$$U = \{(x_1, x_2);\ x_2 > 0\} \cap B(0, d) .$$

Choosing the change of coordinates $(x_1, x_2) \to (x_1, x_2 + h(x_1))$ where the graph of h represents locally $\partial\Omega$, equation (3) becomes

$$\begin{cases} -L\tilde{u} = \frac{1}{\varepsilon^2} \tilde{u}(1 - |\tilde{u}|^2) & \text{on} \quad U , \\ \tilde{u} = \tilde{g} & \text{on} \quad [x_2 = 0] \cap \partial U , \end{cases} \tag{38}$$

where $L = \sum\limits_{i,j=1}^{2} \frac{\partial}{\partial x_j} \left(a_{ij} \frac{\partial}{\partial x_i} \right)$ and

$$a_{11} = 1 , \quad a_{12} = a_{21} = h' , \quad a_{11} = 1 + h'^2 .$$

We use again

$$A = \frac{1}{2} |\nabla \tilde{u}|^2 = \frac{1}{2} \sum_{k=1}^{2} (\tilde{u}_{x_k})^2 ,$$

where ∇ refers to the gradient in the *new* (x_1, x_2) coordinates. We now go over the same computation as in Step A.3, but for the operator L. For simplicity we omit the summation symbol and we write u instead of $\tilde{u}$.

$$LA = a_{ij} u_{x_i x_k} x_{x_j x_k} + u_{x_k} \cdot L(u_{x_k}) . \tag{39}$$

Differentiating the first equation in (38) with respect to x_k we have

$$-L(u_{x_k}) = (a_{ij x_k} u_{x_i})_{x_j} + \frac{1}{\varepsilon^2} u_{x_k} (1 - |u|^2) - \frac{2}{\varepsilon^2} u (u \cdot u_{x_k}) . \tag{40}$$

Inserting this expression into (39) we obtain

$$LA \geq \alpha |D^2 u|^2 + \frac{1}{\varepsilon^2} |\nabla u|^2 (|u|^2 - 1) - C|\nabla u|(|\nabla u| + |D^2 u|), \tag{41}$$

where α denotes the ellipticity constant and C depends on $\|a_{ij}\|_{C^2}$. Therefore

$$\begin{aligned} LA &\geq \frac{\alpha}{2} |D^2 u|^2 + \frac{1}{\varepsilon^2} |\nabla u|^2 (|u|^2 - 1) - C|\nabla u|^2 \\ &\geq \frac{\alpha}{2} |D^2 u|^2 - |\nabla u|^2 \frac{|Lu|}{|u|} - C|\nabla u|^2 . \end{aligned}$$

Since

$$|Lu| \leq C(|D^2 u| + |\nabla u|) \tag{42}$$

we have

$$\begin{aligned} -LA + \frac{\alpha}{2} |D^2 u|^2 &\leq \frac{C}{|u|} |\nabla u|^2 (|D^2 u| + |\nabla u|) + C|\nabla u|^2 \\ &\leq \frac{\alpha}{4} |D^2 u|^2 + C|\nabla u|^4 + C \end{aligned}$$

by Step B.2 and Young's inequality.
Hence we find

$$-LA + \frac{\alpha}{4} |D^2 u|^2 \leq C|\nabla u|^4 + C . \tag{43}$$

Therefore we are led to

$$\frac{\alpha}{4} \int_U \zeta^2 |D^2 u|^2 \leq C \int_U \zeta^2 |\nabla u|^4 + \int_U \zeta^2 LA . \tag{44}$$

Finally we claim

$$\left| \int_U \zeta^2 LA \right| \leq C . \tag{45}$$

Proof of (45). We have

$$\begin{aligned} \int_U \zeta^2 LA = \int_U AL(\zeta^2) + 2 \int_{[x_2=0]} (\zeta^2)_{x_1} a_{12} A + \int_{[x_2=0]} \zeta^2 (a_{12})_{x_1} A \\ - \int_{[x_2=0]} a_{22} \zeta^2 (A)_{x_2} + \int_{[x_2=0]} a_{22} (\zeta^2)_{x_2} A . \end{aligned} \tag{46}$$

All the integrals on the right-hand side of (46) are clearly bounded (since u is bounded in $H^1(\Omega)$ and by Proposition 3) except for $\int_{[x_2=0]} a_{22} \zeta^2 (A)_{x_2}$. To estimate this integral we write

$$(A)_{x_2} = u_{x_1} u_{x_1 x_2} + u_{x_2} u_{x_2 x_2} .$$

On $[x_2 = 0]$ we have, by (38), $Lu = 0$ so that

$$(a_{22} u_{x_2})_{x_2} = -(a_{11} u_{x_1})_{x_1} - (a_{12} u_{x_1})_{x_2} - (a_{21} u_{x_2})_{x_1} .$$

Using the fact that $u_{x_1x_2}u_{x_2} = \frac{1}{2}(u_{x_2}^2)_{x_1}$ we see by a simple computation that $\int\limits_{[x_2=0]} a_{22}\zeta^2 A_{x_2}$ remains bounded (we make several integrations by parts in x_1 and use Proposition 3). This completes the proof of (45).

Finally we go back to (44) which yields, using (45),

$$\frac{\alpha}{4}\int\limits_U \zeta^2|D^2u|^2 \le C\int\limits_U \zeta^2|\nabla u|^4 + C\,.$$

As in the proof of Step A.4 we conclude that

$$\int\limits_U \zeta^2|D^2u|^2 \le C\,.$$

Step B.4: Proof of (11).

