Relaxed Energies for Harmonic Maps

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INTRODUCTION

Let $\Omega \subset \mathbb{R}^3$ be an open bounded set such that $\partial\Omega$ is smooth. Set

$$H^1(\Omega; S^2) = \{u \in H^1(\Omega; \mathbb{R}^3); |u(x)| = 1 \text{ a.e.}\}$$

and

$$H^1_\varphi(\Omega; S^2) = \{u \in H^1(\Omega; S^2); u = \varphi \text{ on } \partial\Omega\},$$

where $\varphi: \partial\Omega \to S^2$ is a given boundary data.

If $u \in H^1(\Omega; S^2)$ is smooth on $\overline{\Omega}$ except at a finite number of point singularities in Ω, and if moreover, $\deg \varphi = \deg(u|_{\partial\Omega}) = 0$, then the length of a minimal connection connecting the singularities has been introduced by Brezis, Coron and Lieb in [BCL] and is given by

$$L(u) = \min_{\sigma \in \sigma_k} \sum_{i=1}^{k} d(p_i, n_{\sigma(i)})$$

where $(p_1, p_2, \ldots, p_k)$ are the singularities of positive degree (counted according to their multiplicity), $(n_1, \ldots, n_k)$ are the singularities of negative degree, d is the geodesic distance in Ω and the minimum is taken over all possible permutations σ of the integer $\{1, 2, \ldots, k\}$. (Since $\deg(u|_{\partial\Omega}) = 0$ the number of positive singularities is the same as the number of negative singularities.)

For any $u \in H^1(\Omega; S^2)$ the vector field $D(u)$, defined as follows,

$$D(u) = (u \cdot u_y \wedge u_z, u \cdot u_z \wedge u_x, u \cdot u_x \wedge u_y)$$

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plays an important role in [BCL]. If u is smooth except at a finite number of singularities (p_i, n_i) we recall (see [BCL], Appendix B) that

$$\operatorname{div} D(u) = 4\pi(\Sigma\delta_{p_i} - \Sigma\delta_{n_i}) \quad \text{in} \quad \mathcal{D}'(\Omega).$$

If in addition $\deg(u|_{\partial\Omega}) = 0$, we also know that (see [BCL], Section IV)

$$L(u) = \sup_{\substack{\xi:\Omega\to\mathbb{R} \\ \|\nabla\xi\|_\infty \le 1}} \left\{ \sum_{i=1}^{k} \xi(p_i) - \sum_{i=1}^{k} \xi(n_i) \right\}.$$

Note that

$$\sum_{i=1}^{k} \xi(p_i) - \sum_{i=1}^{k} \xi(n_i) = \int_\Omega \left(\sum_{i=1}^{k} \delta_{p_i} - \sum_{i=1}^{k} \delta_{n_i} \right) \xi = \frac{1}{4\pi} \int_\Omega \operatorname{div} D(u)\xi =$$
$$= -\frac{1}{4\pi} \int_\Omega D(u) \cdot \nabla\xi + \frac{1}{4\pi} \int_{\partial\Omega} (D(u) \cdot n)\xi \, d\sigma$$

where n denotes the outward normal to $\partial\Omega$. We also recall that $D(u) \cdot n$ depends only on $\varphi = u|_{\partial\Omega}$ and more precisely $D(u) \cdot n = \varphi \cdot \varphi_x \wedge \varphi_y$, where x, y are orthonormal coordinates on $\partial\Omega$ such that $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, n\right)$ is a direct basis. It follows that if u is smooth except at a finite number of point singularities in Ω and $\deg(u|_{\partial\Omega}) = 0$ then

$$(1) \qquad L(u) = \frac{1}{4\pi} \sup_{\substack{\xi:\Omega\to\mathbb{R} \\ \|\nabla\xi\|_\infty \le 1}} \left\{ \int_\Omega D(u) \cdot \nabla\xi - \int_{\partial\Omega} D(u) \cdot n\xi \, d\sigma \right\}.$$

We shall use formula (1) as a *definition for* L. Clearly, it makes sense for any $u \in H^1(\Omega; S^2)$ with $\varphi = u|_{\partial\Omega} \in H^1(\partial\Omega; S^2)$ provided

$$(2) \qquad \deg(u|_{\partial\Omega}) = \frac{1}{4\pi} \int_{\partial\Omega} D(u) \cdot n = 0.$$

(The first equality in (2) is a definition but it coincides with the usual degree when u is C^1 on $\partial\Omega$, and can be extended to maps in $H^1(\partial\Omega; S^2)$ because of the density result of Schoen and Uhlenbeck [SU2]). Throughout Sections 1 and 2 of this paper, we assume (at least) that $\varphi \in H^1(\partial\Omega; S^2) \cap C^0(\partial\Omega; S^2)$ and $\deg(\varphi; \partial\Omega) = 0$. Our first result asserts that L is continuous on H^1_φ.

