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UNIFORM ESTIMATES FOR SOLUTIONS OF $-\Delta u = V(x)u^p$

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Let u be a weak solution of the nonlinear equation

$$(1) \qquad -\Delta u = V(x)\,|u|^{p-1}u \quad \text{in} \quad \Omega \subset \mathbb{R}^n\,,\ n \geq 3$$

possibly with a boundary condition

$$(2) \qquad u = 0 \quad \text{on} \quad \partial\Omega\,.$$

Assume $V \in L^\infty(\Omega)$ (Ω smooth and bounded) and suppose we start with the information $u \in L^\alpha(\Omega)$ for some α. We are concerned with the following questions:

<u>Question 1</u>: Can one conclude that $u \in L^\infty(\Omega)$?

<u>Question 2</u>: Can one <u>estimate</u> $\|u\|_\infty$ in terms of $\|V\|_\infty$ and $\|u\|_\alpha$? More precisely, suppose we have a sequence (u_j) of solutions of

$$\begin{cases} -\Delta u_j = V_j(x)|u_j|^{p-1}u_j & \text{in} \quad \Omega \\ u_j = 0 & \text{on} \quad \partial\Omega\,. \end{cases}$$

with $$\|V_j\|_\infty \leq C_1 \quad \text{and} \quad \|u_j\|_\alpha \leq C_2\,.$$

Can one find a constant C_3 (depending on C_1, C_2, α, p and Ω) such that

$$\|u_j\|_\infty \leq C_3\ ?$$

A natural approach is to use <u>bootstrap</u>. Since $V(x)|u|^{p-1}u \in L^{\alpha/p}(\Omega)$ (assuming $\alpha \geq p$) we deduce from the elliptic regularity theory (see [1]) that $u \in W^{2,\alpha/p}(\Omega)$. Using the Sobolev imbedding (see e.g. [4]) we conclude that

$$u \in L^{\beta}(\Omega) \quad \text{with} \quad \frac{1}{\beta} = \frac{p}{\alpha} - \frac{2}{n}$$

provided $\frac{p}{\alpha} - \frac{2}{n} > 0$ (otherwise we see easily that $u \in L^{\infty}(\Omega)$).

We now distinguish three cases:

<u>Case 1</u>: $\beta > \alpha$, i.e. $\alpha > \frac{n}{2}(p-1)$,

<u>Case 2</u>: $\beta < \alpha$, i.e. $\alpha < \frac{n}{2}(p-1)$,

<u>Case 3</u>: $\beta = \alpha$, i.e. $\alpha = \frac{n}{2}(p-1)$.

In Case 1 bootstrap works well. After a finite number of steps (iterating the above argument) we reach the conclusion that $u \in L^{\infty}(\Omega)$. Moreover the answer to Question 2 is also positive: one can estimate $\|u\|_{\infty}$ in terms of $\|V\|_{\infty}$ and $\|u\|_{\alpha}$.

In Case 2 bootstrap works "backwards" and yields nothing. In general one cannot conclude that $u \in L^{\infty}(\Omega)$. For example, if $p > n/(n-2)$ the function $u(x) = |x|^{-2/(p-1)}$ satisfies (1) in the sense of distributions with $V = \frac{2}{p-1}\left(n-2 - \frac{2}{p-1}\right) > 0$. Moreover $u \in L^{\alpha}(\Omega)$ for any $\alpha < \frac{n}{2}(p-1)$. However $u \notin L^{\infty}(\Omega)$.

Case 3 is interesting because it is a <u>border line case</u>. Bootstrap does not yield any improvement of regularity. However the answer to Question 1 is affirmative and the proof relies on the following result of [2]:

<u>Theorem 0</u>. Assume $u \in H_0^1(\Omega)$ satisfies the linear equation

$$-\Delta u = a(x)\, u \quad \text{in} \quad \Omega \tag{3}$$

with $\qquad a \in L^{n/2}(\Omega)$, $n \geq 3$.

Then

$$u \in L^q(\Omega)\ , \quad \forall q < \infty\ .$$

Applying Theorem 0 to (1) with $a(x) = V(x)|u|^{p-1}$ we see that Case 3 corresponds precisely to the assumption that $a \in L^{n/2}(\Omega)$. Hence we deduce that $u \in L^q(\Omega)$ $\forall q < \infty$ and going back to (1) we conclude that $u \in L^\infty(\Omega)$. Thus the answer to Question 1 is affirmative.