Proof. By Step B.2 we may assume that

$$|u_\varepsilon| \ge \frac{1}{2} \quad \text{on} \quad \Omega\,.$$

Letting

$$\psi = \frac{1}{\varepsilon^2}(1-|u_\varepsilon|^2)$$

we have, as in the proof of Step A.5,

$$-2\varepsilon^2\Delta\psi + \psi \le 4|\nabla u_\varepsilon|^2 \quad \text{on} \quad C\,. \tag{47}$$

Recall (by Step B.3 and Sobolev embedding) that (∇u_ε) is bounded in $L^r(\Omega)$ for every $r<\infty$. Multiplying (47) by ψ^{q-1} we see that, since $\psi=0$ on $\partial\Omega$,

$$\int\limits_\Omega \psi^q \le 4\int\limits_\Omega |\nabla u_\varepsilon|^2\psi^{q-1}.$$

This yields

$$\|\psi\|_{L^q} \le 4\|\nabla u_\varepsilon\|_{L^{2q}}^2 \le C_q\,.$$

In view of (3) we conclude that

$$\|\Delta u_\varepsilon\|_{L^q} \le C_q \quad \text{for every} \quad q<\infty\,.$$

In particular (choosing any $q>2$) we see that

$$\|\nabla u_\varepsilon\|_{L^\infty} \le C\,. \tag{48}$$

Going back to (47) and using the maximum principle we find

$$\|\psi\|_{L^\infty} \le 4\|\nabla u_\varepsilon\|_{L^\infty}^2 \le C\,.$$

This yields (11) since $-\Delta u_\varepsilon = u_\varepsilon\psi$.

Step B.5: Proof of (12).

Since $|u_\varepsilon| \ge 1/2$ on Ω (for ε sufficiently small) we may write

$$u_\varepsilon = \varrho_\varepsilon e^{i\varphi_\varepsilon} \quad \text{with} \quad \varrho_\varepsilon = |u_\varepsilon|\,. \tag{49}$$

Equation (3) becomes

$$\varrho_\varepsilon \Delta\varphi_\varepsilon + 2\nabla\varrho_\varepsilon \nabla\varphi_\varepsilon = 0 \tag{50}$$

i.e.,

$$\operatorname{div}(\varrho_\varepsilon^2 \nabla\varphi_\varepsilon) = 0 \tag{51}$$

and

$$-\Delta\varrho_\varepsilon + \varrho_\varepsilon|\nabla\varphi_\varepsilon|^2 = \frac{1}{\varepsilon^2}\,\varrho_\varepsilon(1-\varrho_\varepsilon^2)\,. \tag{52}$$

We already know by Step B.4 that

$$\|\varrho_\varepsilon - 1\|_{L^\infty(\Omega)} \leq C\varepsilon^2\,. \tag{53}$$

Write (51) as

$$\begin{cases} -\Delta(\varphi_\varepsilon - \varphi_0) = \operatorname{div}((\varrho_\varepsilon^2 - 1)\nabla\varphi_\varepsilon) & \text{on} \quad \Omega\,, \\ \varphi_\varepsilon - \varphi_0 \ = 0 & \text{on} \quad \partial\Omega\,, \end{cases} \tag{54}$$

(recall that $\Delta\varphi_0 = 0$).
It follows from the elliptic estimates that

$$\|\varphi_\varepsilon - \varphi_0\|_{L^\infty} \leq C\|(\varrho_\varepsilon^2 - 1)\nabla\varphi_\varepsilon\|_{L^\infty} \leq C\varepsilon^2 \tag{55}$$

by (53) and (48). Putting together (53) and (55) we obtain (12).

Step B.6: For every integer k we have

$$\|\nabla\varphi_\varepsilon\|_{C^k_{\text{loc}}} \leq C \tag{56}$$

and

$$\left\|\frac{1-\varrho_\varepsilon}{\varepsilon^2}\right\|_{C^k_{\text{loc}}} \leq C \tag{57}$$

The proof is by induction on k. When $k = 0$ these estimates have already been established (even globally on Ω) – see (48) and (53).
Set

$$X_\varepsilon = \frac{1}{\varepsilon^2}(1-\varrho_\varepsilon)\,. \tag{58}$$

We write (52) as

$$-\Delta\varrho_\varepsilon = -\varrho_\varepsilon|\nabla\varphi_\varepsilon|^2 + \varrho_\varepsilon(1+\varrho_\varepsilon)X_\varepsilon\,. \tag{59}$$

The right hand side of (59) remains bounded in C^k_{loc} by (56) and (57). Thus

$$\|\varrho_\varepsilon\|_{W^{k+2,p}_{\text{loc}}} \leq C \quad \forall p < \infty\,. \tag{60}$$

In particular

$$\|\nabla\varrho_\varepsilon\|_{C^k_{\text{loc}}} \leq C\,. \tag{61}$$

By (50) we have

$$-\Delta\varphi_\varepsilon = 2\frac{\nabla\varrho_\varepsilon}{\varrho_\varepsilon}\nabla\varphi_\varepsilon \ \text{ on } \ \Omega\,. \tag{62}$$

From (56), (61), (62) and elliptic estimates we deduce that

$$\|\varphi_\varepsilon\|_{W^{k+2,p}_{\text{loc}}} \leq C \quad \forall p < \infty\,. \tag{63}$$

Using (62) once more together with (60) and (63) we obtain

$$\|\varphi_\varepsilon\|_{W^{k+3,p}_{\mathrm{loc}}} \le C \quad \forall p < \infty$$

which implies by the Sobolev embedding

$$\|\nabla\varphi_\varepsilon\|_{C^{k+1}_{\mathrm{loc}}} \le C\,, \tag{64}$$

i.e., (56) holds with $(k+1)$ instead of k. From the definition of X_ε and (59) we have

$$\varepsilon^2 \Delta X_\varepsilon = -\varrho_\varepsilon|\nabla\varphi_\varepsilon|^2 + \varphi_\varepsilon(1+\varrho_\varepsilon)X_\varepsilon\,. \tag{65}$$

