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SUBLINEAR ELLIPTIC EQUATIONS IN $\mathbb{R}^n$

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1. Introduction:

We are concerned with the question of existence (or nonexistence) and uniqueness of solutions of the problem

(1) $$-\Delta u = \rho(x)u^\alpha \quad \text{in} \quad \mathbb{R}^n, \ n \geq 3$$

with $0 < \alpha < 1$ and $\rho(x) \geq 0$, ρ not identically zero. We shall assume throughout the paper that $\rho \in L^\infty_{loc}$. We look for a solution $u \geq 0$, u not identically zero, so that, by the strong maximum principle, if such a solution exists then $u > 0$ in $\mathbb{R}^n$.

We shall often use the following:

Definition: *We say that a function* $\rho \in L^\infty_{loc}(\mathbb{R}^n)$, $\rho \geq 0$, *has the property* (H) *if the linear problem*

(2) $$-\Delta U = \rho \quad \text{in} \quad \mathbb{R}^n$$

has a bounded solution.

Our main result is

Theorem 1. *Problem (1) has a bounded solution iff* ρ *satisfies* (H). *Moreover there is a minimal positive solution of (1).*

This minimal positive solution of (1) tends to zero at infinity in a sense to be precised later. Moreover it is the unique positive solution of (1) which tends to zero at infinity (see Theorem 2 below).

In Section 2 we prove Theorem 1 and in Section 3 we present uniqueness results for (1). In Appendix I we summarize some properties of the linear Poisson equation (2). In Appendix II we review the uniqueness question for equation (1) in bounded domains.

Problem (1) for <u>bounded domains</u> with zero Dirichlet condition has been extensively studied (even for more general sublinear functions). We refer in particular to Krasnoselskii [10] (Theorem 7.14 and 7.15) and [1] (see also the references therein). Problem (1) in all of space has been considered in [3], [4], and [11] under more restrictive conditions on ρ (ρ is equivalent to a radial function for large $|x|$).

The study of (1) is also related to the asymptotic behavior (as $t \to \infty$) of the solution of

(3) $$\rho(x)\,\frac{\partial u}{\partial t} = \Delta u^m \quad \text{in} \quad \mathbb{R}^n \times (0,\infty)$$

with $m = 1/\alpha > 1$ which has recently been studied by Eidus [5] (see also [6]) for a class of functions ρ tending to zero at infinity. In fact, separating variables, we have a solution $u(x,t)$ of (3) of the form $u(x,t) = C\, v^{1/m}(x)(t + \tau)^{-1/(m-1)}$ provided $v(x)$ is a solution of (1).

2. Proof of Theorem 1

A. *Sufficient condition*:

Let

$$B_R = \{x \in \mathbb{R}^n;\ |x| < R\}$$

and let u_R be the solution of

$$\begin{cases} -\Delta u = \rho\, u^\alpha & \text{in } B_R, \\ u = 0 & \text{on } \partial B_R. \end{cases} \tag{4}$$

It is well-known that u_R exists and is unique (see e.g. [10], [1] or Appendix II). The sequence u_R is increasing with R. Indeed, let $R' > R$. Then $u_{R'}$ is a supersolution for the R-problem. We now construct a subsolution $\underline{u}$ for the R-problem with $\underline{u} \le u_{R'}$. This will imply that there is a solution u for the R-problem between $\underline{u}$ and $u_{R'}$. Since the unique solution is u_R it follows that $u_R \le u_{R'}$ in B_R. For $\underline{u}$ we may take $\varepsilon\varphi_1$ where φ_1 satisfies

$$\begin{cases} -\Delta\varphi_1 = \lambda_1\, \rho\, \varphi_1 & \text{in } B_R\ , \\ \varphi_1 = 0 & \text{on } \partial B_R\ . \end{cases}$$

We now prove that the sequence u_R remains bounded as $R \to \infty$. In fact

$$u_R \le C\,U$$

for some appropriate constant C. Indeed, C U is a supersolution for the R-problem since

$$-\Delta\,(CU) = C\rho \ge \rho\,(CU)^\alpha$$

provided

$$C^{1-\alpha} \ge \|U\|_\infty^\alpha\ .$$

Therefore $u = \lim_{R\to\infty} u_R$ exists and u is a solution of (1) satisfying

$$u \le C\,U\ . \tag{5}$$

Clearly u is the minimal solution; indeed if $\bar{u}$ is another solution of (1) then $u_R \le \bar{u}$ on B_R by the above argument and thus $u \le \bar{u}$.

B. **Necessary condition**

Suppose u is bounded positive solution of (1) and set

$$v = \frac{1}{1-\alpha} u^{1-\alpha}.$$

Then

$$-\Delta v = \alpha u^{-\alpha-1}|\nabla u|^2 + \rho \geq \rho .$$

The solution w_R of the problem

$$\text{(6)} \qquad \begin{cases} -\Delta w_R = \rho & \text{in } B_R , \\ w_R = 0 & \text{on } \partial B_R \end{cases}$$

satisfies $w_R \leq v$. Thus w_R increases as $R \to \infty$ to a bounded solution of (2).