It is instructive to sketch the proof of Theorem 0. Multiplying (3) by $|u|^{k-1}u$ (k to be determined) we obtain

$$k \int |u|^{k-1}|\nabla u|^2 = \int a|u|^{k+1}$$

and thus, using the Sobolev inequality, we find

$$\frac{4kS}{(k+1)^2}\, \|u^{(k+1)/2}\|^2_{2n/(n-2)} \le \int |a|\ |u|^{k+1}. \tag{4}$$

Note that the RHS in (4) can be estimated by $\|a\|_{n/2}\ \|u^{k+1}\|_{n/(n-2)}$. However this is useless since the same expression $\|u^{k+1}\|_{n/(n-2)}$ appears on both sides. The main idea is to split the integral on the RHS of (4) as

$$\int_{[|a|>A]} |a|\ |u|^{k+1} + \int_{[|a|\le A]} |a|\ |u|^{k+1} \tag{5}$$

where A is a constant to be determined. The first integral in (5) is bounded by

$$\left[\int_{[|a|>A]} |a|^{n/2} \right]^{2/n} \|u\|^{k+1}_{(k+1)n/(n-2)}$$

while the second integral is bounded by

$$A \int |u|^{k+1}\ .$$

Choosing A large enough so that

$$\left[\int_{[|a|>A]} |a|^{n/2} \right]^{2/n} \leq \frac{2kS}{(k+1)^2}$$

(this is possible since $a \in L^{n/2}$) we are led to

$$\frac{2kS}{(k+1)^2} \|u\|^{k+1}_{(k+1)n/(n-2)} \leq A \|u\|^{k+1}_{k+1} . \tag{6}$$

We start with some initial value of k (for example $k+1 = 2n/(n-2)$) and then follow the sequence $(k+1)\frac{n}{n-2}$, $(k+1)(\frac{n}{n-2})^2$, $(k+1)(\frac{n}{n-2})^3, \cdots$, until we pass any given q.

<u>Remark 1</u>. At each stage (even at the first step) of the iteration the estimate we obtain for u depends not only on $\|a\|_{n/2}$ but also on the <u>distribution function</u> of a. Hence Theorem 0 <u>does not come with an estimate</u> of the form

$$\|u\|_q \leq F(\|a\|_{n/2}, \|u\|_{2n/(n-2)}) \tag{7}$$

as soon as $q > 2n/(n-2)$. This means that if we apply Theorem 0 to (1) in the Case 3 we cannot estimate $\|u\|_\infty$ in terms of $\|u\|_\alpha$. This is not just a deficiency of the proof. A typical example of lack of L^∞ estimates is given by the sequence

$$u_\varepsilon(x) = \left(\frac{\varepsilon}{\varepsilon^2+|x|^2}\right)^{(n-2)/2} \quad \text{in } \mathbb{R}^n,\ n \geq 3.$$

It satisfies

$$-\Delta u_\varepsilon = B\, u_\varepsilon^{(n+2)/(n-2)}$$

for some positive constant B (depending only on n). Moreover it is easy to see that

$$\|u_\varepsilon\|_{2n/(n-2)} \leq C \quad \text{as} \quad \varepsilon \to 0.$$

Thus we are in Case 3 with $p = (n+2)/(n-2)$ and $\alpha = 2n/(n-2)$. However

$$\|u_\varepsilon\|_\infty = u_\varepsilon(0) = \varepsilon^{-(n-2)/2} \to \infty \quad \text{as} \quad \varepsilon \to 0$$

so that here the answer to Question 2 is negative.

It is a surprising fact that the answer to Question 2 in Case 3 is positive, at least locally, when $p \leq n/(n-2)$. The main result is the following:

<u>Theorem 1</u>. Assume $1 + \frac{2}{n} < p \leq \frac{n}{n-2}$.
Let (u_j) be a sequence of (smooth) solutions of

$$-\Delta u_j = V_j\, u_j^p \quad \text{in} \quad \Omega$$

with

$$u_j \geq 0\ ,\ V_j \geq 0 \quad \text{in} \quad \Omega\ ,$$

(8)
$$\|V_j\|_\infty \leq C_1$$

and

(9)
$$\|u_j\|_\alpha \leq C_2 \quad \text{where} \quad \alpha = \frac{n}{2}\,(p-1)\ .$$

Then (u_j) is bounded in $L^\infty_{loc}(\Omega)$.