By Lemma A.1 in the Appendix applied to $D^k X_\varepsilon$ (where D^k denotes any k^{th} order derivative) we obtain

$$\|D^{k+1}X_\varepsilon\|^2_{L^\infty(\Omega'')} \le C\|D^k X_\varepsilon\|_{L^\infty(\Omega')}(\|D^k X_\varepsilon\|_{L^\infty(\Omega')} + \|D^k \Delta X_\varepsilon\|_{L^\infty(\Omega')}) \tag{66}$$

(with $\overline{\Omega}'' \subset \Omega'$ and $\overline{\Omega}' \subset \Omega$). In view of (57)

$$\|D^k X_\varepsilon\|_{L^\infty(\Omega')} \le C\,.$$

Using (65), (56) and (57) we have

$$\|D^k \Delta X_\varepsilon\|_{L^\infty(\Omega')} \le \frac{C}{\varepsilon^2}\,.$$

Consequently, by (66), we are led to

$$\varepsilon\|D^{k+1}X_\varepsilon\|_{L^\infty_{\mathrm{loc}}} \le C$$

i.e.,

$$\|\varepsilon X_\varepsilon\|_{C^{k+1}_{\mathrm{loc}}} \le C\,. \tag{67}$$

We rewrite (65) as

$$-\varepsilon^2\Delta X_\varepsilon + 2X_\varepsilon = 3\varepsilon^2 X_\varepsilon^2 - \varepsilon^4 X_\varepsilon^3 + \varrho_\varepsilon|\nabla\varphi_\varepsilon|^2 \equiv R_\varepsilon\,. \tag{68}$$

Note that

$$\|R_\varepsilon\|_{C^{k+1}_{\mathrm{loc}}} \le C$$

[this follows from (67), (61) and (64)].
Differentiating (68) at order $(k+1)$ we obtain

$$-\varepsilon^2\Delta(D^{k+1}X_\varepsilon) + 2D^{k+1}X_\varepsilon = D^{k+1}R_\varepsilon \quad \text{on } \Omega'\,. \tag{70}$$

On the other hand

$$\|D^{k+1}X_\varepsilon\|_{L^\infty(\partial\Omega')} \le \frac{C}{\varepsilon} \quad \text{by (67)}\,.$$

Applying Lemma 2 we find

$$\|D^{k+1}X_\varepsilon\|_{L^\infty(\Omega'')} \le C + \frac{C}{\varepsilon}e^{-d/4\varepsilon}\,,$$

where $d = \mathrm{dist}\,(\Omega'', \partial\Omega')$. Consequently

$$\|X_\varepsilon\|_{C^{k+1}_{\mathrm{loc}}} \le C$$

i.e., (57) holds with $(k+1)$ instead of k. This completes the proof of Step B.6.

Step B.7: Proof of (13) *and* (14).

Recall that $\Delta\varphi_0 = 0$. From (62) we deduce that

$$-\Delta(\varphi_\varepsilon - \varphi_0) = 2\frac{\nabla\varrho_\varepsilon}{\varrho_\varepsilon}\nabla\varphi_\varepsilon \quad \text{on } \Omega .$$

Hence, by (55), (56) and (57) we have

$$\|\varphi_\varepsilon - \varphi_0\|_{C^{k+1}_{\text{loc}}} \le C\varepsilon^2 . \tag{72}$$

Therefore

$$u_\varepsilon - u_0 = \varrho_\varepsilon e^{i\varphi_\varepsilon} - u_0 = (\varrho_\varepsilon - 1)e^{i\varphi_\varepsilon} + e^{i\varphi_\varepsilon} - e^{i\varphi_0}$$

satisfies

$$\|u_\varepsilon - u_0\|_{C^k_{\text{loc}}} \le C\varepsilon^2$$

[by (57) and (72)]. This completes the proof of (13).

We now turn to the proof of (14). Returning to (68) we write

$$-\varepsilon^2\Delta\Big(X_\varepsilon - \frac{1}{2}|\nabla u_0|^2\Big) + 2\Big(X_\varepsilon - \frac{1}{2}|\nabla u_0|^2\Big) = |\nabla\varphi_\varepsilon|^2 - |\nabla\varphi_0|^2 + S_\varepsilon , \tag{73}$$

where $S_\varepsilon = 3\varepsilon^2 X_\varepsilon^2 - \varepsilon^4 X_\varepsilon^3 + (\varrho_\varepsilon - 1)|\nabla\varphi_\varepsilon|^2 + \frac{1}{2}\varepsilon^2\Delta(|\nabla u_0|^2)$ (note that $|\nabla u_0| = |\nabla\varphi_0|$ since $u_0 = e^{i\varphi_0}$).
Clearly

$$\|S_\varepsilon\|_{C^k_{\text{loc}}} \le C\varepsilon^2$$

and

$$\||\nabla\varphi_\varepsilon|^2 - |\nabla\varphi_0|^2\|_{C^k_{\text{loc}}} \le C\varepsilon^2$$

[by (56), (57) and (72)]. Applying once more Lemma 2 to $\omega = D^k(X_\varepsilon - \frac{1}{2}|\nabla u_0|^2)$ we are led to

$$\Big\|X_\varepsilon - \frac{1}{2}|\nabla u_0|^2\Big\|_{C^k_{\text{loc}}} \le C\varepsilon^2\Big(1 + \frac{1}{\varepsilon^2}e^{-d/4\varepsilon}\Big) .$$

This completes the proof of (14).