The meaning of Theorem 1 is that if $\rho(x)$ decays fast enough at infinity then Problem (1) has a solution. It need not exist if $\rho(x)$ has a slow decay at infinity. As we see in the next example, if $\rho(x)$ decays like a power, the critical exponent is two.

Example 1: Assume

$$\rho(x) = \frac{1}{1+|x|^p} \qquad \text{with } p > 2$$

or

$$\rho(x) = \frac{1}{(1+|x|^2)|\log(2+|x|)|^p} \qquad \text{with } p > 2$$

then Problem (1) has a bounded solution. Indeed the Poisson integral $\frac{c}{|x|^{n-2}} * \rho$ provides a bounded positive solution of (2) where $c/|x|^{n-2}$ is the fundamental solution of $-\Delta$.

Example 2: Assume

$$\rho(x) = \frac{1}{1+|x|^p} \qquad \text{with } p \leq 2$$

then Problem (1) has no solution. In fact a stronger nonexistence result holds.
Assume

$$\text{(7)} \qquad \int_{|x|\geq 1} \frac{\rho(x)}{|x|^{n-2}} dx = \infty ,$$

then there is no function $u \in L^1_{loc}(\mathbb{R}^n)$ satisfying

$$\text{(8)} \qquad \begin{cases} -\Delta u = \rho u^\alpha & \text{in } \mathscr{D}'(\mathbb{R}^n) \\ u \geq 0 \end{cases}$$

except $u \equiv 0$. Indeed, assume we have a solution of (8). By local regularity, $u \in W^{2,q}_{loc}$ for all $q < \infty$ and if u is not identically zero then $u > 0$ in $\mathbb{R}^n$. As above, set

$$v = \frac{1}{1-\alpha} u^{1-\alpha}$$

so that $-\Delta v \geq \rho$. It follows that

$$w_R \leq v \tag{9}$$

where w_R is defined by (6). As $R \uparrow \infty$, $w_R \uparrow \infty$ because of (7) (see Appendix I). This is impossible by (9).

Remark 1. The minimal solution u obtained in Theorem 1 satisfies

$$u(x) = c \int_{\mathbb{R}^n} \frac{\rho(y)u^{\alpha}(y)}{|x-y|^{n-2}} dy \tag{10}$$

and also

$$\lim_{R\to\infty} ⨍_{S_R} u = 0 \tag{11}$$

where $⨍_{S_R} u$ denotes the average of u on the sphere of radius R (centered at 0).

Indeed, u satisfies (5) for any positive solution U of (2); in particular we can take $U = \frac{c}{|x|^{n-2}} * \rho$. We now apply Lemma A.4 in Appendix I to conclude that (11) holds. As a consequence of (11) we have

$$\liminf_{|x| \to \infty} u(x) = 0 .$$

Next, let $f = \rho u^{\alpha}$. The linear equation $-\Delta v = f$ in $\mathbb{R}^n$ has a unique solution satisfying

$$\lim_{R\to\infty} ⨍_{S_R} v = 0 ,$$

namely $v = \frac{c}{|x|^{n-2}} * f$. Since u satisfies the same equation and also (11) we obtain (10).

Remark 2. The minimal solution u of (1) depends monotonically on ρ. Indeed let $\rho_1 \leq \rho_2$ and let u_1, u_2 be the corresponding minimal solutions of (1). Then u_2 is a supersolution for the equation

$$-\Delta u = \rho_1 u^\alpha \quad \text{in } B_R$$
$$u = 0 \quad \text{on } \partial B_R .$$

Thus $u_{1,R} \le u_2$ in B_R. Passing to the limit as $R \longrightarrow \infty$ we find that $u_1 \le u_2$.

Remark 3. The minimal solution u obtained in Theorem satisfies $C_1 U^{1-\alpha} \le u \le C_2 U$. In general these bounds are sharp. For example if ρ has compact support then both u and U behave at infinity like the fundamental solution. However if $\rho(x) \sim |x|^{-p}$ at infinity with $2+(n-2)(1-\alpha) < p < n$ then a simple computation shows that $U(x) \sim |x|^{-(p-2)}$ and $u(x) \sim |x|^{-(p-2)(1-\alpha)}$

3. Uniqueness

As we have noted the minimal solution u constructed above satisfies

$$\liminf_{|x| \to \infty} u(x) = 0 . \tag{12}$$

Our main uniqueness result is

Theorem 2. *Assuming ρ has property* (H), *then there is exactly one bounded positive solution of (1) satisfying (12).*

Remark 4. There exist other bounded positive solutions of (1) which do not satisfy (12). In fact, given any positive constant a, there exists a solution of (1) satisfying

$$\liminf_{|x| \to \infty} u(x) = a.$$

Indeed, consider the problem

$$\begin{cases} -\Delta u = \rho u^\alpha & \text{in } B_R , \\ u = a & \text{on } \partial B_R . \end{cases} \tag{13}$$

As subsolution for (13) we may take a and as supersolution we may take $(CU + a)$ where $U = \frac{c}{|x|^{n-2}} * \rho$ with C is large enough. We then let $R \longrightarrow \infty$.