<u>Proof</u>. It is inspired by some of the techniques introduced in [3]. Let K be a compact subset of Ω and consider the "concentration function":

$$\varphi_j(t) = \operatorname*{Max}_{a \in K} \int_{B_t(a)} V_j^{\alpha/p}\, u_j^\alpha$$

defined for $t < \text{dist}\,(K, \partial\Omega) = d$, where $B_t(a) = \{x \in \mathbb{R}^n;\ |x-a| < t\}$.

<u>Lemma 1</u>. With the same assumptions as in Theorem 1, given any $\varepsilon > 0$ there is some $t \in (0, d)$ such that

(10)
$$\limsup_{j \to \infty}\ \varphi_j(t) < \varepsilon\ .$$

Proof of Lemma 1.

We argue by contradiction. If (10) does not hold, there is some $\varepsilon_0 > 0$ and sequences $R_j \to 0$, (a_j) in K such that

$$\int_{B_{R_j}(a_j)} V_j^{\alpha/p} u_j^{\alpha} \geq \varepsilon_0$$

(along a subsequence still denoted u_j).

Set $g_j = V_j\, u_j^p\, \chi_{B_{R_j}(a_j)}$, where χ denotes the characteristic function. Let v_j be the solution of

$$\begin{cases} -\Delta v_j = g_j & \text{in } \Omega, \\ v_j = 0 & \text{on } \partial\Omega. \end{cases}$$

By the maximum principle $u_j \geq v_j$.

On the other hand

$$v_j \geq \frac{C}{|x|^{n-2}} * g_j + 0(1)$$

so that

$$v_j(x) \geq C \int_{B_{R_j}(a_j)} \frac{1}{|x-y|^{n-2}}\, g_j(y)dy + 0(1)$$

and in particular

$$v_j(x) \geq \frac{C}{(|x-a_j|+R_j)^{n-2}} \int_{B_{R_j}(a_j)} g_j(y) + 0(1).$$

Since $p \leq n/(n-2)$ we have $\alpha \leq p$ and by Hölder's inequality

$$\int_{B_{R_j}(a_j)} V_j^{\alpha/p}\, u_j^{\alpha} \leq \left[\int_{B_{R_j}(a_j)} V_j\, u_j^p \right]^{\alpha/p} C\, R_j^{n(1-\alpha/p)}.$$

It follows that

$$\int_{B_{R_j}(a_j)} V_j\, u_j^p \geq C\varepsilon_0^{p/\alpha}\; R_j^{n(1-p/\alpha)} .$$

We conclude that

$$u_j(x) \geq \frac{C\;\; R_j^{n(1-p/\alpha)}}{(|x-a_j|+R_j)^{n-2}} \;+\; 0(1)$$

for some positive constant C (depending on ε_0). It follows that

$$\int_{B_d(a_j)} u_j^\alpha \geq C \int_0^d R_j^{n(\alpha-p)} \frac{r^{n-1}}{(r+R_j)^{\alpha(n-2)}}\, dr \;+\; 0(1) .$$

But

$$\int_0^d R_j^{n(\alpha-p)} \frac{r^{n-1}}{(r+R_j)^{\alpha(n-2)}}\, dr \;=\; \int_0^{d/R_j} R_j^{-np+n+2\alpha}\, \frac{s^{n-1} ds}{(s+1)^{\alpha(n-2)}}$$

$$= \begin{cases} \log \frac{1}{R_j} + 0(1) & \text{if} \quad \alpha = n/(n-2) \\ CR_j^{-np+n\alpha} + 0(1) & \text{if} \quad \alpha < n/(n-2) \end{cases} .$$

In both cases the integral tends to $+\infty$ (note that $\alpha \leq n/(n-2)$ since $p \leq n/(n-2)$ and either $p = \alpha = n/(n-2)$ or $\alpha < p < n/(n-2)$). This is impossible since $\|u_j\|_\alpha \leq C_2$ by assumption (9). Hence we have proved Lemma 1.

