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# Regularity of Minimizers of Relaxed Problems for Harmonic Maps

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We prove that every minimizer on $H^1(\Omega; S^2)$ of the relaxed energy $\int |\nabla u|^2 + 8\pi\lambda L(u)$, where $0 \leqslant \lambda < 1$ and $L(u)$ is the length of a minimal connection connecting the singularities of $u$, is smooth except at a finite number of points.


## INTRODUCTION

Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain. Set

$$H^1(\Omega; S^2) = \{u \in H^1(\Omega; \mathbb{R}^3);\ u(x) \in S^2 \text{ a.e.}\},$$

where $S^2$ is the unit sphere in $\mathbb{R}^3$. Given a boundary data $\varphi\colon \partial\Omega \to S^2$ we define

$$H^1_\varphi(\Omega; S^2) = \{u \in H^1(\Omega; S^2);\ u = \varphi \text{ on } \partial\Omega\}.$$

If $u \in H^1_\varphi(\Omega; S^2)$ is smooth except at a finite number of singularities in $\Omega$ and if moreover $\deg \varphi = 0$, then the length of a minimal connection connecting the singularities has been introduced in [BCL] and is given by

$$L(u) = \operatorname{Min} \sum_{i=1}^{k} d(p_i, n_{\sigma(i)}),$$

where $(p_1, p_2, ..., p_k)$ are the singularities of positive degree (counted



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according to their multiplicity), $(n_1, n_2, ..., n_k)$ are the singularities of negative degree, $d$ is the geodesic distance in $\Omega$, and the minimum is taken over all permutations of the integers $\{1, 2, ..., k\}$. (Since $\deg \varphi_{|\partial\Omega} = 0$ the number of positive singularities is the same as the number of negative singularities.)

For any $u \in H^1(\Omega; S^2)$ the vector field $D(u)$ defined as follows

$$D(u) = (u.u_y \wedge u_z, u.u_z \wedge u_x, u.u_x \wedge u_y)$$

plays an important role (see [BCL]). Set

$$R = \{u \in H^1(\Omega; S^2); u \text{ is smooth except at a finite number of singularities}\}$$

and

$$R_\varphi = R \cap H^1_\varphi(\Omega; S^2).$$

Recall (see [BZ]) that $R$ is dense in $H^1(\Omega; S^2)$ and $R_\varphi$ is dense in $H^1_\varphi(\Omega; S^2)$. If $u \in R$ (with singularities $(a_i)$) then

$$\operatorname{div} D(u) = 4\pi \sum \deg(u, a_i) \delta_{a_i}.$$

If $u \in R_\varphi$ then (see [BCL])

$$L(u) = \frac{1}{4\pi} \sup_{\substack{\zeta: \Omega \to \mathbb{R} \\ \|\nabla\zeta\|_\infty \leqslant 1}} \left\{ \int_\Omega D(u).\nabla\zeta \, dx - \int_{\partial\Omega} (\operatorname{Jac} \varphi)\zeta \, d\sigma \right\}. \tag{1}$$

Clearly this makes sense for any $u \in H^1_\varphi(\Omega; S^2)$ and we shall use formula (1) as a definition of $L$ for a general $u \in H^1_\varphi(\Omega; S^2)$. By a result of [BBC], $L$ is continuous (and even locally Lipschitz) on $H^1_\varphi$.

The functional

$$F_\lambda(u) = \int_\Omega |\nabla u|^2 + 8\pi\lambda L(u), \qquad \lambda \in [0, 1] \tag{2}$$

introduced in [BBC] has some remarkable properties. In particular, it is weakly lower semicontinuous for the weak topology on $H^1$. Thus

$$\min_{u \in H^1_\varphi} F_\lambda(u) \text{ is achieved} \tag{3}$$

and we recall that minimizers of $F_\lambda$ are (weakly) harmonic maps, i.e., they are weak solutions of

$$-\Delta u = u \, |\nabla u|^2 \qquad \text{on } \Omega.$$

For $\lambda = 0$ it is known (see [SU1, SU2]) that mimizers of (3) are smooth except at a finite number of points. Our main result asserts that this is still true for $\lambda \in [0, 1)$.

THEOREM 1. *Every minimizer of $F_\lambda$ is smooth on $\bar{\Omega}$ except at isolated singularities.*

For $\lambda = 0$ it is known (see [BCL]) that the singularities of minimizers have a simple form; i.e., if $x_0$ is a singularity then, for some rotation $R$, $u(x) \simeq \pm R((x - x_0)/|x - x_0|)$ as $x \to x_0$. In particular all singularities have degree $\pm 1$. This last property can be established for $\lambda$ small, but it is an open problem for $\lambda$ large.

Unfortunately, our arguments do not give any information about the nature of singularities when $\lambda = 1$. The case $\lambda = 1$ is very important (because it corresponds to the relaxed energy; see [BBC]), and it would be extremely interesting to decide whether minimizers of $F_1$ are smooth. Partial regularity results for minimizers of $F_1$ have been obtained in [GMS2].