The proof of Theorem 2 is divided into 3 steps:

Step 1. Assume $\rho_1 \le \rho_2$ and that they satisfy property (H). Given any bounded positive solution u ofthen there exists a bounded positive solution u_2 of

$$(15) \qquad \begin{cases} -\Delta u_2 = \rho_2 u_2^{\alpha} & \text{in } \mathbb{R}^n \\ \lim_{R\to\infty} \fint_{S_R} u_2 = 0 \end{cases}$$

such that $u_1 \le u_2$.

Proof. Clearly u_1 is a subsolution for (15) in the sense that

$$-\Delta u_1 \le \rho_2 u_1^{\alpha} .$$

Since u_1 is bounded we have

$$-\Delta u_1 \le C \rho_2$$

and by Lemma A.6 we find that

$$u_1 \le C \left(\frac{1}{|x|^{n-2}} * \rho_2\right) .$$

The right–hand side is a supersolution for (15) provided C is large enough. Using the standard monotone iteration technique (directly in $\mathbb{R}^n$) we obtain a solution u_2 of (15) such that

$$u_1 \le u_2 \le C\left(\frac{1}{|x|^{n-2}} * \rho_2\right) .$$

The only difference with the usual case of bounded domains is that the Dirichlet condition is replaced by the condition at infinity $\lim_{R\to\infty} \fint_{S_R} u = 0$. The standard maximum principle is replaced at each stage by Lemma A.6.

We shall now show that it suffices to prove Theorem 2 in the case $\rho > 0$.

Step 2. Assume we have proved uniqueness for any $\rho > 0$, then we also have uniqueness for a general $\rho \ge 0$.

Proof. Let $\rho_\varepsilon = \rho + \varepsilon h$ where $h \in C^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $h > 0$. Let u_ε be the unique solution of

$$(16) \qquad \begin{cases} -\Delta u_\varepsilon = \rho_\varepsilon u_\varepsilon^{\alpha} & \text{in } \mathbb{R}^n , \\ \lim_{R\to\infty} \fint_{S_R} u_\varepsilon = 0 & . \end{cases}$$

Let u be any solution of

(17)
$$\begin{cases} -\ \Delta u = \rho\, u^{\alpha} & \text{in } \mathbb{R}^n \\ \lim\limits_{R\to\infty} ⨍_{S_R} u = 0 \ . \end{cases}$$

By Step 1 (and by the uniqueness of u_ε) we know that

(18)
$$u \le u_\varepsilon$$

We prove that, as $\varepsilon \downarrow 0$, $u_\varepsilon \downarrow \underline{u}$ where $\underline{u}$ is the minimal solution constructed in Theorem 1. Indeed let $u_{\varepsilon,R}$ and u_R be the positive solutions of

(19)
$$-\Delta u_{\varepsilon,R} = \rho_\varepsilon\, u_{\varepsilon,R}^{\alpha} \quad \text{in } B_R$$

with
$$u_{\varepsilon,R} = 0 \quad \text{on } \partial B_R$$

and

(20)
$$-\Delta u_R = \rho\, u_R^{\alpha} \quad \text{in } B_R$$

with
$$u_R = 0 \quad \text{on } \partial B_R \ .$$

We now use the same device as in Appendix II (method II), namely, we multiply (19) by u_R and (20) by $u_{\varepsilon,R}$. Integrating by parts we find

$$\int_{B_R} \rho\, u_{\varepsilon,R}^{\alpha}\, u_R^{\alpha}(u_{\varepsilon,R}^{1-\alpha} - u_R^{1-\alpha}) = \int_{B_R} (\rho_\varepsilon - \rho) u_R\, u_{\varepsilon,R}^{\alpha}$$

and thus
$$\int_{B_R} \rho\, u_{\varepsilon,R}^{\alpha}\, u_R^{\alpha}(u_{\varepsilon,R}^{1-\alpha} - u_R^{1-\alpha}) \le C\varepsilon$$

where C is independent of R. Passing to the limit as $R \to \infty$ (and using Fatou) we obtain

$$\int_{\mathbb{R}^n} \rho\, u_\varepsilon^{\alpha}\, \underline{u}^{\alpha}\, (u_\varepsilon^{1-\alpha} - \underline{u}^{1-\alpha}) \le C\varepsilon \ .$$

Using (18) we have
$$\int \rho\, u^{\alpha}\, \underline{u}^{\alpha}\, (u^{1-\alpha} - \underline{u}^{1-\alpha}) = 0$$

and thus $\rho u^{\alpha} = \rho \underline{u}^{\alpha}$. Hence $\Delta(u - \underline{u}) = 0$ and therefore $u = \underline{u}$ (by the condition at infinity).

The last step involves the use of parabolic equations as in [8]. As we already mentioned in the Introduction if $u(x)$ is a solution of (1) then

$$v(x,\ t) = \frac{C_m u^{1/m}(x)}{(t\ +\ \tau)^{1/(m-1)}}$$

satisfies

(21)
$$\rho\,\frac{\partial v}{\partial t} = \Delta v^m$$

where $m = 1/\alpha$ and $C_m = (m-1)^{-\,1/(m-1)}$. Our proof of uniqueness for problem (1), (12) relies heavily on existence, uniqueness and comparison properties of solution of (21).

Step 3. We recall first a well-known fact about bounded domains (see e.g. [2]).