<u>Lemma 2</u>. Let (u_j) be a sequence of (smooth) solutions of

(11) $$-\Delta u_j = V_j\, u_j^p \quad \text{in} \quad B_R$$

such that

$$u_j \geq 0 \quad \text{in} \quad B_R ,$$

$$\|V_j\|_{L^\infty(B_R)} \leq C_1,$$

and

$$\|u_j\|_{L^\alpha(B_R)} \le C_2 \quad \text{with} \quad \alpha = \tfrac{n}{2}(p-1) .$$

Then there is some $\varepsilon_0 > 0$ (depending only on n, C_1 and p) such that

$$\int V_j^{\alpha/p} u_j^\alpha \le \varepsilon_0$$

implies

$$\|u_j\|_{L^\infty(B_{R/2})} \le C_3$$

where C_3 depends on C_1, C_2, R, n, p and ε_0.

<u>Proof</u>. This is a localized version of the proof of Theorem 0. Fix a function $\zeta \in C_0^\infty(B_R)$ with $\zeta = 1$ on $B_{R/2}$. Multiplying (11) by $\zeta^2 u_j^k$ (k to be determined later) we obtain

$$I = \int \nabla u_j \, \nabla(\zeta^2 u_j^k) = \int V_j u_j^{p-1} \zeta^2 u_j^{k+1} = II.$$

We have

$$I = k \int |\nabla u_j|^2 u_j^{k-1} \zeta^2 + \frac{1}{k+1} \int \nabla(u_j^{k+1}) \nabla\zeta^2$$

$$= \frac{4k}{(k+1)^2} \int |\nabla u_j^{(k+1)/2}|^2 \zeta^2 - \frac{1}{k+1} \int u_j^{k+1} \Delta\zeta^2$$

$$= \frac{4k}{(k+1)^2} \int |\nabla(\zeta u_j^{(k+1)/2})|^2 + \frac{1}{(k+1)^2} \int ((k-1)\Delta\zeta^2 - 4k|\nabla\zeta|^2) u_j^{k+1} .$$

Therefore, using Sobolev's inequality, we obtain

$$I \ge \frac{4k}{(k+1)^2} S \, \|\zeta u_j^{(k+1/2)}\|_{2n/(n-2)} - C \int u_j^{k+1}$$

with a constant C depending only on R. We estimate II by Hölder's inequality:

$$II \le \|V_j u_j^{p-1}\|_{n/2} \, \|\zeta^2 u_j^{k+1}\|_{n/(n-2)} .$$

Note that

$$\|V_j\, u_j^{p-1}\|_{n/2}^{n/2} = \int V_j^{n/2}\, u_j^{\alpha} \le C_1^{(n/2)-\alpha/p} \int V_j^{\alpha/p}\, u_j^{\alpha} \le \varepsilon_0\, C_1^{(n/2)-\alpha/p}\,.$$

Hence we find

$$\frac{4k}{(k+1)^2}\, S\, \|\zeta^2\, u_j^{k+1}\|_{n/(n-2)} \le C_1^{1/p}\varepsilon_0^{2/n}\, \|\zeta^2\, u_j^{k+1}\|_{n/(n-2)} + C \int u_j^{k+1}.$$

We start with $k = \alpha-1$; using the smallness of ε_0 we obtain an estimate for $\|\zeta^2\, u_j^{\alpha}\|_{n/(n-2)}$. Iterating we find local estimates for u_j in L^q, $q = \alpha(\frac{n}{n-2})$, $\alpha(\frac{n}{n-2})^2$, etc... After a <u>finite</u> number of steps we reach some $q > p\frac{n}{2}$. Therefore we have an estimate for u_j^p in $L_{loc}^{(n/2)+\varepsilon}$ and thus (by (11) we have an estimate for u_j in L_{loc}^{∞}.