The proof of Theorem 1 is divided into several steps.

*Step* 1. Minimizers of $F_\lambda$ satisfy a reverse Hölder inequality.

*Step* 2. This is used to prove an "ε-regularity lemma."

*Step* 3. One concludes by a blow-up technique (similar to the one used by [SU1]) that singularities are isolated.

## 1. A Reverse Hölder Inequality

The usefulness of "reverse Hölder inequality" was originally discovered by Gehring. It has been extensively used to establish partial regularity (see, e.g., Giaquinta's book [G] and references therein). We shall follow a technique recently introduced by [HKL] in variational problems involving $S^2$-valued maps.

In what follows we fix $\lambda \in [0, 1)$ and some minimizer $u$ of $F_\lambda$ on $H^1_\varphi(\Omega; S^2)$.

THEOREM 2. *There exist constants $q > 2$ and $C$ (depending only on $\lambda$) such that*

$$\left( \fint_{B_r} |\nabla u|^q \right)^{1/2} \leqslant C \left( \fint_{B_{2r}} |\nabla u|^2 \right)^{1/2} \tag{4}$$

*for every ball $B_r$ such that $B_{2r} \subset \Omega$.*

We shall use the following:

LEMMA 1. *Let $B_r$ be a ball contained in $\Omega$. Then*

$$\int_{B_r} |\nabla u|^2 \leqslant \frac{1+\lambda}{1-\lambda} \int_{B_{2r}} |\nabla v|^2,$$
$$\forall v \in H^1(B_r; S^2) \quad \text{such that } v = u \text{ on } \partial B_r. \tag{5}$$

*Proof.* Set

$$w = \begin{cases} v & \text{on} \quad Br \\ u & \text{on} \quad \Omega \setminus B_r. \end{cases}$$

Since $w \in H^1_\varphi$ we have

$$F_\lambda(u) \leqslant F_\lambda(w),$$

i.e.,

$$\int_\Omega |\nabla u|^2 + 8\pi\lambda L(u) \leqslant \int_\Omega |\nabla w|^2 + 8\pi\lambda L(w)$$

and therefore

$$\int_{B_r} |\nabla u|^2 + 8\pi\lambda L(u) \leqslant \int_{B_r} |\nabla v|^2 + 8\pi\lambda L(w). \tag{6}$$

On the other hand,

$$L(u) = \frac{1}{4\pi} \sup_{\substack{\zeta: \Omega \to \mathbb{R} \\ \|\nabla\zeta\|_\infty \leqslant 1}} \left\{ \int_0 D(u).\nabla\zeta \, dx - \int_{\partial\Omega} (\text{Jac } \varphi)\zeta \, d\sigma \right\}$$

and

$$L(w) = \frac{1}{4\pi} \sup_{\substack{\zeta: \Omega \to \mathbb{R} \\ \|\nabla\zeta\|_\infty \leqslant 1}} \left\{ \int_\Omega D(w).\nabla\zeta \, dx - \int_{\partial\Omega} (\text{Jac } \varphi)\zeta \, d\sigma \right\}.$$

It follows that

$$\begin{aligned} L(w) - L(u) &\leqslant \frac{1}{4\pi} \sup_{\substack{\zeta: \Omega \to \mathbb{R} \\ \|\nabla\zeta\|_\infty \leqslant 1}} \left\{ \int_\Omega (D(w) - D(u)).\nabla\zeta \, dx \right\} \\ &= \frac{1}{4\pi} \sup_{\substack{\zeta \\ \|\nabla\zeta\|_\infty \leqslant 1}} \left\{ \int_{B_r} (D(w) - D(u)).\nabla\zeta \, dx \right\} \\ &\leqslant \frac{1}{8\pi} \left\{ \int_{B_r} (|\nabla v|^2 + |\nabla u|^2) \right\}. \end{aligned}$$

Combining this with (6) we obtain

$$\int_{B_r} |\nabla u|^2 - \int_{B_r} |\nabla v|^2 \leqslant \lambda \left[ \int_{B_r} |\nabla u|^2 + \int_{B_r} |\nabla v|^2 \right].$$

We also recall the following result of [HKL].

LEMMA 2. *There is a universal constant $C$ such that, for every $\xi \in \mathbb{R}^3$,*

$$\operatorname*{Min}_{\substack{v \in H^1(B_r; S^2) \\ v = u \text{ on } \partial B_r}} \int_{B_r} |\nabla v|^2 \leqslant C \, \|\nabla_T u\|_{L^2(\partial B_r)} \, \|u - \xi\|_{L^2(\partial B_r)}. \tag{7}$$

For the proof we refer to [HKL, Appendix]. The idea is to consider the usual harmonic extension $\bar{u}$ of $u_{|\partial B_r}$ (with values in $B^3$) and then some appropriate radial projection of $\bar{u}$ on $S^2$.

