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Integrability for the Jacobian of Orientation Preserving Mappings

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Assume F is a map from Ω into $\mathbb{R}^n$ where Ω is a bounded domain in $\mathbb{R}^n$ such that $|Df| \in L^n(\log L)^{-s}$ with $0 \leqslant s \leqslant 1$, i.e., $\int_\Omega |Df|^n [\log(1+|Df|)]^{-s} < \infty$, then $J \in L(\log L)^{1-s}(K)$ for any compact subset $K \subset \Omega$, where $J = \det(Df)$. When $s=0$ we recover a well-known result of Müller (*J. Reine Angew. Math.* **412** (1990), 20–34), while the case $s=1$ was obtained by Iwaniec and Sbordone (*Arch. Rational Mech. Anal.*, to appear). © 1993 Academic Press, Inc.

1. INTRODUCTION

Throughout this paper we assume that Ω is a bounded open set in $\mathbf{R}^n$ and that $f: \Omega \to \mathbf{R}^n$, $n \geqslant 2$, is a mapping with nonnegative Jacobian $J = J(x, f) = \det Df$.

In [M] S. Müller proved that, if $|Df| \in L^n(\Omega)$, then $J \in L \log L(K)$ for any compact subset $K \subset \Omega$ (see also [CLMS]).

In [IS] T. Iwaniec and C. Sbordone proved that

(A) if $|Df| \in L^n(\log L)^{-1}(\Omega)$, then $J \in L^1_{\mathrm{loc}}(\Omega)$

(B) if $|Df| \in L^{n,\infty}(\Omega)$, then $J \in L^1_{\mathrm{loc}}(\Omega)$.

In this paper we interpolate between Müller's result and (A); namely, we prove:

THEOREM 1. *If $|Df| \in L^n(\log L)^{-s}(\Omega)$, $0 \leqslant s \leqslant 1$, then $J \in L(\log L)^{1-s}(K)$ for any $K \subset \Omega$.*

This theorem is proved in Section 3.

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It would be also natural to interpolate between Müller's result and (B), and to try to prove that if $|Df|$ belongs to the Lorentz space $L^{n,q}(\Omega)$, $n<q<\infty$, then $J \in L(\log L)^{n/q}(K)$ for any $K \subset \Omega$. Unfortunately this is not true. However, the maximal function MJ of J satisfies $MJ \in L^{1,q/n}_{\mathrm{loc}}$, which is a weaker statement than saying $J \in L(\log L)^{n/q}(K)$.

This is discussed in Section 4.

2. Some Preliminary Results

Denote by g^* the decreasing rearrangement on the interval $(0, +\infty)$ of a measurable function g defined on some open set $\Omega \subset \mathbf{R}^n$.

The Lorentz space $L^{p,q}(\Omega)$ $(1<p<\infty,\ 1 \leqslant q \leqslant \infty)$ consists of all measurable functions g on Ω for which

$$\| g \|_{p,q} = \begin{cases} \left\{ \int_0^\infty [t^{1/p} g^*(t)]^q \dfrac{dt}{t} \right\}^{1/q} & q<\infty \\ \sup\limits_{t>0} t^{1/p} g^*(t) & q=\infty \end{cases}$$

is finite. The spaces $L^{p,q}$ increase with q; in particular,

$$L^{p,1} \subset L^{p,p} = L^p \subset L^{p,\infty}$$

(see, e.g., [Z]).

The Zygmund space $L^p(\log L)^\alpha(\Omega)$ $(1 \leqslant p<\infty,\ -\infty<\alpha<\infty)$ consists of all measurable functions g on Ω for which

$$\int_\Omega |g|^p \log^\alpha \left(e + \frac{|g|}{|g|_\Omega} \right) dx < \infty,$$

where $|g|_\Omega = (1/|\Omega|) \int_\Omega |g|\, dx$.