<u>Proof of Theorem 1 completed</u>. Let $\varepsilon_0 > 0$ be as in Lemma 2. By Lemma 1 there is some $t > 0$ such that

$$\limsup_{j\to\infty}\ \varphi_j(t) < \varepsilon_0\,.$$

Hence for $j \ge j_0$ we have

$$\int_{B_t(a)} V_j^{\alpha/p}\, u_j^{\alpha} \le \varepsilon_0 \quad \forall a \in K.$$

By Lemma 2 we have an estimate for u_j in $L^{\infty}(B_{t/2}(a))$. Therefore we have a bound for (u_j) in $L^{\infty}(K)$.

<u>Some counterexamples</u>.

1. <u>No estimate up to the boundary</u>.

The conclusion of Theorem 1 is sharp. In general (u_j) <u>cannot be estimated in L^{∞} up to the boundary</u>, even if $u_j = 0$ on $\partial\Omega$.

Using an idea of [3] we shall construct sequences (u_j) and (V_j) such that

$$\begin{cases} -\Delta u_j = V_j\, u_j^p & \text{in} \quad \Omega \\ u_j = 0 & \text{on} \quad \partial\Omega \end{cases}$$

with $u_j \geq 0$, $V_j \geq 0$ in Ω such that

$$\|V_j\|_{L^\infty(\Omega)} \leq C_1, \quad \|u_j\|_{L^\alpha(\Omega)} \leq C_2, \quad \alpha = \tfrac{n}{2}(p-1)$$

and

$$\|u_j\|_{L^\infty(\Omega)} \longrightarrow \infty.$$

Let Ω be the unit ball centered at $(1, 0, 0, \cdots)$.

Let $a_\varepsilon = (d_\varepsilon, 0, 0, \cdots)$ with $d_\varepsilon = 2\varepsilon < 1$ and let

$$f_\varepsilon = \begin{cases} \dfrac{2N}{\varepsilon^\theta} & \text{in} \quad B_\varepsilon(a_\varepsilon) \\ 0 & \text{otherwise} \end{cases},$$

with $\theta = 2p/(p-1)$.

Let u_ε be the solution of

$$\begin{cases} -\Delta u_\varepsilon = f_\varepsilon & \text{in} \quad \Omega, \\ u_\varepsilon = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Set

$$V_\varepsilon = f_\varepsilon / u_\varepsilon^p,$$

so that u_ε satisfies

$$\begin{cases} -\Delta u_\varepsilon = V_\varepsilon\, u_\varepsilon^p & \text{in} \quad \Omega, \\ u_\varepsilon = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

We claim that

(12) $$\|V_\varepsilon\|_\infty \leq C_1$$

(13) $$\|u_\varepsilon\|_\alpha \leq C_2$$

and

(14) $$\|u_\varepsilon\|_\infty \longrightarrow \infty \quad \text{as} \quad \varepsilon \longrightarrow 0 .$$

<u>Verification of (12)</u>. Let v_ε be the solution of

$$\begin{cases} -\Delta v_\varepsilon = f_\varepsilon & \text{in} \quad B_{d_\varepsilon}(a_\varepsilon) \\ v_\varepsilon = 0 & \text{on} \quad \partial B_{d_\varepsilon} \end{cases} .$$

By the maximum principle $v_\varepsilon \le u_\varepsilon$ in $B_{d_\varepsilon}(a_\varepsilon)$ so that $(1/u_\varepsilon^p) \le (1/v_\varepsilon^p)$. But v_ε is given explicitly by

$$v_\varepsilon(x) = \begin{cases} \dfrac{1}{\varepsilon^\theta} r^2 + \dfrac{n}{(n-2)\varepsilon^{\theta-2}} - \dfrac{2\ \varepsilon^{n-\theta}}{(n-2)d_\varepsilon^{n-2}} & \text{if} \quad r < \varepsilon \\ \dfrac{2}{n-2}\varepsilon^{n-\theta}\left(\dfrac{1}{r^{n-2}} - \dfrac{1}{d_\varepsilon^{\,n-2}}\right) & \text{if} \quad \varepsilon < r < d_\varepsilon \end{cases}$$

with $r = |x-a_\varepsilon|$.