*Proof of Theorem* 2. Combining Lemma 1 and Lemma 2 we obtain

$$\int_{B_r} |\nabla u|^2 \leqslant C_\lambda [\|\nabla_T u\|_{L^2(\partial B_r)} \, \|u - \xi\|_{L^2(\partial B_r)}].$$

We now follow the argument of [HKL, Theorem 4.1] to deduce that for every $\delta > 0$

$$\int_{B_r} |\nabla u|^2 \leqslant \delta \int_{B_{2r}} |\nabla u|^2 + \frac{C}{\delta r^2} \left( \int_{B_{2r}} |\nabla u|^p \right)^{2/p}$$

with $p = 6/5$.

This implies, using Hölder's inequality, that

$$\int_{B_r} |\nabla u|^2 \leqslant 2\delta \int_{B_{2r}} |\nabla u|^2 + \frac{C}{\delta r^3} \left( \int_{B_{2r}} |\nabla u| \right)^2$$

i.e.,

$$⨍_{B_r} |\nabla u|^2 \leqslant 16\delta ⨍_{B_{2r}} |\nabla u|^2 + \frac{C}{\delta} \left( ⨍_{B_{2r}} |\nabla u| \right)^2.$$

Fixing $\delta < 1/16$ we may now apply this reverse-Hölder inequality to conclude the existence of a $q > 2$ such that

$$\left( ⨍_{B_r} |\nabla u|^q \right)^{1/q} \leqslant C \left( ⨍_{B_{2r}} |\nabla u|^2 \right)^{1/2}.$$

## 2. $\varepsilon$-REGULARITY

In what follows, we fix $\lambda \in [0, 1)$ and some minimizer $u$ of $F_\lambda$ on $H^1_\varphi(\Omega; S^2)$.

THEOREM 3. *There is some $\varepsilon_0 > 0$ (depending only on $\lambda$) such that if*

$$\frac{1}{r}\int_{B_r} |\nabla u|^2 < \varepsilon_0 \qquad \textit{for some ball } B_r,$$

*then $u$ is smooth on $B_{r/4}$ (and $u$ is a minimizing harmonic map on $B_{r/4}$ in the usual sense).*

The proof relies on several lemmas.
Set for $\mu \in (0, 1)$

$$W_\mu^- = \{\sigma \in S^2, |P\sigma| < \mu\}$$
$$W_\mu^+ = \{\sigma \in S^2, |P\sigma| > \mu\},$$

where $P\colon S^2 \to \mathbb{C}$ denotes the stereographic projection. Note that, as $\mu \to 0$, area $(W_\mu^-) \sim \pi\mu^2$.

LEMMA 3. *Let $G \subset \mathbb{R}^3$ be a smooth bounded domain. There exists $\delta > 0$ (depending only on $\lambda \in [0, 1)$ and not on $G$) such that if $\psi\colon \partial G \to S^2$ and $\psi(\partial G) \subset W_\delta^-$ then every $v \in H^1_\psi(G; S^2)$ satisfying*

$$\int_G |\nabla v|^2 \leqslant \operatorname*{Min}_{w \in H^1_\psi(G; S^2)} \int_G |\nabla w|^2 + 8\pi\lambda L(v) \tag{8}$$

*is smooth on $G$.*

*Proof of Lemma* 3. For $\mu \in (0, 1)$ (to be determined later) let

$$G_\mu^\pm = \{x \in G; u(x) \in W_\mu^\pm\}.$$

The proof is divided into three steps.

*Step* 1. We have

$$\int_{G_\mu^+} |\nabla v|^2 \geqslant 8\pi(1 - C\mu^2) L(v) \tag{9}$$

for every $v \in H^1_\psi(G; S^2)$ with $\psi(\partial G) \subset W_\mu^-$ ($C$ is some universal constant).

*Proof of Step* 1. By density (see [BZ]) we may always assume that $v$

has just a finite number of singularities. Following [ABL], we have, using Federer's coarea formula,

$$\int_{G_\mu^+} |\nabla v|^2 \geqslant 2 \int_{G_\mu^+} |D(v)| = 2 \int_{W_\mu^+} \mathscr{H}^1(v^{-1}(\sigma))\, d\sigma.$$

For a.e. $\sigma \in W_\mu^+$, $v^{-1}(\sigma)$ consists of curves connecting the singularities (and possibly some closed loops). Note that there are no curves connecting the singularities to $\psi(\partial G) \subset W_\mu^+$. Thus

$$\mathscr{H}^1(v^{-1}(\sigma)) \geqslant L(v) \qquad \text{for} \quad \text{a.e. } \sigma \in W_\mu^+,$$

and consequently

$$\begin{aligned} \int_{G_\mu^+} |\nabla v|^2 &\geqslant 2(\text{area } W_\mu^+) L(v) \\ &\geqslant 8\pi(1 - C\mu^2) L(v). \end{aligned}$$

*Step* 2. Suppose as above that $\psi \colon \partial G \to W_\delta^-$. Then

$$\operatorname*{Min}_{w \in H_\psi^1(G; S^2)} \int_G |\nabla w|^2 \leqslant \mu^2 \int_{G_\mu^+} |\nabla v|^2 + \int_{G_\mu^-} |\nabla v|^2 \tag{10}$$

for every $v \in H_\varphi^1(G; S^2)$ and every $\mu > \sqrt{\delta}$.