Theorem 1. *We have the following inequality*

$$(3) \qquad |L(u) - L(v)| \le C\|\nabla u - \nabla v\|_2(\|\nabla u\|_2 + \|\nabla v\|_2) \quad \forall\, u, v \in H^1_\varphi.$$

The quantity L is very useful when dealing with questions of approximation of maps in $H^1(\Omega; S^2)$ by smooth maps from Ω into S^2. We recall that smooth maps from Ω into S^2 are not dense in $H^1(\Omega; S^2)$, (see [SU2], [BeZ], [Be1], [Be2]). We have the following result:

Theorem 2. *Let u be in $H^1_\varphi(\Omega; S^2)$. Then there exists a sequence of maps $v_n \in H^1_\varphi(\Omega; S^2) \cap C^0(\overline{\Omega}; S^2)$ such that:*

$$v_n \rightharpoonup u \quad \text{weakly in} \quad H^1$$

and

$$\overline{\lim} \int_\Omega |\nabla v_n|^2 \leq \int_\Omega |\nabla u|^2 + 8\pi L(u). \tag{4}$$

If, in addtion, $\varphi \in C^\infty(\partial\Omega; S^2)$ then one may take $v_n \in C^\infty(\overline{\Omega}; S^2)$.

As direct consequences of Theorem 2 we have the following corollaries

Corollary 1. *With the same assumptions as in Theorem 2 we have:*

$$\inf_{v \in H^1_\varphi \cap C^0 (resp.\ C^\infty)} \int_\Omega |\nabla(u - v)|^2 \leq 8\pi L(u). \tag{5}$$

Corollary 2. *We have:*

$$\inf_{u \in H^1_\varphi} \left\{ \int_\Omega |\nabla u|^2 + 8\pi L(u) \right\} = \inf_{v \in H^1_\varphi \cap C^0 (resp.\ C^\infty)} \int_\Omega |\nabla v|^2. \tag{6}$$

Clearly we have

$$\mu_\varphi \equiv \min_{u \in H^1_\varphi} \int_\Omega |\nabla u|^2 \leq \inf_{v \in H^1_\varphi \cap C^0} \int |\nabla v|^2 \equiv \overline{\mu}_\varphi.$$

As pointed out by Hardt and Lin [HL] (see also [B1]) one can construct smooth maps

$$\mu_\varphi < \overline{\mu}_\varphi.$$

They also raised the very interesting question (which is still unanswered) whether the infimum defining $\overline{\mu}_\varphi$ is achieved. The main difficulty comes from the fact that if (v_n) is a minimizing sequence for $\overline{\mu}_\varphi$ and $v_n \rightharpoonup v$

weakly in H^1, then v need not be continuous on $\overline{\Omega}$ (and hence it is not clear whether v is a minimizer for $\overline{\mu}_\varphi$). In trying to attack this problem and also in view of Theorem 2 it seems natural to introduce the following "energy":

$$F(u) = E(u) + 8\pi L(u), \quad \text{for } u \text{ in } H^1_\varphi \tag{7}$$

where $E(u) = \int_\Omega |\nabla u|^2 dx$ is the usual energy. Obviously F coincides with E on smooth maps, and by Corollary 2 we have

$$\inf_{u \in H^1_\varphi} F(u) = \inf_{v \in H^1_\varphi \cap C^0 \, (resp.\, C^\infty)} E(v). \tag{8}$$

The main interest of F lies in the following property:

Theorem 3. *F is sequentially lower semi-continuous on H^1_φ for the weak H^1 topology.*

As a consequence, we obtain:

Corollary 3. *We have*

$$\inf_{u \in H^1_\varphi} F(u) \quad \textit{is achieved.} \tag{9}$$

Also, every minimizing sequence for $\overline{\mu}_\varphi$ converges (up to a subsequence) weakly in H^1 to some minimizer for F on H^1_φ.

In view of Theorems 2 and 3, the function F is the largest sequentially lower semi-continuous function on H^1_φ which is less than E on $H^1_\varphi \cap C^0$ (resp. C^∞). This means that F plays the role of a "*relaxed energy*."