3 The case of a boundary condition depending on ε

We now return to the minimization problem (2) but we allow g to depend on ε. More precisely, we have a family of boundary conditions $g_\varepsilon : \partial\Omega \to \mathbb{C}$ and we consider the problem

$$\underset{H^1_{g_\varepsilon}(\Omega;\mathbb{C})}{\text{Min}} \left\{ \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_\Omega (|u|^2 - 1)^2 \right\} . \tag{74}$$

In what follows we denote by u_ε any minimizer of (74).

We make the following assumptions

$$\|g_\varepsilon\|_{L^\infty(\partial\Omega)} \le 1 , \tag{75}$$

$$\|g_\varepsilon\|_{H^1(\partial\Omega)} \le C \tag{76}$$

and

$$\int_{\partial\Omega} (|g_\varepsilon| - 1)^2 \leq C\varepsilon^2 \tag{77}$$

(note that we do *not* assume that g_ε takes its values into S^1).

We also assume that

$$g_\varepsilon \to g \quad \text{uniformly on } \partial\Omega \tag{78}$$

so that, by (77), $|g| = 1$ and hence $\deg(g, \partial\Omega)$ is well defined. We assume that

$$\deg(g, \partial\Omega) = 0\,. \tag{79}$$

As in Section 1 we write

$$g = e^{i\varphi_0} \quad \text{on} \quad \partial\Omega$$

for some harmonic function φ_0.

Set

$$u_0 = e^{i\varphi_0} \quad \text{in} \quad \Omega\,.$$

Our main result is the following

Theorem 2. *Under the assumptions* (75)–(79) *we have,*

$$u_\varepsilon \to u_0 \quad \text{strongly in} \quad H^1(\Omega)\,, \tag{80}$$

$$u_\varepsilon \to u_0 \quad \text{uniformly on} \quad \overline{\Omega}\,, \tag{81}$$

$$u_\varepsilon \to u_0 \quad \text{in} \quad C^k_{\text{loc}}(\Omega) \quad \forall k \tag{82}$$

and

$$\frac{1 - |u_\varepsilon|^2}{\varepsilon^2} \to |\nabla u_0|^2 \quad \text{in} \quad C^k_{\text{loc}}(\Omega) \quad \forall k\,. \tag{83}$$

We split the proof into 3 steps.

Step 1. We have

$$u_\varepsilon \to u_0 \quad \text{strongly in} \quad H^1(\Omega) \tag{80}$$

and

$$\frac{1}{\varepsilon^2}\int_\Omega (|u_\varepsilon|^2 - 1)^2 \to 0\,. \tag{84}$$

Proof. We use a special comparison function of the form

$$v_\varepsilon = \eta_\varepsilon e^{i\psi_\varepsilon}\,, \tag{85}$$

where η_ε is the solution of

$$\begin{cases} -\varepsilon^2 \Delta\eta_\varepsilon + \eta_\varepsilon = 1 & \text{on} \quad \Omega\,, \\ \eta_\varepsilon = |g_\varepsilon| & \text{on} \quad \partial\Omega\,, \end{cases} \tag{86}$$

and ψ_ε is the solution of

$$\begin{cases} \Delta\psi_\varepsilon = 0 & \text{on} \quad \Omega\,, \\ \psi_\varepsilon = \varphi_\varepsilon & \text{on} \quad \partial\Omega\,, \end{cases} \tag{87}$$

where $\varphi_\varepsilon : \partial\Omega \to \mathbb{R}$ is defined by

$$e^{i\varphi_\varepsilon} = \frac{g_\varepsilon}{|g_\varepsilon|}$$

(this is always possible since $\deg(g_\varepsilon, \partial\Omega) = 0$ for ε sufficiently small). In view of (78) we may choose φ_ε such that $\varphi_\varepsilon \to \varphi_0$ uniformly on $\partial\Omega$. We claim that

$$\int_\Omega |\nabla \eta_\varepsilon|^2 \le C\varepsilon \tag{88}$$

and

$$\frac{1}{\varepsilon^2}\int_\Omega (\eta_\varepsilon - 1)^2 \le C\varepsilon\,. \tag{89}$$

Proof of (88) *and* (89). Note that η_ε is a minimizer for

$$\int_\Omega |\nabla\eta|^2 + \frac{1}{\varepsilon^2}\int_\Omega (\eta-1)^2 \quad \text{on} \quad H^1_{|g_\varepsilon|}(\Omega;\mathbb{R})\,.$$

We use as comparison function

$$\overline{\eta}_\varepsilon(x_1,x_2) = (|g_\varepsilon(x_1)| - 1)\gamma(x_2) + 1$$

written in local coordinates assuming $\Omega = \{(x_1,x_2);\ x_2 > 0\}$ near a boundary point, and γ is a smooth function with small support near 0 with $\gamma(0) = 1$. Note that

$$\int_\Omega |\nabla\overline{\eta}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\int_\Omega (\overline{\eta}_\varepsilon - 1)^2 \le C \tag{90}$$

(here we use (76) and (77)). Hence

$$\int_\Omega |\nabla\eta_\varepsilon|^2 + \frac{1}{\varepsilon^2}\int_\Omega (\eta_\varepsilon - 1)^2 \le C\,. \tag{91}$$

Next, we multiply (86) – as in the proof of Proposition 3 – by $V\cdot\nabla(\eta_\varepsilon - 1)$. This yields

$$\int_{\partial\Omega} \left|\frac{\partial\eta_\varepsilon}{\partial n}\right|^2 \le C \tag{92}$$

[the computation relies on (91), (76) and (77)]. Finally, we multiply (86) by $(\eta_\varepsilon - 1)$ and we obtain

$$\begin{aligned}\varepsilon^2\int_\Omega |\nabla\eta_\varepsilon|^2 + \int_\Omega (\eta_\varepsilon - 1)^2 &\le \varepsilon^2\int_{\partial\Omega}\left|\frac{\partial\eta_\varepsilon}{\partial n}\right||\eta_\varepsilon - 1| \\ &\le \varepsilon^2\left\|\frac{\partial\eta_\varepsilon}{\partial n}\right\|_{L^2(\partial\Omega)} \||g_\varepsilon| - 1\|_{L^2(\partial\Omega)} \\ &\le C\varepsilon^3\,.\end{aligned}$$

Thus we have proved (88) and (89).