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, $\rho \in L^\infty(\Omega)$, $\rho \geq \delta > 0$ on Ω. Then given any $v_0 \geq 0$ on Ω, $v_0 \in L^\infty(\Omega)$, there exists a unique solution $v(x,t)$ of the problem

(22)
$$\begin{cases} \rho\,\frac{\partial v}{\partial t} - \Delta v^m = 0 & \text{in } \Omega \times (0,\infty) \\ v = 0 & \text{on } \partial\Omega \times (0,\infty) \\ v(x,0) = v_0(x) & \text{in } \Omega \end{cases}$$

Moreover if there is another solution $\tilde{v}(x,t)$ of (22) with $\tilde{v}(x,t) \geq 0$ on $\partial\Omega \times (0,\infty)$ and $\tilde{v}(x,0) \geq v_0(x)$ then $\tilde{v}(x,t) \geq v(x,t)$.

Let $\underline{u}$ be the minimal positive solution of (1) in the sense of Theorem 1. Let u be any bounded positive solution of (1) satisfying (12). By Appendix I we know that

$$\lim_{R\to\infty} \fint_{S_R} u = 0\ .$$

Let v_R be the solution of

$$\begin{cases} \rho\,\frac{\partial v_R}{\partial t} - \Delta {v_R}^m = 0 & \text{in } B_R \times (0,\infty) \\ v_R(x,t) = 0 & \text{on } \partial B_R \times (0,\infty) \\ v_R(x,0) = C_m u^{1/m}(x) & \text{in } B_R \end{cases}\ .$$

By comparison in bounded domains we see that

(23)
$$v_R(x,t) \leq \frac{C_m u^{1/m}(x)}{(t\ +\ 1)^{1/(m-1)}}$$

and also

(24)
$$v_R(x,t) \leq \frac{C_m \underline{u}^{1/m}(x)}{t^{1/(m-1)}}\ .$$

As $R \uparrow \infty$ the sequence v_R increases to some limit $v_\infty(x,t)$ which satisfies

(25) $$\rho \frac{\partial v_\infty}{\partial t} - \Delta v_\infty{}^m = 0 \qquad \text{in } \mathbb{R}^n \times (0, \infty)$$

and

(26) $$v_\infty(x,0) = C_m u^{1/m}(x) .$$

Moreover we have

(27) $$v_\infty(x,t) \le \frac{C_m u^{1/m}(x)}{(t+1)^{1/(m-1)}} .$$

We already have a solution of (25), (26) namely $\dfrac{C_m u^{1/m}(x)}{(t+1)^{1/(m-1)}}$. We claim that

(28) $$v_\infty(x,t) = \frac{C_m u^{1/m}(x)}{(t+1)^{1/(m-1)}} \equiv \hat{v}(x,t) .$$

For this purpose we multiply

$$\rho \frac{\partial}{\partial t}(\hat{v} - v_\infty) - \Delta(\hat{v}^m - v_\infty^m) = 0$$

by the function $K(x) = c\left[\frac{1}{|x|^{n-2}} - \frac{1}{R^{n-2}}\right]$ and integrate over $B_R \times (0,T)$. We find

$$\int_{B_R} \rho(x)\,(\hat{v} - v_\infty)K(x)dx_{|t=T} + \int_0^T (\hat{v}^m - v_\infty{}^m)dt_{|x=0}$$

$$= -\int_0^T \int_{\partial B_R} (\hat{v}^m - v_\infty{}^m)\,\frac{\partial K}{\partial \nu}\,dS\,dt .$$

The integral on the right hand side is bounded by

$$C\,T \fint_{S_R} u$$

which tends to zero as $R \to \infty$. Thus $\hat{v} = v_\infty$ (since $\rho > 0$). Passing to the limit in (24) we find

$$\frac{C_m u^{1/m}(x)}{(t+1)^{1/(m-1)}} \le \frac{C_m \underline{u}^{1/m}(x)}{t^{1/(m-1)}} .$$

Letting $t \to \infty$ we conclude that $u \le \underline{u}$.

Remark 5. Assume ρ has property (H). As we know from Appendix I

$$\lim_{R\to\infty} \fint_{S_R} U = 0$$

where $U = \frac{c}{|x|^{n-2}} * \rho$, and thus $\liminf_{|x|\to\infty} U = 0$. It may happen that $U(x)$ does not tend to zero as $|x| \to \infty$. Here is a simple example for $n \ge 4$. Let $\psi(x')$ be the solution of

$$\begin{cases} -\Delta_{x'} \psi = \rho(x') & \text{in } \mathbb{R}^{n-1} \\ \lim\limits_{|x'| \to \infty} \psi = 0 \end{cases}$$

where $\rho \in C_0^\infty(\mathbb{R}^{n-1})$, $\rho \geq 0$ and ρ not identically zero. Then

$$U(x) = \psi(x') \qquad x = (x_1, x')$$

provides such an example since $U(x_1, 0) = \psi(0)$ does not tend to zero as $|x_1| \longrightarrow \infty$. In such a situation there is no solution u of (1) which tends to zero at infinity because of the estimate from below $\underline{u}^{1-\alpha} \geq (1 - \alpha)\, U$ (see the proof of necessary condition in Theorem 1).