*Proof of Step* 2. Fix a map $\Phi \colon S^2 \to S^2$ satisfying

$$\begin{aligned} &\Phi(S^2) \subset W_{\mu^2}^- \\ &\Phi = Id \qquad \text{on } W_{\mu^2}^- \\ &|\nabla\Phi| \leqslant 1 \qquad \text{on } S^2 \\ &|\nabla\Phi| \leqslant \mu^2 \qquad \text{on } W_\mu^+. \end{aligned}$$

To construct such a $\Phi$ one can, using sterographic projection, define $\Phi \colon \mathbb{C} \to \mathbb{C}$ by

$$\begin{aligned} \Phi(z) &= z \qquad &\text{for} \quad |z| < \mu^2 \\ \Phi(z) &= \frac{\mu^4}{\bar{z}} \qquad &\text{for} \quad |z| \geqslant \mu^2. \end{aligned}$$

It is clear that $|\nabla\Phi| \leqslant 1$ everywhere and $|\nabla\Phi| \leqslant \mu^4/|z|^2$ for $|z| \geqslant \mu^2$. In particular $|\nabla\Phi| \leqslant \mu^2$ for $|z| > \mu$.

Given $v \in H_\varphi^1$, set

$$w = \Phi \circ v$$

so that $w \in H^1_\varphi(G; S^2)$ (since $\mu > \sqrt{\delta}$). We have

$$\int |\nabla w|^2 = \int_{G_\mu^+} |\nabla w|^2 + \int_{G_\mu^-} |\nabla w|^2$$

$$\leqslant \mu^2 \int_{G_\mu^+} |\nabla v|^2 + \int_{G_\mu^-} |\nabla v|^2.$$

This proves Step 2.

*Step* 3. *Proof of Lemma* 3 *Completed.* Let $v$ satisfy (8). By (10) we have

$$\int_G |\nabla v|^2 \leqslant \mu^2 \int_{G_\mu^+} |\nabla v|^2 + \int_{G_\mu^-} |\nabla v|^2 + 8\pi\lambda L(v)$$

and thus

$$(1-\mu^2) \int_{G_\mu^+} |\nabla v|^2 \leqslant 8\pi L(v).$$

Combining this with Step 1 we have

$$8\pi(1-\mu^2)(1-C\mu^2) L(v) \leqslant 8\pi\lambda L(v).$$

Now choose $\mu > 0$ small enough so that

$$(1-\mu^2)(1-C\mu^2) > \lambda$$

and then choose $0 < \delta < \mu^2$. It follows that

$$L(v) = 0.$$

Going back to (8) we see that $v$ is a minimizing harmonic map. We may now invoke [HKW] to conclude that $v$ is smooth (since $\psi(\partial G)$ is contained in a hemisphere). Alternatively, we may also invoke [SU1] together with the fact that $L(v) = 0$.

This concludes the proof of Lemma 3. With its help we are going to prove Theorem 3.

*Proof of Theorem* 3. We split the proof in two steps.

*Step* 1. Let $B \subset \Omega$ be a ball such that

$$u_{|\partial B} \in H^1 \qquad \text{and} \qquad u(\partial B) \subset W_\delta^-, \tag{11}$$

where $\delta$ is defined in Lemma 3. Then

$$\int_B |\nabla v|^2 \leqslant \min_{\substack{w \in H^1_\psi(B;S^2) \\ v=u \text{ on } \partial B}} \int |\nabla v|^2 + 8\pi\lambda L_B(u), \tag{12}$$

where $L_B$ denotes the length of a minimal connection connecting the singularities of $u$ *in* $B$ (without connections to the boundary). Moreover $u$ is smooth in $B$.