It can be shown (see [BS, p. 252]) that $g \in L^p(\log L)^\alpha(\Omega)$ iff

$$\int_0^\infty (1+|\log t|)^\alpha g^*(t)^p\, dt < \infty.$$

We denote by M_Ω the local Hardy–Littlewood maximal function, defined for $g \in L^1_{\mathrm{loc}}(\Omega)$ by

$$M_\Omega g(x) = \sup_{x \in Q \subset \Omega} \fint_Q |g|\, dy.$$

The following result is well known (see e.g. [BR]).

LEMMA 1. *For $p>1$, $0\leqslant s\leqslant 1$, M_Ω maps boundedly $L^p(\log L)^{-s}$ into itself, namely*

$$\| M_\Omega g \|_{L^p(\log L)^{-s}(\Omega)} \leqslant c \| g \|_{L^p(\log L)^{-s}(\Omega)}$$

with $c=c(n,p,s)$.

When the exponent in the Zygmund space is $p=1$ the behaviour of the maximal function is different; namely the following result holds (see [S] for $\alpha=1$).

LEMMA 2. *For $0<\alpha\leqslant 1$, the following statements are equivalent,*

(a) $g\in L(\log L)^\alpha$

(b) $M_\Omega g\in L(\log L)^{\alpha-1}$.

In particular, there exists $c=c(n,\alpha)$ such that

$$\int_\Omega |g| \log^\alpha\left(e+\frac{|g|}{|g|_\Omega}\right) dx \leqslant c\int_\Omega M_\Omega g \log^{\alpha-1}\left(e+\frac{M_\Omega g}{|g|_\Omega}\right) dx. \tag{1}$$

The equivalence between (a) and (b) is essentially due to Bennett [B, Theorem 4.1]. Here we prove inequality (1).

Proof. Suppose $|g|_\Omega=1$. Then it is well known that $\forall t>1$,

$$\frac{1}{t}\int_{|g|>t} |g| \leqslant 2^n |\{M_\Omega g>t\}|.$$

We have by integration by parts

$$\begin{aligned}
&\int_{|g|>1} |g| \log^\alpha(e+|g|) \\
&\quad = -\int_1^\infty \log^\alpha(e+t)\left(\int_{|g|>t} |g|\, dx\right)' dt \\
&\quad = -\left[\log^\alpha(e+t)\int_{|g|>t} |g|\, dx\right]_{t=1}^{t=\infty} \\
&\qquad + \alpha\int_1^\infty \frac{\log^{\alpha-1}(e+t)}{e+t}\int_{|g|>t} |g|\, dx\, dt \\
&\quad \leqslant \log^\alpha(1+e)\int_{|g|>1} |g|\, dx + 2^n\alpha\int_1^\infty \log^{\alpha-1}(e+t)\, |\{M_\Omega g>t\}|\, dt.
\end{aligned}$$

If we set $\phi(t) = t \log^{\alpha-1}(e+t)$ we have $\phi'(t) \geqslant \alpha \log^{\alpha-1}(e+t)\ \forall t > 0$. And so

$$\int_{|g|>1} |g| \log^{\alpha}(e+|g|)$$

$$\leqslant \log^{\alpha}(1+e) \int_{|g|>1} |g|\, dx + 2^n \int_1^{\infty} \phi'(t)\, |\{M_\Omega g > t\}|\, dt$$

$$= \log^{\alpha}(1+e) \int_{|g|>1} |g|\, dx + 2^n \int_{M_\Omega g > 1} \phi(M_\Omega g).$$

In fact

$$\int_{|h|>1} \phi(|h|)\, dx = \int_1^{\infty} \phi'(t)\, |\{|h| > t\}|\, dt.$$

So we have

$$\int_\Omega |g| \log^{\alpha}(e+|g|) \leqslant \log^{\alpha}(1+e) \int_\Omega |g|\, dx + 2^n \int_\Omega M_\Omega g \log^{\alpha-1}(e+M_\Omega g).$$

From this inequality one gets the result when $|g|_\Omega = 1$ since, $\forall \varepsilon > 0$ and $\forall t > 0$,

$$t \leqslant \varepsilon t \log^{\alpha}(e+t) + C(\varepsilon, \alpha)\, t \log^{\alpha-1}(e+t).$$

The general case then follows by homogeneity.