Assuming one is able to show that a minimizer for (9) is continuous, this would answer the Hardt–Lin problem. Unfortunately we have only partial regularity results for the minimizers of F. We shall return to the question of regularity in a forthcoming paper. In Section 2 we discuss some properties of the minimizers for (9), in particular we have:

Theorem 4. *Every minimizer u for F in H^1_φ satisfies*

$$-\Delta u = u|\nabla u|^2 \quad \text{in} \quad \mathcal{D}'(\Omega; \mathbf{R}^3) \tag{10}$$

that is, u is weakly harmonic.

Assume that φ is a smooth map. By a result of [SU2] one knows that minimizers for E in H^1_φ are smooth except at a finite number of points.

Moreover, by [BCL], each singularity has degree ± 1 and the behavior of u near each singularity is well understood. In interesting situations the singular set of every minimizer of E on H^1_φ is not empty (otherwise the Hardt–Lin problem is irrelevant). We denote this property by (P_φ) and we have:

Theorem 5. *Property (P_φ) holds if and only if $\mu_\varphi < \overline{\mu}_\varphi$. In this case the set of minimizers for E on H^1_φ and the sets of minimizers for F on H^1_φ are disjoint.*

Note that Theorem 5 provides, in interesting situations, the existence of non-minimizing weakly harmonic maps with given boundary data φ. In fact, we shall prove in Section 3 the following:

Theorem 6. *Assume* $\deg(\varphi; \partial Omega) = 0$ *and P_φ holds or* $\deg(\varphi; \partial\Omega) \neq 0$, *then there exist infinitely many weakly harmonic maps having φ as boundary value.*

Finally, let us mention that a weaker form of the Hardt–Lin problem is still open: it is not known whether every smooth map φ (of degree zero) admits a smooth harmonic extension in Ω.

Some of the results in this paper answer questions raised in [B2].

1. Basic Properties of L and F

We start with:

Proof of Theorem 1. It is useful to set, for u, v in H^1_φ,

$$L(u,v) = \sup_{\substack{\xi:\Omega\to\mathbb{R} \\ \|\nabla\xi\|_\infty \le 1}} \int_\Omega (D(u) - D(v))\nabla\xi = L(v,u). \tag{11}$$

Clearly we have, for u, v in H^1_φ

$$\int_\Omega D(u)\cdot\nabla\xi - \int_{\partial\Omega} D(u)\cdot n\,\xi d\sigma = \int_\Omega D(v)\cdot\nabla\xi - \int_{\partial\Omega} D(v)\cdot n\,\xi d\sigma + \int_\Omega (D(u) - D(v))\cdot\nabla\xi$$

(since $D(u)\cdot n$ depends only on φ) and therefore

$$|L(u) - L(v)| \le L(u,v). \tag{12}$$

In order to estimate $L(u,v)$ we proceed as follows:

$$(D(u) - D(v)) \cdot \nabla\xi = \mathrm{I} + \mathrm{II} + \mathrm{III}$$

where

$$\begin{aligned}
\mathrm{I} &= (u-v)\cdot[(u_y \wedge u_z)\xi_x + (u_z \wedge u_x)\xi_y + (u_x \wedge u_y)\xi_z] \\
\mathrm{II} &= v \cdot [(u_y - v_y) \wedge u_z\xi_x + (u_z - v_z) \wedge u_x\xi_y + (u_x - v_x) \wedge u_y\xi_z] \\
\mathrm{III} &= v \cdot [v_y \wedge (u_z - v_z)\xi_x + v_z \wedge (u_x - v_x)\xi_y + v_x \wedge (u_y - v_y)\xi_z].
\end{aligned}$$

By the Cauchy–Schwarz inequality, we clearly have:

$$\left|\int_\Omega \mathrm{II}\right| \le C\|\nabla(u-v)\|_2\|\nabla u\|_2$$

$$\left|\int_\Omega \mathrm{III}\right| \le C\|\nabla(u-v)\|_2\|\nabla v\|_2.$$

We now turn to the first term which we estimate using the following identity:

$$\text{(13)} \qquad \begin{cases} \displaystyle\int_\Omega \mathrm{I} = \frac{1}{2}\int_\Omega u\cdot[(u-v)_y \wedge u_z + u_y \wedge (u-v)_z]\xi_x \\ \displaystyle\qquad + \frac{1}{2}\int_\Omega u\cdot[(u-v)_z \wedge u_x + u_z \wedge (u-v)_x]\xi_y \\ \displaystyle\qquad + \frac{1}{2}\int_\Omega u\cdot[(u-v)_x \wedge u_y + u_x \wedge (u-v)_y]\xi_z. \end{cases}$$