We claim that

$$\frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\int_\Omega (|u_\varepsilon|^2 - 1)^2 \le \frac{1}{2}\int_\Omega |\nabla\psi_\varepsilon|^2 + C\varepsilon\,. \tag{93}$$

Indeed, we use the fact that u_ε is a minimizer for (74) and the comparison function v_ε defined in (85). Note that

$$\frac{1}{\varepsilon^2}\int_\Omega (|v_\varepsilon|^2-1)^2 = \frac{1}{\varepsilon^2}\int_\Omega (\eta_\varepsilon^2-1)^2 \le C\varepsilon$$

by (89). On the other hand

$$\int_\Omega |\nabla v_\varepsilon|^2 = \int_\Omega |\nabla \eta_\varepsilon|^2 + \eta_\varepsilon^2|\nabla\psi_\varepsilon|^2 \le C\varepsilon + \int |\nabla \psi_\varepsilon|^2$$

since $\eta_\varepsilon \le 1$. This proves (93).

Finally we observe that $\psi_\varepsilon \to \varphi_0$ strongly in $H^1(\Omega)$. Indeed φ_ε is bounded in $H^1(\partial\Omega)$ and $\varphi_\varepsilon \to \varphi_0$ uniformly on $\partial\Omega$ imply that $\varphi_\varepsilon \to \varphi_0$ strongly in $H^{1/2}(\partial\Omega)$. By (87) we deduce that $\psi_\varepsilon \to \varphi_0$ strongly in $H^1(\Omega)$.

From (93) we know that (u_ε) is bounded in H^1 and thus

$$u_{\varepsilon_n} \rightharpoonup u \quad \text{weakly in} \quad H^1 .$$

By (93) and lower semi-continuity we see that

$$\int_\Omega |\nabla u|^2 \le \int_\Omega |\nabla \varphi_0|^2 = \int_\Omega |\nabla u_0|^2 . \tag{94}$$

On the other hand

$$\int_\Omega (|u_\varepsilon|^2-1)^2 \le C\varepsilon^2$$

and therefore $|u| = 1$ a.e. Hence $u \in H^1_g(\Omega; S^1)$ and in view of (94) u is a minimizer for (5), i.e., $u = u_0$. The strong convergence $u_\varepsilon \to u_0$ in H^1 follows from the fact that

$$\overline{\lim_{\varepsilon\to 0}} \int |\nabla u_\varepsilon|^2 \le \int |\nabla u_0|^2$$

and the uniqueness of u_0. Going back to (93) and using the strong convergence of $u_\varepsilon \to u_0$ in H^1 we obtain (84). This completes the proof of Step 1.

Step 2: Proof of (81).

Steps A.1 and A.2 in Sect. 2 hold without any modification and therefore $|u_\varepsilon| \to 1$ uniformly on every compact subset of Ω.

We shall now prove that

$$|u_\varepsilon| \to 1 \quad \text{uniformly on} \quad \overline{\Omega} . \tag{95}$$

We argue by contradiction, i.e., we assume that there are sequences $\varepsilon_n \to 0$, $a_n \in \Omega$ such that

$$|u_{\varepsilon_n}(a_n)| \le 1-\delta \tag{96}$$

for some $\delta > 0$. We may also assume that $a_n \to a$ and $a \in \partial\Omega$. We set $u_n = u_{\varepsilon_n}$ and $d_n = \operatorname{dist}(a_n, \partial\Omega)$.

We claim that

$$\frac{d_n}{\varepsilon_n} \to 0 . \tag{97}$$

Proof of (97). Let $r_n \le \frac{1}{2}d_n$ be a sequence of positive numbers to be chosen later. By Lemma A.1 we know that

$$|\nabla u_n(x)|^2 \le C\left(\frac{1}{\varepsilon_n^2} + \frac{1}{\text{dist}^2(x,\partial\Omega)}\right) \quad \forall x \in \Omega\,,$$

where C is some universal constant. In particular we have

$$|\nabla u_n(x)| \le C\left(\frac{1}{\varepsilon_n} + \frac{1}{d_n}\right) \quad \forall x \in B(a_n, r_n)\,.$$

Therefore

$$|u_n(x) - u_n(a_n)| \le Cr_n\left(\frac{1}{\varepsilon_n} + \frac{1}{d_n}\right) \quad \forall x \in B(a_n, r_n)$$

and consequently

$$|u_n(x)| \le |u_n(a_n)| + Cr_n\left(\frac{1}{\varepsilon_n} + \frac{1}{d_n}\right) \quad \forall x \in B(a_n, r_n)\,.$$

Thus

$$1 - |u_n(x)| \ge \delta - Cr_n\left(\frac{1}{\varepsilon_n} + \frac{1}{d_n}\right) \quad \forall x \in B(a_n, r_n)\,.$$

We shall choose r_n in such a way that

$$\delta - Cr_n\left(\frac{1}{\varepsilon_n} + \frac{1}{d_n}\right) \ge \frac{\delta}{2}\,.$$

Hence

$$(1 - |u_n|^2)^2 \ge \frac{\delta^2}{4} \quad \text{on} \quad B(a_n, r_n)\,.$$

It follows that

$$\int_\Omega (1 - |u_n|^2)^2 \ge \frac{\delta^2}{4}\pi r_n^2\,.$$

On the other hand we know by (84) that

$$\int_\Omega (1 - |u_n|^2)^2 = \varepsilon_n^2 o(1)$$

and we deduce that

$$\frac{r_n}{\varepsilon_n} \to 0\,. \tag{98}$$

We now choose r_n so that all the requirements are satisfied, i.e.,

$$\frac{r_n}{d_n} \le \frac{1}{2}\,, \quad \frac{r_n}{\varepsilon_n} \le \frac{\delta}{4C}\,, \quad \frac{r_n}{d_n} \le \frac{\delta}{4C}\,.$$

For example we take

$$r_n = \min\left\{\frac{d_n}{2}\,, \frac{d_n\delta}{4C}\,, \frac{\varepsilon_n\delta}{4C}\right\}.$$

Using (98) we see that (97) holds.