*Proof.* Let $u_0$ be a minimizer for

$$\min_{\substack{v \in H^1_\psi(B;S^2) \\ v=u \text{ on } \partial B}} \int |\nabla v|^2.$$

By (11) and [HKW], $u_0$ is smooth inside $B$. Set

$$w = \begin{cases} u_0 & \text{on } B \\ u & \text{on } \Omega \setminus B. \end{cases}$$

Since $u$ is a minimizer for $F_\lambda$ on $H^1_\varphi(\Omega; S^2)$ we have

$$\int |\nabla u|^2 + 8\pi\lambda L(u) \leqslant \int |\nabla w|^2 + 8\pi\lambda L(w).$$

Thus

$$\int_B |\nabla u|^2 + 8\pi\lambda L(u) \leqslant \int_B |\nabla u_0|^2 + 8\pi\lambda L(w). \tag{13}$$

On the other hand (using (1)),

$$L(w) - L(u) \leqslant L(w, u), \tag{14}$$

where

$$L(w, u) = \frac{1}{4\pi} \sup_{\substack{\zeta: \Omega \to \mathbb{R} \\ \|\nabla\zeta\|_\infty \leqslant 1}} \int_\Omega (D(w) - D(u)).\nabla\zeta$$

$$= \frac{1}{4\pi} \sup_{\substack{\zeta: \Omega \to \mathbb{R} \\ \|\nabla\zeta\|_\infty \leqslant 1}} \int_B [D(u_0) - D(u)].\nabla\zeta.$$

But

$$\int_B D(u_0).\nabla\zeta = \int_{\partial B} (D(u_0).n)\zeta.$$

(Since $u_0(B) \subset W_\delta^-$ by [HKW] and so we may approximate $u_0$ by a sequence of smooth maps converging to $u_0$ in $H^1(B; S^2)$ (and also in $H^1(\partial B; S^2)$).) Therefore

$$L(w, u) = \frac{1}{4\pi} \sup_{\substack{\zeta : \Omega \to \mathbb{R} \\ \|\nabla\zeta\|_\infty \leqslant 1}} \left\{ \int_{\partial B} (D(u_0).n)\zeta - \int D(u).\nabla\zeta \right\}$$

$$= L_B(u) \tag{15}$$

(since $u = u_0$ on $\partial B$).

Combining (13), (14), (15) we find

$$\int_B |\nabla u|^2 \leqslant \int_B |\nabla u_0|^2 + 8\pi\lambda L_B(u).$$

This completes the proof of Step 1.

*Step* 2. *Proof of Theorem* 3 *Completed.* Suppose

$$\frac{1}{r} \int_{B_r} |\nabla u|^2 < \varepsilon_0$$

for some ball $B_r \subset \Omega$ and some $\varepsilon_0$ (to be determined later). By Theorem 2 we have

$$\frac{1}{r^3} \int_{B_{r/2}} |\nabla u|^q \leqslant C \left( \frac{1}{r^2} \varepsilon_0 \right)^{q/2}.$$

Thus

$$\int_{r/4}^{r/2} d\rho \int_{S_\rho} |\nabla u|^q \, d\sigma \leqslant C r^{3-q} \varepsilon_0^{q/2}$$

and hence there is some $r_0 \in [r/4, r/2]$ such that

$$\int_{S_{r_0}} |\nabla u|^q \, d\sigma \leqslant C r^{2-q} \varepsilon_0^{q/2},$$

that is,

$$\left( \int_{S_{r_0}} |\nabla u|^q \, d\sigma \right)^{1/q} \leqslant C r^{(2/q)-1} \varepsilon_0^{1/2}.$$

By the Sobolev imbedding we conclude that

$$u(S_{r_0}) \subset W_\mu^-,$$

where

$$\mu = C\varepsilon_0^{1/2}.$$

We now choose $\varepsilon_0$ such that $C\varepsilon_0^{1/2} < \delta$ ($\delta$ given by Lemma 3). By Step 1 above (with $B = B_{r_0}$) we conclude that $u$ is smooth in $B_{r_0}$ and thus in $B_{r/4}$. This concludes the proof of Theorem 3.

*Remark* 1. We may now assert that $\mathscr{H}^1_{\mathrm{loc}}(Z) = 0$, where $Z$ is the singular set of $u$ (as above $u$ is a minimizer of $F_\lambda$ on $H_\varphi$). Let

$$\theta(a) = \limsup_{r \to 0} \frac{1}{r} \int_{B_r(a)} |\nabla u|^2$$

and

$$\tilde{Z} = \{a \in \Omega; \theta(a) > 0\}.$$

Note that if $a \in \Omega \backslash \tilde{Z}$ then $u$ is smooth in a neighborhood of $a$ (by Theorem 3). Thus $Z \subset \tilde{Z}$. Since $\tilde{Z}$ is obviously contained in $Z$ we conclude that $Z = \tilde{Z}$. Since $u \in H^1(\Omega; S^2)$ it follows by a standard covering argument that $\mathscr{H}^1_{\mathrm{loc}}(Z) = 0$. In fact, since $u \in W^{1,q}_{\mathrm{loc}}$ we conclude that $\mathscr{H}^{3-q}_{\mathrm{loc}}(Z) = 0$ (see, e.g., [HKL]).

## 3. The Singular Set Consists of Isolated Points

We prove here that $u$ has only isolated singularities by a variant of the blow-up technique of [SU1, SU2]. Here we rely on a new monotonicity formula of [GMS2]. Let $Z$ be the complement of the largest open set on which $u$ is smooth.