LEMMA 3. *If* $g \in L^n(\log L)^{-s}(\Omega)$, $0 < s \leqslant 1$, $g \geqslant 0$, *then*

$$\lim_{\varepsilon \to 0} \varepsilon^s \fint_\Omega g^{n-\varepsilon}(x)\, dx = 0.$$

Proof. We have

$$\varepsilon^s \fint_\Omega g^{n-\varepsilon}(x)\, dx \leqslant \varepsilon^s + \frac{1}{|\Omega|} \int_{g \geqslant 1} \frac{\varepsilon^s g^n}{e^{\varepsilon \log g}}\, dx.$$

The second integral converges to zero by dominated convergence. Clearly the integrand converges to zero pointwise and in addition

$$\frac{\varepsilon^s g^n}{e^{\varepsilon \log g}} \leqslant \frac{\varepsilon^s g^n}{1+\varepsilon \log g} \leqslant \frac{g^n}{(\log g)^s}.$$

3. Proof of Theorem 1

Let us recall the following inequality which holds for $f \in W^{1,n-\varepsilon}(\Omega; \mathbf{R}^n)$, $-\infty < \varepsilon < 1$, and $Q \subset Q_0/2$, Q_0 a cube contained in Ω.

$$\fint_Q |Df_1|^{-\varepsilon} J(x,f) \leqslant c(n)|\varepsilon| \fint_{2Q} |Df|^{n-\varepsilon} + c(n)\left[\fint_{2Q} |Df|^{n(n-\varepsilon)/(n+1)}\right]^{(n+1)/n}$$

(see [IS]). Since $|Df| \in L^n(\log L)^{-s}$, if $0 < s \leqslant 1$ we can pass to the limit as $\varepsilon \to 0$ and use Lemma 3 to obtain, for any cube $Q \subset Q_0/2$,

$$\fint_Q J \leqslant c(n)\left[\fint_{2Q} |Df|^{n^2/(n+1)}\right]^{(n+1)/n}. \tag{2}$$

If we denote by M the local maximal function associated to the cube $Q_0/2$, i.e., $M = M_{Q_0/2}$, and by $\mathcal{M}$ the local maximal function associated to the cube Q_0, i.e., $\mathcal{M} = M_{Q_0}$, by (1) we deduce for $x \in Q_0/2$,

$$MJ(x) \leqslant c(n)[\mathcal{M}(|Df|^{n^2/(n+1)})(x)]^{(n+1)/n}. \tag{3}$$

Since $|Df| \in L^n(\log L)^{-s}$, it is easy to check that $|Df|^{n^2/(n+1)}$ belongs to $L^{(n+1)/n}(\log L)^{-s}$ and then, by Lemma 1, $\mathcal{M}(|Df|^{n^2/(n+1)})$ belongs also to the same space. From this it follows that $[\mathcal{M}(|Df|^{n^2/(n+1)})]^{(n+1)/n}$ and also MJ, by (3), belongs to $L(\log L)^{-s}$ $(Q_0/2)$. Finally, from Lemma 2 we deduce that $J \in \mathrm{L}(\log L)^{1-s}(K)$ $\forall K \subset \Omega$.

Remark. Using inequalities (1) and (3) and the fact that

$$\int_{Q_0/2} \frac{|\mathcal{M}(|h|)|^{(n+1)/n}}{\log^s(e + \mathcal{M}(|h|))} \leqslant c \int_{Q_0/2} \frac{|h|^{(n+1)/n}}{\log^s(e + |h|)}$$

for any $h \in L^{(n+1)/n}(\log L)^{-s}(Q_0/2)$ (see [FS], Prop. 1.2), one can easily prove the inequality

$$\int_{Q_0/2} J \log^{1-s}\left(e + \frac{J}{J_{Q_0/2}}\right) \leqslant c \int_{Q_0} |Df|^n \log^{-s}\left(e + \frac{|Df|}{|Df|_{Q_0}}\right)$$

$\forall Q_0 \subset \Omega$.