It follows that

$$\|V_\varepsilon\|_\infty \le \frac{C}{\varepsilon^\theta}\varepsilon^{(\theta-2)p} = C .$$

<u>Verification of (13)</u>. Let G be the half-space

$$G = \{(x_1, x');\ x_1 > 0\}$$

and let w_ε be the solution of

$$\begin{cases} -\Delta w_\varepsilon = f_\varepsilon & \text{in} \quad G , \\ w_\varepsilon = 0 & \text{on} \quad \partial G . \end{cases}$$

By the maximum principle $u_\varepsilon \le w_\varepsilon$ and thus

$$\|u_\varepsilon\|_{L^\alpha(\Omega)} \le \|w_\varepsilon\|_{L^\alpha(\Omega)} .$$

But w_ε is given explicitly by

$$w_\varepsilon(x) = \begin{cases} -\dfrac{1}{\varepsilon^\theta} r^2 + \dfrac{n}{(n-2)\varepsilon^{\theta-2}} - \dfrac{2\,\varepsilon^{n-\theta}}{(n-2)\,|x-a'_\varepsilon|^{n-2}} & \text{if } |x-a_\varepsilon| < \varepsilon \\ \dfrac{2\,\varepsilon^{n-\theta}}{(n-2)} \left(\dfrac{1}{|x-a_\varepsilon|^{n-2}} - \dfrac{1}{|x-a'_\varepsilon|^{n-2}}\right) & \text{otherwise} \end{cases}$$

where $r = |x-a_\varepsilon|$ and $a'_\varepsilon = -a_\varepsilon$. Note that

$$w_\varepsilon(x) \le \frac{C}{\varepsilon^{\theta-2}} \quad \text{if } |x-a_\varepsilon| < \varepsilon$$

so that

$$\int_{|x-a_\varepsilon|<\varepsilon} w_\varepsilon^\alpha \le \frac{C}{\varepsilon^{(\theta-2)\alpha}} \varepsilon^n = C$$

since

$$(\theta-2)\alpha = \frac{2}{p-1}\alpha = n .$$

On the other hand we claim that

$$(15) \qquad w_\varepsilon(x) \le \frac{C\,\varepsilon^{n-\theta+1}}{|x-a_\varepsilon|^{n-1}} \quad \text{if } x \in G \text{ and } |x-a_\varepsilon| > \varepsilon .$$

Indeed we have

$$\frac{1}{|x-a_\varepsilon|^{n-2}} - \frac{1}{|x-a'_\varepsilon|^{n-2}} \le \frac{C}{|x-a_\varepsilon|^{n-1}} (|x-a'_\varepsilon| - |x-a_\varepsilon|) \le \frac{C\,d_\varepsilon}{|x-a_\varepsilon|^{n-1}}$$

which implies (15). Therefore

$$\|w_\varepsilon\|_\alpha^\alpha \le C\,\varepsilon^{(n-\theta+1)\alpha} \int_\varepsilon^2 \frac{r^{n-1}dr}{r^{(n-1)\alpha}} \le C$$

since $(n-1)\alpha > n$ (this follows from the assumption $p > (n+1)/(n-1)$) and $\alpha(\theta-2) = 1$.

<u>Verification of (14)</u>. We have

$$u_\varepsilon(a_\varepsilon) \ge v_\varepsilon(a_\varepsilon) = \frac{C}{\varepsilon^{\theta-2}} \to \infty \quad \text{as } \varepsilon \to 0 .$$

2. <u>No L^∞ local estimate when $p > n/(n-2)$</u>.

As we have already pointed out, there are no L^∞ local estimates when $p = (n+2)/(n-2)$ and $\alpha = 2n/(n-2)$. In fact, the same holds for any $p > n/(n-2)$ and $\alpha = \frac{n}{2}(p-1)$.

Using an idea of [3] we shall construct in $\Omega = \{x \in \mathbb{R}^n;\ |x| < 1\}$ sequences (u_j) and (V_j) satisfying

$$\begin{cases} -\Delta u_j = V_j\, u_j^p & \text{in} \quad \Omega, \quad \text{with} \quad p > n/(n-2) \\ u_j = 0 & \text{on} \quad \partial\Omega \end{cases}$$

$$u_j \geq 0\ ,\ V_j \geq 0 \quad \text{in} \quad \Omega$$

$$\|V_j\|_{L^\infty(\Omega)} \leq C_1\ , \quad \|u_j\|_{L^\alpha(\Omega)} \leq C_2\ , \quad \alpha = \tfrac{n}{2}(p-1)$$

and

$$u_j(0) \longrightarrow \infty\ .$$

Indeed, set, as above, $\theta = 2p/(p-1)$ and let

$$f_\varepsilon(x) = \begin{cases} \dfrac{2n}{\varepsilon^\theta} & \text{if} \quad |x| < \varepsilon\ , \\ 0 & \text{otherwise} \quad . \end{cases}$$