The uniqueness question becomes easier under a stronger assumption

Theorem 2′. *Assume there is a solution* U *of (2) such that*

$$\lim_{|x| \to \infty} U(x) = 0 . \tag{29}$$

Then there exists a unique positive solution u *of (1) such that*

$$\lim_{|x| \to \infty} u(x) = 0 .$$

Proof. The existence part is clear since we already know that there is a solution u of (1) such that $u \leq CU$. For the uniqueness we could invoke Theorem 2 but we present instead a simple argument due to Louis Nirenberg.

First we change the unknown. As in the proof of Theorem 1 we set

$$v = \frac{1}{1 - \alpha} \quad u^{1-\alpha}$$

so that we find

$$-\Delta v - \frac{C}{v} |\nabla v|^2 = \rho \tag{30}$$

for some positive constant C (depending on α). Uniqueness holds for (30) since the function $1/v$ is decreasing in v. More precisely, suppose we have two solutions v_1, v_2 of (30) with $\lim\limits_{|x| \to \infty} v_1 = \lim\limits_{|x| \to \infty} v_2 = 0$. Then $w = v_1 - v_2$ satisfies

$$-\Delta w - \frac{C}{v_1} \nabla(v_1 + v_2) \cdot \nabla w + \frac{C}{v_1 v_2} |\nabla v_2|^2 w = 0 .$$

Since the coefficient of w is nonnegative we may use the maximum principle to conclude that $w = 0$.

Remark 6. Clearly if ρ is a radial function satisfying (H) then (29) holds. It also holds if ρ is bounded by a radial function satisfying (H).

4. Some generalization

Our methods extend to more general problems of the form

$$-\Delta u = \rho(x)\ f(u) \qquad \text{in } \mathbb{R}^n$$

under suitable assumptions of f and in particular $f(u)$ behaves like u^α near $u = 0$. For simplicity we restrict our attention to the model problem
$f(u) = u^\alpha(1-u)$, i.e.

$$-\Delta u = \rho(x)\ u^\alpha(1-u) \qquad \text{in } \mathbb{R}^n \ . \tag{31}$$

Theorem 3. *Assume* ρ *satisfies (H). Then there is a unique solution* u, $0 < u < 1$ *of (31) such that*

$$\liminf_{|x| \to \infty} u(x) = 0 \ . \tag{32}$$

Proof. For the existence part we proceed as in the proof of Theorem 1 (sufficient condition). We obtain a minimal solution $\underline{u}$ with $\underline{u} \le 1$ and $\underline{u} \le CU$. For the uniqueness we proceed in two steps.
Step 1. Let u be any solution of (31), (32). Then there exists some $\varepsilon > 0$ such that

$$\varepsilon\, u \le \underline{u} \ . \tag{33}$$

It is useful to introduce the unique positive solution v of the problem

$$\begin{cases} -\Delta v = \rho\ v^\alpha & \text{in } \mathbb{R}^n \\ \liminf\limits_{|x| \to \infty} v(x) = 0 \end{cases} \tag{34}$$

Note that u is a subsolution for (34) since

$$u^\alpha(1-u) \le u^\alpha$$

and therefore, by monotone iteration and uniqueness of v, we obtain

$$u \le v \ .$$

Next, we note that for $\varepsilon > 0$ small enough εv is a subsolution for (31) since

$$-\Delta(\varepsilon v) = \varepsilon \rho v^\alpha \le \rho(\varepsilon v)^\alpha(1-\varepsilon v).$$

It follows that $\varepsilon v \le \underline{u}$, the minimal solution of (31) (to justify this we use comparison in

B_R and then let $R \to \infty$). Thus (33) holds.

Step 2. We now follow the same technique as in Method III of Appendix II. Let u be any solution of (31), (32) and let

$$\Lambda = \{t \in [0, 1]; \ \ tu \leq \underline{u}\} .$$

We claim that $1 \in \Lambda$. Suppose not, that

$$t_0 = \sup \Lambda < 1.$$

By Step 1 we know that $t_0 > 0$. Fix K large enough so that the function $f(t) + Kt$ is increasing on $[0, 1]$. We have

$$(35) \qquad - \Delta(\underline{u} - t_0 u) + K\rho(\underline{u} - t_0 u) \geq \rho[f(t_0 u) - t_0 f(u)].$$

Choose $\varepsilon>0$ small enough so that

$$t_0^\alpha - t_0 \geq \varepsilon(K+1) .$$

We claim that

$$(36) \qquad - \Delta(\underline{u}-t_0u-\epsilon u) + K\rho(\underline{u}-t_0u-\epsilon u) \geq 0 .$$

Indeed we have by (35)

$$-\Delta(\underline{u}-t_0u-\epsilon u) + K\rho(\underline{u}-t_0u-\epsilon u) \geq \rho[f(t_0u) - t_0f(u) - \varepsilon f(u) - \varepsilon Ku] .$$

But

$$\begin{aligned} f(t_0u) - t_0f(u) - \varepsilon f(u) - \varepsilon Ku &= (t_0^\alpha-t_0-\epsilon)u^\alpha + (t_0-t_0^{\alpha+1}+\varepsilon)u^{\alpha+1}-\epsilon Ku \\ &\geq \varepsilon Ku^\alpha - \varepsilon Ku \geq 0 \end{aligned}$$

since $u\leq 1$.