THEOREM 4. *$Z$ consists of isolated points (in $\Omega$).*

*Step* 1. *A Monotonicity Formula*

Recall that for a (standard) minimizing harmonic map $u$ we have the well-known monotonicity formula

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\int_{B_r} |\nabla u|^2\right) \geqslant 0.$$

This is not true any more for minimizers of $F_\lambda$ but we have a variant of this formula due to Giaquinta, Modica, and Souček [GMS2].

Let $u \in R_\varphi$ and let $\mu$ be the one-dimensional Hausdorff measure uniformly distributed over a minimal connection (warning: the minimal connection need not be unique, and so $\mu$ is not uniquely determined by $u$). Set

$$E(r)=\int_{B_r(x_0)} |\nabla u|^2 + 8\pi\lambda \int_{B_r(x_0)} d\mu$$

$$\alpha(r)=\frac{d}{dr}\left[2\int_{B_r(x_0)} \left|\frac{\partial u}{\partial r}\right|^2\right] + 8\pi\lambda \int_{B_r(x_0)} [1-(\zeta . \nu)^2]\, d\mu,$$

where $\zeta$ is the unit vector tangent to the minimal connection at $\nu=(x-x_0)/|x-x_0|$.

Using the formalism of currents (see [GMS1]) these expressions make sense for every $u\in H^1_\varphi$. We have

LEMMA 4. *For every minimizer of $F_\lambda$ we have*

$$\frac{d}{dr}\left(\frac{1}{r}E(r)\right) \doteq \frac{1}{r}\alpha(r).$$

*In particular $(1/r)E(r)$ is nondecreasing in $r$.*

This formula has been established by [GMS2] when $\lambda=1$, but the same argument holds for any $\lambda\in(0,1)$.

*Step 2. The Blow-Up*

Let $x_0\in\Omega$ be any point in $\Omega$ and let $u$ be a minmizer of $F_\lambda$ on $H^1_\varphi$. For simplicty we take $x_0=0$ and we write $B_R=B_R(0)$. By Lemma 4, $(1/r)E(r)$ remains bounded as $r\to 0$ and so does $(1/r)\int_{B_r}|\nabla u|^2$. Set

$$u_\sigma(x)=u(\sigma x), \qquad \text{for} \quad x\in B^3.$$

Then

$$\int_{B_1} |\nabla u_\sigma|^2 = \frac{1}{\sigma}\int_{B_\sigma} |\nabla u_\sigma|^2 \leqslant C \qquad \text{as} \quad \sigma\to 0. \tag{16}$$

Therefore

$$u_{\sigma_n} \rightharpoonup v \qquad \text{weakly in } H^1(B_1).$$

We claim that

$$\frac{\partial v}{\partial r}=0.$$

Indeed we have, by Lemma 4

$$\int_0^{\sigma_n} \alpha(r)\frac{dr}{r} = \int_0^{\sigma_n} \frac{d}{dr}\left(\frac{1}{r}E(r)\right) dr$$
$$\leqslant \frac{1}{\sigma_n} E(\sigma_n) - \lim_{r \to 0} \frac{E(r)}{r} \to 0$$

as $\sigma_n \to 0$.

On the other hand

$$\int_0^{\sigma_n} \frac{\alpha(r)}{r} dr \geqslant 2 \int_0^{\sigma_n} \frac{1}{r} \frac{d}{dr} \int_{B_r} \left|\frac{\partial u}{\partial r}\right|^2$$
$$= 2 \int_0^{\sigma_n} \frac{1}{r} \int_{S_r} \left|\frac{\partial u}{\partial r}\right|^2$$
$$= 2 \int_{B_{\sigma_n}} \frac{1}{r} \left|\frac{\partial u}{\partial r}\right|^2 dr.$$

It follows that $\int_{B_{\sigma_n}} (1/r)\, |\partial u/\partial r|^2 \to 0$. But

$$\int_{B_1} \frac{1}{r} \left|\frac{\partial}{\partial r} u_{\sigma_n}\right|^2 = \int_{B_{\sigma_n}} \frac{1}{r} \left|\frac{\partial u}{\partial r}\right|^2$$

and the claim follows. Therefore we have

$$v(x) = \psi\left(\frac{x}{|x|}\right) \tag{17}$$

for some $\psi \in H^1(S^2, S^2)$. Following the strategy of [SU1, Part 4, Proposition 4.6] we now prove that $u_{\delta_n} \to v$ strongly in $H^1(B_1)$. Since we have

$$-\Delta u_\sigma = u_\sigma\, |\nabla u_\sigma|^2 \qquad \text{on } B_1$$

and

$$\int_{B_1} |\nabla u_\sigma|^2 \leqslant C$$

we deduce from standard elliptic estimates that $\nabla u_\sigma$ is relatively compact in $L^1_{\text{loc}}(B_1)$ and therefore for an appropriate subsequence we may assume that $\nabla u_{\sigma_n} \to \nabla v$ a.e. on $B_1$. In order to conclude that $\nabla u_{\sigma_n} \to \nabla v$ in $L^2$ it suffices to know that