4. Inequalities in Lorentz Spaces

Here is one positive result.

Theorem 2. *If $n < q < \infty$, $|Df| \in L^{n,q}(\Omega)$, and M denotes the local maximal function associated to a cube $Q \Subset \Omega$, then we have $MJ \in L^{1,q/n}(Q)$.*

Proof. First we recall (see, e.g., [BR]) that the maximal function maps $L^{p,r}$ into itself and the following inequality holds:

$$\| Mh \|_{L^{p,r}} \leqslant c \, \| h \|_{L^{p,r}}. \tag{4}$$

We will also use the relation

$$\| h^{\alpha} \|_{L^{p,r}} = \| h \|^{\alpha}_{L^{p\alpha, r\alpha}}. \tag{5}$$

As in the proof of Theorem 1 we have (2), since $|Df|$ belongs to $L^{n,q}$ and $L^{n,q} \subset L^n/\log L$ (see [BR, Theorem 9.3]). Hence we have also (3). Next we apply (4) to $h = |Df|^{n^2/(n+1)}$, with $p = (n+1)/n$, $r = (n+1)\, q/n^2$, and rewrite the right-hand side, using (5) with $\alpha = n^2/(n+1)$. So this proves that

$$MJ \in L^{1,q/n}(Q). \tag{6}$$

In view of inequality (6) it would be natural to expect that $J \in L(\log L)^{n/q}(K)$ for $K \subset \Omega$. However this is not true as it can be seen by the following.

EXAMPLE. Consider the function

$$f(x) = \frac{x}{|x|} \, |\log |x||^{-1/q} \, (\log |\log |x||)^{-1/n},$$

where $|x| < a < 1$, $q > n$. We claim that

$$|Df| \in L^{n,q} \tag{7}$$

$$J \notin L(\log L)^{n/q}. \tag{8}$$

Verification of (7). It is easy to check that $|Df|$ is equivalent to $(1/|x|)\, |\log |x||^{-1/q} \, (\log |\log |x||)^{-1/n}$. Then the claim follows since

$$\int_0^a \left[\frac{1}{|\log r|^{1/q} \, (\log |\log r|)^{1/n}} \right]^q \frac{dr}{r} < \infty.$$

Verification of (8). J is now equivalent to $(1/|x|^n)\, |\log |x||^{-1-n/q}$ $(\log |\log |x||)^{-1}$ and this function does not belong to $L(\log L)^{n/q}$.

REFERENCES

[B] C. BENNETT, Intermediate spaces and the class $L \log^+ L$, *Ark. Mat.* **11** (1973), 215–228.

[BR] C. BENNETT AND K. RUDNICK, On Lorentz–Zygmund spaces, *Dissertationes Math.* **175** (1980), 1–72.

[BS] C. BENNETT AND R. SHARPLEY, "Interpolation of Operators," Academic Press, New York/London, 1988.

[CLMS] R. R. COIFMAN, P. L. LIONS, Y. MEYER, AND S. SEMMES, Compensated compactness and Hardy spaces, preprint, 1991.

[FS] N. FUSCO AND C. SBORDONE, Higher integrability of the gradient of minimizers of functionals with nonstandard growth conditions, *Comm. Pure Appl. Math.* **43** (1990), 673–683.

[IS] T. IWANIEC AND C. SBORDONE, On the integrability of the Jacobian under minimal hypotheses, *Arch. Rational Mech. Anal.*, to appear.

[M] S. MÜLLER, Higher integrability of determinants and weak convergence in L^1, *J. Reine Angew. Math.* **412** (1990), 20–34.

[S] E. M. STEIN, Note on the class $L \log L$, *Studia Math.* **31** (1969), 305–310.

[Z] W. P. ZIEMER, "Weakly Differentiable Functions," Springer, New York, 1989.