By Kato's inequality (see [9]) we have

$$\Delta(t_0u+\varepsilon u-\underline{u})^+ \geq \Delta(t_0u+\varepsilon u-\underline{u}) \ \ \text{sign}^+ \ (t_0u+\varepsilon u-\underline{u}) .$$

Using (36) we deduce that

$$\Delta(t_0u+\varepsilon u-\underline{u})^+ \geq 0,$$

i.e., the function $\varphi = (t_0u+\varepsilon u-\underline{u})^+$ is subharmonic. It follows that, for any x_0,

$$\varphi(x_0) \leq \fint_{S_R(x_0)} \varphi$$

where $S_R(x_0)$ denotes the sphere of radius R centered at x_0. But $\varphi \leq (t_0+\varepsilon)u \leq (t_0+\varepsilon)v$ and we know (see Remark 1) that

$$\lim_{R\to\infty} \fint_{S_R(x_0)} v = 0$$

(since the origin may be shifted to any point x_0). We conclude that $\varphi \equiv 0$ and thus $(t_0+\varepsilon)u \leq \underline{u}$. Hence $t_0+\varepsilon \in \Lambda$, which contradicts the maximality of t_0.

Appendix I

Throughout the paper we have often used the property (H), namely that the equation

$$-\Delta U = f \quad \text{in } \mathbb{R}^n \tag{A.1}$$

has a bounded solution. We discuss here some equivalent forms and some consequences. In what follows we always assume that $f \in L^\infty_{loc}(\mathbb{R}^n)$, $f \geq 0$ a.e. and that f is not identically zero. Let u_R be the solution of

$$\begin{cases} -\Delta u_R = f & \text{in } B_R \\ \quad u_R = 0 & \text{on } \partial B_R \end{cases} . \tag{A.2}$$

Note that u_R is a nondecreasing sequence of positive functions (in B_R) for R large enough. Moreover u_R is given by

$$u_R(x) = \int_{B_R} G_R(x,y)\, f(y)\, dy \tag{A.3}$$

where G_R is the Green's function relative to B_R and zero boundary condition.
Let

$$u_\infty(x) = \lim_{R\uparrow\infty} u_R(x) \qquad \text{(possibly } +\infty).$$

Note that, by monotone convergence of G_R,

$$u_\infty(x) = c \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} f(y)dy = \frac{c}{|x|^{n-2}} * f$$

(possibly $+\infty$), where $c/|x|^{n-2}$ is the fundamental solution. Remark that there are only two possibilities, either $u_\infty(x) = +\infty$ $\forall x$ or $u_\infty(x) < +\infty$ $\forall x$. Indeed suppose for example that $u_\infty(0) < +\infty$.
Write

$$u_\infty(x) = c \int_{|y| \leq 2|x|} \frac{f(y)}{|x-y|^{n-2}} + c \int_{|y| > 2|x|} \frac{f(y)}{|x-y|^{n-2}} .$$

The first integral is finite (for each fixed x) while the second integral is bounded by $2^{n-2}\, c \int \frac{f(y)}{|y|^{n-2}}\, dy$. Hence $u_\infty(x) < \infty$.

If we make the assumption that

$$u_\infty(0) = c \int \frac{f(y)}{|y|^{n-2}}\, dy < \infty$$

then $u_\infty(x)$ is finite for each fixed x but it need not be uniformly bounded on $\mathbb{R}^n$.

Lemma A.1. f *satisfies property (H) iff*

$$\frac{c}{|x|^{n-2}} * f \in L^\infty(\mathbb{R}^n) \ . \tag{A.4}$$

Proof. Suppose first that (H) holds. By adding a constant we may always assume that $U \geq 0$ in $\mathbb{R}^n$. By the maximum principle

$$u_R \leq U \qquad \text{on } B_R$$

and therefore

$$u_\infty = \frac{c}{|x|^{n-2}} * f \leq U \ . \tag{A.5}$$

Conversely, the function $\frac{c}{|x|^{n-2}} * f$ provides a bounded solution of (A.1).

Since U could be any nonnegative solution of (A.1) we have

Corollary A.2. *If (H) holds then* u_∞ *is the minimal positive solution of (A.1).*

As a consequence of minimality we have

Corollary A.3. *If (H) holds then*

$$\liminf_{|x| \to \infty} u_\infty(x) = 0 \ .$$

In fact, any bounded solution U of (A.1) such that

$$\liminf_{|x| \to \infty} U(x) = 0$$

coincides with u_∞. This follows from the fact that the difference of any two bounded solutions of (A.1) is a bounded harmonic function and thus it is a constant.

A stronger way of expressing that u_∞ tends to zero at infinity is the following

Lemma A.4. *Suppose* $u_\infty(x) < \infty \quad \forall x \in \mathbb{R}^n$ *then*

$$\lim_{R \to \infty} \fint_{S_R} u_\infty = 0$$

where $⨍_{S_R}$ *denotes the average on the sphere of radius* R.