$$\int_{B_1} |\nabla u_\sigma|^q \leqslant C$$

for some $q > 2$. This is a consequence of Theorem 2. Indeed, we have by (4)

$$\left[\frac{1}{\sigma^3}\int_{B_\sigma} |\nabla u|^q\right]^{1/q} \leqslant C\left(\frac{1}{\sigma^3}\int_{B_{2\sigma}} |\nabla u|^2\right)^{1/2} \leqslant \frac{C}{\sigma},$$

i.e.,

$$\frac{1}{\sigma^{3-q}}\int_{B_\sigma} |\nabla u|^q \leqslant \int_{B_1} |\nabla u_\sigma|^q \leqslant C.$$

Thus we have established that $u_{\sigma_n} \to \psi$ strongly in $H^1$.

Finally we show that the singularities of $u$ are isolated (in any compact subset of $\Omega$). As in [SU1], we follow the dimension reduction argument of Federer. Let $(x_n)$ be a subsequence of singularities such that $x_n \to x_0 = 0$, $x_n/|x_n| \to l \in S^2$. We choose $\sigma_n = 2\,|x_n|$. Note that $u_{\sigma_n}$ has a singularity at the point $x_n/(2\,|x_n|) \to l/2$. By Theorem 3, there is some $\varepsilon_0$ such that

$$\forall r,\ \frac{1}{r}\int_{B_r(x_n/(2|x_n|))} |\nabla u_{\sigma_n}|^2 \geqslant \varepsilon_0 \tag{18}$$

(otherwise $u_{\sigma_n}$ would be regular at $x_n/(2\,|x_n|)$).

Since $u_{\sigma_n} \to \psi$ strongly in $H^1$, we may pass to the limit in (18) and conclude that

$$\frac{1}{r}\int_{B_r(l/2)} |\nabla v|^2 \geqslant \varepsilon_0, \qquad \text{for every } r. \tag{19}$$

Since $v(x) = \psi(x/|x_n|)$ the left-hand side in (16) is of the order of $\int_{S^2 \cap B_{2r}(l)} |\nabla_T \psi|^2$. This is impossible since $\psi \in H^1(S^2)$. This completes the proof of Theorem 4.

## 4. Boundary Regularity

Here we complete the proof of Theorem 1 by showing that every minimizer of $F_\lambda$ is smooth in some neighborhood of $\partial\Omega$. This follows essentially the same pattern as above with the following modifications:

(a) *Reverse Hölder Inequality near $\partial\Omega$*

THEOREM 2′. *There exist constants $q > 2$ and $C_1, C_2$ (depending only on $\psi$) such that*

$$\left(\fint_{B_r(x_0)\cap\Omega} |\nabla u|^q\right)^{1/q} \leqslant C_1\left(\fint_{B_{2r}(x_0)\cap\Omega} |\nabla u|^2\right)^{1/2} + C_2\,\|\nabla\varphi\|_{L^\infty(\partial\Omega)}$$

*for any $x_0 \in \partial\Omega$.*

To prove Theorem 2′ we adapt an idea of Jost and Meier [JM], namely we use as testing function in the inequality

$$F_\lambda(u) \leqslant F_\lambda(w)$$

the map

$$w = \Pi_a(u - \eta(x)(u - \bar{\varphi})),$$

where $\bar{\varphi}$ is the usual harmonic extension of $\varphi$, $\eta$ is some appropriate cut-off function with support in $B_s$ (as in [JM, Lemma 1]), and $\Pi_a$ is the radial projection with vertex at some appropriate $a$ (as in [HKL]). As in Lemma 1 of Section 1 we have

$$\int_{B_s} |\nabla u|^2 \leqslant \frac{1+\lambda}{1-\lambda} \int_{B_s} |\nabla w|^2 \leqslant C\left(\frac{1+\lambda}{1-\lambda}\right) \int_{B_s} |\nabla(u - \eta(u - \bar{\varphi}))|^2.$$

We then proceed as in [JM] to derive the conclusion of Theorem 2′.

(b) *$\varepsilon$-Regularity*

The counterpart of Theorem 3 is

THEOREM 3′. *There is some $\varepsilon_1 > 0$ (depending only on $\lambda$) and $r_0$ (depending only on $\varphi$) such that if*

$$\frac{1}{r} \int_{B_r(x_0) \cap \Omega} |\nabla u|^2 < \varepsilon_1 \qquad \textit{for some} \quad r < r_0 \quad \textit{and} \quad x_0 \in \partial\Omega,$$

*then $u$ is smooth on $B_{r/4}(x_0) \cap \Omega$ (and $u$ is a minimizing harmonic map on $B_{r/4}(x_0) \cap \Omega$ in the usual sense).*

The proof is essentially the same as the proof of Theorem 3, except that $r_0$ is chosen so small that $\varphi(\partial\Omega \cap B_r(x_0)) \subset W_{\delta_1}^-$ for some suitable $\delta_1$.