Let u_ε be the solution of

$$\begin{cases} -\Delta u_\varepsilon = f_\varepsilon & \text{in} \quad \Omega = \{x \in \mathbb{R}^n;\ |x| < 1\}\ , \\ u_\varepsilon = 0 & \text{on} \quad \partial\Omega \quad . \end{cases}$$

Set

$$V_\varepsilon = f_\varepsilon / u_\varepsilon^p$$

so that u_ε satisfies

$$\begin{cases} -\Delta u_\varepsilon = V_\varepsilon\, u_\varepsilon^p & \text{in} \quad \Omega\,, \\ u_\varepsilon = 0 & \text{on} \quad \partial\Omega\,. \end{cases}$$

We claim that

(16) $$\|V_\varepsilon\|_\infty \le C_1$$

(17) $$\|u_\varepsilon\|_\alpha \le C_2$$

and

(18) $$u_\varepsilon(0) \longrightarrow \infty \quad \text{as} \quad \varepsilon \longrightarrow 0\,.$$

<u>Verification of (16)</u>. As above we have an explicit formula for u_ε:

$$u_\varepsilon(x) = \begin{cases} -\dfrac{1}{\varepsilon^\theta} r^2 + \dfrac{n}{(n-2)\varepsilon^{\theta-2}} - \dfrac{2\,\varepsilon^{n-\theta}}{(n-2)} & \text{if} \quad r < \varepsilon \\ \dfrac{2\,\varepsilon^{n-\theta}}{(n-2)} \left(\dfrac{1}{r^{n-2}} - 1\right) & \text{if} \quad \varepsilon < r < 1 \end{cases}$$

with $r = |x|$. It follows that

$$\|V_\varepsilon\|_\infty \le \frac{C}{\varepsilon^\theta}\, \varepsilon^{(\theta-2)p} = C\,.$$

<u>Verification of (17)</u>. We have

$$\|u_\varepsilon\|_\alpha^\alpha \le \frac{C}{\varepsilon^{(\theta-2)\alpha}}\, \varepsilon^n + C\, \varepsilon^{(n-\theta)\alpha} \int_\varepsilon^1 \frac{r^{n-1}\, dr}{r^{(n-2)\alpha}}\,.$$

But $\alpha > n/(n-2)$ (since $p > n/(n-2)$) and thus

$$\|u_\varepsilon\|_\alpha^\alpha \le C\, \frac{\varepsilon^n}{\varepsilon^{(\theta-2)\alpha}} = C\,.$$

<u>Verification of (18)</u>. We have

$$u_\varepsilon(0) = \frac{n}{(n-2)\varepsilon^{\theta-2}}\,(1 + o(1))$$

and thus $u_\varepsilon(0) \longrightarrow \infty$ since $\theta > 2$.

3. <u>No L^∞ local estimate if V changes sign</u>.

S. Wang [5] has given an example showing that when, $p = n/(n-2)$, the assumption $V_j \geq 0$ in Theorem 1 is essential. He constructs sequences (u_j) and (V_j) in Ω, the unit ball of $\mathbb{R}^n$, such that

$$\begin{cases} -\Delta u_j = V_j\, u_j^p & \text{in} \quad \Omega\,, \\ u_j = 0 & \text{on} \quad \partial\Omega\,, \end{cases}$$

$$u_j \geq 0 \quad \text{in} \quad \Omega, \quad \|V_j\|_\infty \leq C_1, \quad \|u_j\|_\alpha \leq C_2$$

and

$$u_j(0) \longrightarrow \infty$$

with

$$p = \alpha = \frac{n}{n-2}\,.$$

On the other hand, S. Wang [5] also proves that, when $1 + \frac{2}{n} < p < \frac{n}{n-2}$, the conclusion of Theorem 1 still holds without the assumption $V_j \geq 0$.

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