Proof. By Fubini we have

$$⨍_{S_R} u_\infty(y)\, dS_y = c \int_{\mathbb{R}^n} f(x) \frac{1}{R^{n-1}} \left[\int_{|y|=R} \frac{dS_y}{|x-y|^{n-2}} \right] dx \ .$$

Note that

$$I(x) = \int_{|y|=R} \frac{dS_y}{|x-y|^{n-2}} = \begin{cases} CR(\frac{R}{|x|})^{n-2} & \text{if } |x| > R \\ I(0) & \text{if } |x| < R \end{cases}$$

with

$$I(0) = \int_{|y|=R} \frac{dS_y}{|y|^{n-2}} = CR$$

(this is a consequence of the fact that $I(x)$ is harmonic in $|x| < R$ and in $|x| > R$; moreover $I(x) = I(|x|)$ and in addition $I(\infty) = 0$).
Hence we have

$$⨍_{S_R} u_\infty = \frac{C}{R^{n-2}} \int_{|x|<R} f(x)\, dx + c \int_{|x|>R} \frac{f(x)}{|x|^{n-2}}\, dx \ .$$

Clearly the second integral tends to zero as $R \to \infty$. We estimate the first one by

$$\frac{C}{R^{n-2}} \int_{|x|<R_0} f(x)\, dx + C \int_{R_0<|x|<R} \frac{f(x)}{|x|^{n-2}}\, dx \ .$$

We first choose R_0 so that

$$C \int_{R_0<|x|} \frac{f(x)}{|x|^{n-2}}\, dx < \varepsilon$$

and then R large enough so that

$$\frac{C}{R^{n-2}} \int_{|x|<R_0} f(x)\, dx < \varepsilon \ .$$

Lemma A5. *Any bounded solution* U *of (A.1) such that*

$$\fint_{S_R} U \longrightarrow 0 \qquad \text{as } R \to \infty$$

coincides with u_∞.

This is clear since the difference of two bounded solutions of (A.1) is a constant.

Lemma A.6. *Assume (H). Let* $U \in L^\infty$ *be a function with* $\Delta U \in L^\infty_{loc}$ *satisfying*

$$-\Delta U \leq f \qquad \text{in } \mathbb{R}^n$$

and

$$\fint_{S_R} U \longrightarrow 0 \qquad \text{as } R \longrightarrow \infty .$$

Then $U \leq u_\infty$.

Proof. Set

$$g = -\Delta(u_\infty - U) \geq 0 .$$

Since

$$\fint_{S_R} (u_\infty - U) \longrightarrow 0 \qquad \text{as } R \longrightarrow \infty$$

we may apply Lemma A.5 to conclude that

$$u_\infty - U = \frac{c}{|x|^{n-2}} * g \geq 0.$$

Appendix II

Here we briefly review several proofs of uniqueness for the problem

$$\text{(A.6)} \qquad \begin{cases} -\Delta u = \rho(x)\, f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = \psi \geq 0 & \text{on } \partial\Omega \end{cases}$$

under the assumptions that $\frac{f(t)}{t}$ is decreasing, Ω is a smooth <u>bounded</u> domain and $\rho \geq 0$

Method I. This is the method introduced in [1]. Let u_1 and u_2 be two solutions of (A.6). We have

$$\text{(A.7)} \qquad -\frac{\Delta u_1}{u_1} + \frac{\Delta u_2}{u_2} = \rho\left(\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2}\right)$$

Multiplying (A.7) by $(u_1^2 - u_2^2)$ we obtain

$$\int |\nabla u_1 - \frac{u_1}{u_2} \nabla u_2|^2 + |\nabla u_2 - \frac{u_2}{u_1} \nabla u_1|^2 = \int \rho (\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2})(u_1^2 - u_2^2) \ .$$

It follows that $u_1 = u_2$ on the set $[\rho > 0]$. In particular $\rho f(u_1) = \rho f(u_2)$ on Ω. Going back to (A.6) we see that $u_1 = u_2$.

Method II. Let u_1 and u_2 be two solution of (A.6). We have

$$-(\Delta u_1)\, u_2 + (\Delta u_2)\, u_1 = \rho u_1 u_2 (\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2}) \ . \tag{A.8}$$

Integrating (A.8) on the set $[u_1 > u_2] = E$ we obtain formally

$$-\int_{\partial E} \frac{\partial u_1}{\partial \nu} u_2 + \int_{\partial E} \frac{\partial u_2}{\partial \nu} u_1 = \int_E \rho\, u_1 u_2 (\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2})$$

Note that $u_1 = u_2$ and $\frac{\partial}{\partial \nu}(u_1 - u_2) \le 0$ on ∂E. Thus the lefthand side is nonnegative while the integrand on the righthand side is nonpositive. Similarly, using $F = [u_1 < u_2]$, we are led to

$$\int_\Omega \rho\, u_1 u_2\, |\frac{f(u_1)}{u_1} \ \frac{f(u_2)}{u_2}| = 0 \ .$$

We conclude as above.