(c) *Monotonicity Formula*

Set $E(r)$ and $\alpha(r)$ as in Section 4 (Step 1) except that $B_r$ is replaced by $B_r(x_0) \cap \Omega$ with $x_0 \in \partial\Omega$. The counterpart of Lemma 4 is

LEMMA 4′.

$$\frac{d}{dr}\left(\frac{1}{r} E(r)\right) \geqslant \frac{1}{r} \alpha(r) - C$$

*for some constant $C$ depending only on $\|\nabla\varphi\|_{L^\infty(\partial\Omega)}$.*

The proof has the same ingredients as in [GMS2] together with Lemma 1.3 of [SU2].

(d) *Blow-Up and Conclusion*

Let $x_0 \in \partial\Omega$ and let $u$ be a minimizer of $F_\lambda$ on $H^1_\varphi$. For simplicity assume that $\partial\Omega$ is flat near $x_0$ with outward normal $(0, 0, -1)$. Set

$$u_\sigma(x) = u(\sigma x)$$

for $x \in B^3_+ = \{(x_1, x_2, x_3) \in B^3;\ x_3 \geqslant 0\}$. As in Step 2 of Section 3, $u_{\sigma_n} \rightharpoonup v$ weakly in $H^1(B^3_+)$ and $\partial v/\partial r = 0$. Therefore $v(x) = \psi(x/|x|)$ for some $\psi \in H^1(S^2_+; S^2)$. As above $u_{\sigma_n}$ is bounded in $L^q(B^3_+)$ for some $q > 2$ and $u_{\sigma_n} \to v$ strongly in $H^1(B^3_+)$. In particular $v \in W^{1,q}(B^3_+; S^2)$ and so $\psi \in W^{1,q}(S^2_+; S^2)$. On the other hand $v$ is (weakly) harmonic and constant on $\partial B^3_+ \cap [x_3 = 0]$. Hence $\psi$ is weakly harmonic from from $S^2_+$ into $S^2$ and $\psi$ is constant on $\partial S^2_+$. Since $\psi \in W^{1,q}(S^2_+)$ with $q > 2$, it follows (by bootstrap) that $\psi$ is smooth on $S^2_+$. Using a result of [L] we deduce that $\psi$ is constant on $S^2_+$. Hence

$$\int_{B_1^+} |\nabla u_\sigma|^2 < \varepsilon_1 \qquad \text{for } \sigma \text{ small enough,}$$

i.e.,

$$\frac{1}{r}\int_{B_r(x_0)\cap\Omega} |\nabla u|^2 < \varepsilon_1 \qquad \text{for } r \text{ small enough.}$$

By Theorem 3′ we conclude that $u$ is smooth on $B_{r/4}(x_0) \cap \Omega$.

## 5. A Variant of the Relaxed Energy

Here we assume that $\varphi\colon \partial\Omega \to S^2$ is given and smooth but deg $\varphi$ need not be zero. We *fix* $v \in H^1_\varphi(\Omega; S^2)$ and we set

$$L(u, v) = \frac{1}{4\pi} \sup_{\substack{\zeta\colon \Omega \to \mathbb{R} \\ \|\nabla\zeta\|_\infty \leqslant 1}} \left\{ \int_\Omega D(u) - D(v)).\nabla\zeta \, dx \right\}.$$

(Note that $L(u, v) = L(u)$ if deg $\varphi = 0$ and $v$ is smooth.) The functional

$$\Phi_\lambda(u) = \int |\nabla u|^2 + 8\pi\lambda L(u, v), \qquad \lambda \in [0, 1]$$

introduced in [BBC] is also weakly lower semicontinuous for the weak topology on $H^1$ and minimizers of $\Phi_\lambda$ on $H^1_\varphi$ are weakly harmonic maps.

THEOREM 5. *Assume $v$ is smooth on $\bar{\Omega}$ except at isolated singularities then any minimizer $u$ of $\Phi_\lambda$ is smooth except at isolated singularities.*

*Sketch of Proof.* Let $S$ be the singular set of $v$. Using the same arguments as in the proof of Theorem 1 it is easy to see that on every compact subset of $\bar{\Omega} \setminus S$, $u$ has only isolated singularities. It could still happen that singularities of $u$ accumulate on $S$. This is excluded by a blow-up analysis centered at a point on $S$.

*Remark* 2. We recall that in [BBC] we have proved that if $\deg \varphi \neq 0$ the minimizers of $\Phi_\lambda(u)$ are distinct for a sequence $(\lambda_n)$. By Theorem 5 these minimizers are smooth except at isolated points. For example, if $\varphi(x) = x$ we find infinitely many distinct harmonic maps with isolated singularities and such that $u = \varphi$ on $\partial\Omega$.

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