Proof of (95) *completed.* We use a blow-up argument.
Set

$$v_n(y) = u_n(d_n y + a_n), \quad \text{for} \quad y \in G_n = \frac{1}{d_n}(\Omega - a_n).$$

Modulo a rotation, we may always assume that

$$G_n \to G = (-1, +\infty) \times \mathbb{R}.$$

Clearly v_n satisfies

$$-\Delta v_n = \left(\frac{d_n}{\varepsilon_n}\right)^2 v_n(1 - |v_n|^2) \quad \text{on} \quad G_n, \tag{99}$$

and

$$\int_{G_n} |\nabla v_n|^2 = \int_{\Omega} |\nabla u_n|^2 \leq C.$$

Passing to a subsequence, we may also assume that $v_n \to v$ uniformly on compact subsets of G where v satisfies

$$\Delta v = 0 \quad \text{in } G \quad [\text{by (97) and (99)}]$$

and

$$\int_G |\nabla v|^2 < +\infty.$$

Finally, we also see easily – since $g_\varepsilon \to g$ uniformly on $\partial\Omega$ – that

$$v = g(a) \quad \text{on} \quad \partial G.$$

It follows that $v \equiv g(a)$ on G. On the other hand, $v_n(0) = u_n(a_n)$, and thus $|v_n(0)| \leq 1-\delta$. Hence $|v(0)| \leq 1-\delta$; this contradicts the fact that $|v(0)| = |g(a)| = 1$. The proof of (95) is complete.

Proof of (81). We write, as in Sect. 2,

$$u_\varepsilon = \varrho_\varepsilon e^{i\varphi_\varepsilon}.$$

We have just proved that $\varrho_\varepsilon \to 1$ uniformly on $\overline{\Omega}$. Next we write, using (51),

$$-\operatorname{div}(\varrho_\varepsilon^2 \nabla(\varphi_\varepsilon - \varphi_0)) = \operatorname{div}((\varrho_\varepsilon^2 - 1)\nabla\varphi_0). \tag{100}$$

The equation is uniformly elliptic since $\varrho_\varepsilon \to 1$ uniformly on $\overline{\Omega}$. It follows from elliptic estimates (see [10] or [3]) that

$$\|\varphi_\varepsilon - \varphi_0\|_{L^\infty(\Omega)} \leq C(\|\varphi_\varepsilon - \varphi_0\|_{L^\infty(\partial\Omega)} + \|(\varrho_\varepsilon^2 - 1)\nabla\varphi_0\|_{L^p(\Omega)})$$

with any $p > 2$. Note that $\varphi_0 \in H^{3/2}(\Omega)$ (since $g \in H^1(\partial\Omega)$), and thus $\nabla\varphi_0 \in H^{1/2}(\Omega) \subset L^4(\Omega)$. We conclude that $\varphi_\varepsilon \to \varphi_0$ uniformly on $\overline{\Omega}$ and this completes the proof of (81).

Step 3: Proof of (82) *and* (83)

We follow the same argument as in Section 2 The proofs in Steps A.3, A.4, and A.5 are unchanged. They yield:

$$u_\varepsilon \quad \text{is bounded in} \quad H^2_{\text{loc}} \tag{101}$$

$$\nabla u_\varepsilon \quad \text{is bounded in} \quad L^\infty_{\text{loc}} \tag{102}$$

$$\frac{1}{\varepsilon^2}(1 - |u_\varepsilon|) \quad \text{is bounded in} \quad L^\infty_{\text{loc}} \tag{103}$$

$$\Delta u_\varepsilon \quad \text{is bounded in} \quad L^\infty_{\text{loc}} \tag{104}$$

Next we prove that, for every integer k,

$$\|\nabla \varphi_\varepsilon\|_{C^k_{\text{loc}}} \leq C \tag{105}$$

$$\left\| \frac{1 - \varrho_\varepsilon}{\varepsilon^2} \right\|_{C^k_{\text{loc}} k} \leq C \,. \tag{106}$$

For $k = 0$, these estimates have already been established [see (102), (103)]. The induction argument presented in Step B.6 can be repeated without any modification. From (105) and (106) we deduce that

$$\varphi_\varepsilon \to \varphi_0 \quad \text{in} \quad C^k_{\text{loc}} \tag{107}$$

and

$$\varrho_\varepsilon \to 1 \quad \text{in} \quad C^k_{\text{loc}} \,. \tag{108}$$

This implies that $u_\varepsilon = \varrho_\varepsilon e^{i\varphi_\varepsilon}$ converges to u_0 in C^k_{loc}, i.e., we have proved (82).

The proof of (83) follows the same argument as the proof of (14) in Step B.7. We use again (73). We know that

$$|\nabla \varphi_\varepsilon|^2 - |\nabla \varphi_0|^2 \to 0 \quad \text{in} \quad C^k_{\text{loc}}$$

and

$$S_\varepsilon \to 0 \quad \text{in} \quad C^k_{\text{loc}}$$

by (107), (106) and (108). On the other hand X_ε is bounded in C^k_{loc} by (106). Applying once more Lemmma 2 to $\omega = D^k(X_\varepsilon - \frac{1}{2}|\nabla u_0|^2)$ we have that

$$\left\| X_\varepsilon - \frac{1}{2}|\nabla u_0|^2 \right\|_{C^k_{\text{loc}}} \to 0 \,.$$

This completes the proof of (83).