To make this argument rigorous we proceed as follows. Let θ be a smooth nondecreasing function such that $\theta(0) = 0$ and $\theta(t) = 1$ for $t \ge 1$, $\theta(t) = -1$ for $t \le -1$. Set

$$\theta_\varepsilon(t) = \theta(t/\varepsilon)$$

Multiplying (A.8) by $\theta_\varepsilon(u_1 - u_2)$ and integrating we obtain

$$\begin{cases} \int [(\nabla u_1)\cdot u_2 - (\nabla u_2)\cdot u_1]\ \theta'_\varepsilon(u_1 - u_2)\cdot\nabla(u_1 - u_2) \\ \quad = \int \rho\, u_1 u_2\, (\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2})\ \theta_\varepsilon(u_1 - u_2) \ . \end{cases} \tag{A.9}$$

Clearly $\qquad \text{LHS} \ge \int (\nabla u_2)(u_2 - u_1)\ \theta'_\varepsilon(u_1 - u_2)\cdot\nabla(u_1 - u_2)$

Note that

$$\int \nabla u_2 (u_2 - u_1) \theta'_\varepsilon(u_1 - u_2) \nabla(u_1 - u_2) = -\int \nabla u_2 \nabla \gamma_\varepsilon(u_1 - u_2)$$

where $$\gamma_\varepsilon(t) = \int_0^t s\ \theta'_\varepsilon(s)\ ds\ .$$

Since $|\gamma_\varepsilon(t)| \le C\ \varepsilon$ and $\Delta u_2 \in L^\infty$ we see that

$$\text{LHS} \ge -\ C\ \varepsilon\ .$$

Going back to (A.9) we obtain, as $\varepsilon \to 0$,

$$\int \rho\ u_1 u_2\ |\frac{f(u_1)}{u_1} \frac{f(u_2)}{u_2}| = 0\ .$$

Method III. This is a variant of Krasnoselkii's method [10]. Let u_1 and u_2 be two solutions. Let

$$\Lambda = \{t \in [0,1];\ \ tu_1 \le u_2 \ \text{ on }\ \Omega\}\ .$$

Clearly Λ contains a neighbourhood of 0. We claim that $1 \in \Lambda$. Suppose not, that
$$t_0 = \sup \Lambda < 1\ .$$

Then

$$-\ \Delta(u_2 - t_0 u_1) = \rho\ f(u_2) - t_0\ \rho\ f(u_1).$$

Fix a positive constant K large enough so that $f(t) + Kt$ is increasing on $[0,\text{Max}\ u_2]$. Then

$$-\ \Delta(u_2 - t_0u_1) + K\rho(u_2 - t_0u_1) = \rho[f(u_2) + Ku_2 - t_0(f(u_1) + Ku_1)]$$

$$\ge \rho[f(t_0u_1) + Kt_0u_1 - t_0(f(u_1) + Ku_1)] = \rho[f(t_0u_1) - t_0f(u_1)] \ge 0$$

(the last inequality follows from the fact that $f(u)/u$ is decreasing). On $\partial\Omega$ we have $u_2 - t_0u_1 = (1 - t_0)\ \varphi \ge 0$.

We distinguish two cases:

Case 1: $\varphi \equiv 0$. Using the strong maximum principle we see that either $u_2 - t_0u_1 > 0$ on Ω with $\frac{\partial}{\partial \nu}(u_2 - t_0u_1) < 0$ on $\partial\Omega$. Then, clearly there is some $\varepsilon > 0$ such that $u_2 - t_0u_1 \ge \varepsilon\ u_1$. Thus $t_0 + \varepsilon \in \Lambda$. Impossible. Or $u_2 - t_0u_1 \equiv 0$. This case is also impossible since we would have, by the equation $\rho f(u_2) = t_0\ \rho f(u_1)$, but $f(t_0u_1) > t_0f(u_1)$.

Case 2: φ is not identically zero. We claim that there is some $\varepsilon > 0$ such that
$$w \equiv u_2 - t_0u_1 \ge \varepsilon u_1\ .$$

Suppose not, that for every $\varepsilon > 0$ there is some point $x_\varepsilon \in \bar{\Omega}$ such that

$$w(x_\varepsilon) < \varepsilon u_1(x_\varepsilon) .$$

Clearly $x_\varepsilon \notin \partial\Omega$ (for ε small). Choosing a point of minimum for the function $(w - \varepsilon u_1)$ we may also assume that

$$\nabla w(x_\varepsilon) = \varepsilon \nabla u_1(x_\varepsilon) .$$

As $\varepsilon \to 0$ (through an appropriate sequence) $x_\varepsilon \to x_0 \in \bar{\Omega}$ such that

$$w(x_0) \leq 0 \quad \text{and} \quad \nabla w(x_0) = 0 .$$

It follows that $w(x_0) = 0$ and thus $x_0 \in \partial\Omega$. This contradicts the strong maximum principle since we have

$$\begin{cases} -\Delta w + K\rho w \geq 0 & \text{in } \Omega , \\ w \geq 0 & \text{on } \partial\Omega , \\ w \text{ not identically zero} . \end{cases}$$

Method IV. This is a variant of Nirenberg's method already presented in the proof of Theorem 2′. It requires further restrictions on f, namely, f is positive, concave and $\int_0^\delta \frac{dt}{f(t)} < \infty$.

We use the new unknown

$$v = \int_0^u \frac{dt}{f(t)} ,$$

or in other words $u = h(v)$ where h satisfies

$$h'(s) = f(h(s)) .$$

The equation for v becomes

$$-\Delta v - f'(h(v))|\nabla v|^2 = \rho .$$

Uniqueness holds provided the function $f'(h(v))$ is nonincreasing in v (see the proof of Theorem 2′). This follows from the assumptions on f.

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