Appendix

Some interpolation – type inequalities

The following results are interpolation estimates in the spirit of the Gagliardo-Nirenberg inequalities (see e.g. [8]); they are presumably known to the experts but we present the proofs for the convenience of the reader.

Lemma A.1. *Assume u satisfies*

$$-\Delta u = f \quad \text{on} \quad \Omega \subset \mathbb{R}^N .$$

Then

$$|\nabla u(x)|^2 \le C(\|f\|_{L^\infty(\Omega)}\|u\|_{L^\infty(\Omega)} + \frac{1}{\text{dist}^2(x, \partial\Omega)} \|u\|^2_{L^\infty(\Omega)}) \quad \forall x \in \Omega , \tag{A.1}$$

where C is some constant depending only on N.

Proof. Assume for simplicity that $0 \in \Omega$ and set $d = \text{dist}(0, \partial\Omega)$. We shall prove that (A.1) holds at $x = 0$. Let $0 < \lambda \le d$ be a constant to be determined later. The function

$$v(y) = u(\lambda y)$$

is defined on the ball $B(0, 1) = B_1$ since $\lambda \le d$ and it satisfies

$$-\Delta v(y) = \lambda^2 f(\lambda y) \quad \text{in} \quad B_1 . \tag{A.2}$$

It follows from standard elliptic estimates in B_1 that

$$|\nabla v(0)| \le C(\lambda^2 \|f(\lambda y)\|_{L^\infty(B_1)} + \|v\|_{L^\infty(B_1)}) ,$$

where C depends only on N. In particular we have

$$\lambda |\nabla u(0)| \le C(\lambda^2 \|f\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)}) . \tag{A.3}$$

We now distinguish two cases:

Case 1:

$$\left(\frac{\|u\|_{L^\infty}}{\|f\|_{L^\infty}}\right)^{1/2} \le d .$$

In this case we apply (A.3) with

$$\lambda = \left(\frac{\|u\|_{L^\infty}}{\|f\|_{L^\infty}}\right)^{1/2} .$$

We deduce that

$$|\nabla u(0)| \le 2C \|f\|^{1/2}_{L^\infty} \|u\|^{1/2}_{L^\infty}$$

and thus (A.1) holds at $x = 0$.

Case 2:

$$\left(\frac{\|u\|_{L^\infty}}{\|f\|_{L^\infty}}\right)^{1/2} > d .$$

We now apply (A.3) with $\lambda = d$ and we find

$$\begin{aligned} |\nabla u(0)| &\le C\left(d\|f\|_{L^\infty} + \frac{1}{d}\|u\|_{L^\infty}\right) \\ &\le C\left(\|f\|^{1/2}_{L^\infty}\|u\|^{1/2}_{L^\infty} + \frac{1}{d}\|u\|_{L^\infty}\right) . \end{aligned}$$

This yields (A.1) at $x = 0$.

Lemma A.2. *Assume u satisfies*

$$\begin{cases} -\Delta u = f & on \quad \Omega \subset \mathbb{R}^N \\ u = 0 & on \quad \partial\Omega , \end{cases} \tag{A.4}$$

where Ω is a smooth bounded domain. Then

$$\|\nabla u\|^2_{L^\infty(\Omega)} \le C\|f\|_{L^\infty(\Omega)}\|u\|_{L^\infty(\Omega)} , \tag{A.5}$$

where C is a constant depending only on Ω.

Proof. From the elliptic theory we know that

$$\|u\|_{L^\infty(\Omega)} \le C\|f\|_{L^\infty(\Omega)} . \tag{A.6}$$

On the other hand if K is a compact subset of Ω we may use (A.1) together with (A.6) to conclude that

$$\|\nabla u\|^2_{L^\infty(K)} \le C_K\|f\|_{L^\infty(\Omega)}\|u\|_{L^\infty(\Omega)} . \tag{A.7}$$

Therefore we have only to estimate ∇u near the boundary. After a local change of coordinates near a boundary point x_0 equation (A.4) becomes

$$\begin{cases} -\sum_{i,j} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial}{\partial x_i}\right) = f & \text{on} \quad B_R^+ = \{x \in B_R;\ x_N > 0\} , \\ u = 0 & \text{on} \quad B_R \cap \{x_N = 0\} , \end{cases} \tag{A.8}$$

where $a_{ij}(x)$ are smooth and uniformly elliptic coefficients (they depend only on $\partial\Omega$) and R may be *fixed* independent of x_0.
Set

$$v(y) = u(\lambda y + \xi) \quad \text{in} \quad B_1^+ ,$$

where $0 < \lambda \le R/2$ will be determined later and ξ is an arbitrary point on $B_{R/2} \cap \{y_N = 0\}$. The function v satisfies

$$\begin{cases} -\sum_{i,j} \frac{\partial}{\partial y_j}\left(a_{ij}(\lambda y + \xi)\frac{\partial v}{\partial y_i}(y)\right) = \lambda^2 f(\lambda y + \xi) & \text{on} \quad B_1^+ , \\ v = 0 & \text{on} \quad B_1 \cap \{y_N = 0\} , \end{cases}$$

Standard elliptic estimates in B_1^+ imply that

$$\|\nabla v\|_{L^\infty(B^+_{1/2})} \le C(\lambda^2\|f(\lambda y + \xi)\|_{L^\infty(B_1^+)} + \|v\|_{L^\infty(B_1^+)}) , \tag{A.9}$$

where C depends on the ellipticity constant of $a_{ij}(\lambda y+\xi)$ and on $\|a_{ij}(\lambda y+\xi)\|_{C^1(B_1^+)}$. Since all these quantities are controlled independently of λ and ξ when $\lambda \le R/2$ and $|\xi| \le R/2$ we deduce that

$$\lambda\|\nabla u\|_{L^\infty(\xi + B^+_{\lambda/2})} \le C(\lambda^2\|f\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)}) . \tag{A.10}$$

We distinguish two cases:

Case 1:

$$\left(\frac{\|u\|_{L^\infty}}{\|f\|_{L^\infty}}\right)^{1/2} \leq \frac{R}{2}.$$

In this case we apply (A.10) with $\lambda = \left(\frac{\|u\|_{L^\infty}}{\|f\|_{L^\infty}}\right)^{1/2}$.

This yields

$$\|\nabla u\|_{L^\infty(\xi+B^+_{\lambda/2})} \leq C\|f\|^{1/2}_{L^\infty(\Omega)}\|u\|^{1/2}_{L^\infty(\Omega)}.$$

Since ξ is arbitrary with $|\xi| \leq R/2$ we deduce that

$$|\nabla u(x)| \leq C\|f\|^{1/2}_{L^\infty(\Omega)}\|u\|^{1/2}_{L^\infty(\Omega)} \quad \forall x = (x', x_N) \quad \text{with } |x'| \leq \frac{R}{2} \text{ and } 0 \leq x_N \leq \frac{\lambda}{2}.$$

Going back to u on Ω we have proved that

$$|\nabla u(x)| \leq C\|f\|^{1/2}_{L^\infty(\Omega)}\|u\|^{1/2}_{L^\infty(\Omega)} \quad \forall x \in \Omega \quad \text{with} \quad \text{dist}(x, \partial\Omega) \leq \frac{\lambda}{K},$$

where K is some large constant depending only on Ω. On the other hand if $\text{dist}(x, \partial\Omega) > \lambda/K$ we may apply (A.1) and conclude that

$$\begin{aligned}|\nabla u(x)|^2 &\leq C\Big(\|f\|_{L^\infty(\Omega)}\|u\|_{L^\infty(\Omega)} + \frac{K^2}{\lambda^2}\|u\|^2_{L^\infty(\Omega)}\Big)\\ &= C(1+K^2)\|f\|_{L^\infty(\Omega)}\|u\|_{L^\infty(\Omega)}.\end{aligned}$$

In both situations we see that (A.5) holds.

Case 2:

$$\left(\frac{\|u\|_{L^\infty}}{\|f\|_{L^\infty}}\right)^{1/2} \geq \frac{R}{2}.$$

In this case we apply (A.10) with $\lambda = R/2$ and $\xi = 0$. This yields

$$\begin{aligned}\|\nabla u\|_{L^\infty(B^+_{R/2})} &\leq C\Big(R\|f\|_{L^\infty(\Omega)} + \frac{1}{R}\|u\|_{L^\infty(\Omega)}\Big)\\ &\leq C\Big(2\|f\|^{1/2}_{L^\infty(\Omega)}\|u\|^{1/2}_{L^\infty(\Omega)} + \frac{1}{R}\|u\|_{L^\infty(\Omega)}\Big)\\ &\leq C\|f\|^{1/2}_{L^\infty(\Omega)}\|u\|^{1/2}_{L^\infty(\Omega)}.\end{aligned}$$

Going back to u on Ω we see that

$$\|\nabla u\|_{L^\infty(U)} \leq C\|f\|^{1/2}_{L^\infty(\Omega)}\|u\|^{1/2}_{L^\infty(\Omega)}$$

for some fixed neighbourhood U of $\partial\Omega$. This completes the proof since we already have the interior estimate (A.7).

Acknowledgements. Part of this work was done while the first author (F.B.) and the third author (F.H.) were visiting Rutgers University. They thank the Mathematics Department for its support and hospitality; their work was also partially supported by a Grant of the French Ministry of Research and Technology (MRT Grant 90S0315).

References

1. Bethuel, F., Brezis, H., Hélein, F.: Limite singulière pour la minimisation de fonctionnelles du type Ginzburg-Landau. C.R. Acad. Sci. Paris **314**, 891–895 (1992)
2. Bethuel, F., Brezis, H., Hélein, F.: Ginzburg-Landau vortices (to appear)
3. Gilbarg, D., Trudinger, N.: Elliptic partial differential equations of second order. Berlin, Heidelberg, New York: Springer 1982
4. Hélein, F.: Régularité des applications faiblement harmoniques entre une surface et une sphère. C.R. Acad. Sci. Paris **311**, 519–524 (1990)
5. Kosterlitz, J.M., Thouless, D.J.: Two dimensional physics. In: Brewer, D.F. (ed.) Progress in low temperature physics, vol. VIIB. Amsterdam: North-Holland 1978
6. Morrey, C.B.: The problem of Plateau on a Riemannian manifold. Ann. Math. **49**, 807–851 (1948); Morrey, C.B.: Multiple integrals in the calculus of variations. (Grundlehren math. Wiss., vol. 130) Berlin, Heidelberg, New York: Springer 1966
7. Nelson, D.R.: Defect mediated phase transitions. In: Domb, C., Lebowitz, J.L. (eds) Phase transitions and critical phenomena, vol. 7. New York: Academic Press 1983
8. Nirenberg, L.: On elliptic partial differential equations. Ann. Sc. Norm. Sup. Pisa **13**, 116–162 (1958)
9. Saint-James, D., Sarma, G., Thoma, E.J.: Type II superconductivity. New York: Pergamon Press 1969
10. Stampacchia, G.: Equations elliptiques du second ordre avec coéfficients discontinus. Presses de l'Université de Montréal 1966