This identity is easily established first when u, v are in $C^\infty(\overline{\Omega};\mathbf{R}^3)$, $u - v \in C_0^\infty(\Omega;\mathbf{R}^3)$ and $\xi \in C^\infty(\overline{\Omega})$: write, $(u_y \wedge u_z) = \frac{1}{2}(u \wedge u_z)_y + \frac{1}{2}(u_y \wedge u)_z$ etc..., integrate by parts, and note that:

$$(u\wedge u_z)\xi_{xy} + (u_y\wedge u)\xi_{xz} + (u\wedge u_x)\xi_{yz} + (u_z\wedge u)\xi_{xy} + (u\wedge u_y)\xi_{xz} + (u_x\wedge u)\xi_{yz} = 0.$$

The general case ($u, v \in H^1_\varphi$ and $\nabla\xi \in L^\infty$) follows by density. We deduce from (13) that

$$\left|\int_\Omega \mathrm{I}\right| \le C\|\nabla(u-v)\|_2\|\nabla u\|_2.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Let us first assume that $\varphi \in C^\infty$. Set

$$R^\infty_\varphi = \left\{u \in H^1_\varphi(\Omega;S^2); \exists a_1, a_2, \ldots, a_N \in \Omega, u \in C^\infty\left(\overline{\Omega} \setminus \bigcup_{i=1}^N \{a_i\}; S^2\right)\right\},$$

that is, R_φ^∞ is the subset of maps in H_φ^1 which are smooth except at most at a finite number of points. We recall (see [BeZ], Theorem 4bis) that R_φ^∞ is dense in H_φ^1.

Given $u \in H_\varphi^1$ and given n, we fix some $u_n \in R_\varphi^\infty$ such that

$$\|\nabla(u_n - u)\|_2 < 1/n.$$

It follows that $\text{meas}\{u \in \Omega;\ |u_n(x) - u(x)| > n^{-1/2}\} \leq C/n$. We now apply Theorem 2 of [Be1] (see also the remark at the end of Section II.3 in [Be1]) to u_n and we find some map $v_n \in C^\infty(\overline{\Omega}; S^2)$ with $v_n = \varphi$ on $\partial\Omega$ such that:

$$\begin{cases} E(v_n) \leq E(u_n) + 8\pi L(u_n) + 1/n, \\ \text{meas}\{x \in \Omega; v_n(x) \neq u_n(x)\} < 1/n. \end{cases} \tag{14}$$

Since $u_n \to u$ in H_φ^1 it follows that $L(u_n) \to L(u)$ (by Theorem 1) and therefore v_n is bounded in H^1. On the other hand

$$\text{meas}\{x \in \Omega; |v_n(x) - u(x)| > n^{-1/2}\} \leq (C+1)/n$$

and thus $v_n \to u$ a.e. We deduce that $v_n \rightharpoonup u$ weakly in H^1. Passing to the limit in (14) we obtain

$$\overline{lim}_{n\to+\infty} E(v_n) \leq E(u) + 8\pi L(u).$$

In the case where $\varphi \in C^0$ the proof is exactly the same except that R_φ^∞ is replaced by:

$$R_\varphi^0 = \left\{ u \in H_\varphi^1(\Omega; S^2); \exists a_1, a_2, \ldots, a_N \in \Omega, u \in C^0\left(\overline{\Omega} \setminus \sum_{i=1}^N \{a_i\}; S^2\right)\right\}.$$

Remark 1. Inequality (5) in Corollary 1 is optimal in the sense that given any $\delta > 0$ there is some $\varphi \in C^\infty(\partial\Omega; S^2)$ and some $u \in H_\varphi^1(\Omega; S^2)$ such that:

$$\inf_{v \in H_\varphi^1 \cap C^\infty} \int |\nabla(u - v)|^2 \geq 8\pi(1 - 2\delta)L(u) \tag{15}$$

with $L(u) = 2 - \delta^2$ (when Ω is the unit ball). Here is an example.

Let Ω be the unit ball with north pole N and south pole S. Along the NS axis we place two ε-dipoles with the same orientation: $[p_1, n_1]$ is centered at N, $[p_2, n_2]$ is centered at S (for more details, see the Example

in Section 2 of [B1]). For the map u obtained by gluing these dipoles we have, on Ω, $L(u) = 2 - \varepsilon$, and $E(u) \leq 16\pi\varepsilon + 2\varepsilon$.

On the other hand we claim that

$$E(u) + E(v) \geq 8\pi L(u), \quad \forall u \in H^1_\varphi,\ \forall v \in H^1_\varphi \cap C^0. \tag{16}$$

Indeed, since $v \in H^1_\varphi \cap C^0$, div $D(v) = 0$ and thus

$$\int_{\partial\Omega} D(u) \cdot n\xi d\sigma = \int_{\partial\Omega} D(v) \cdot n\xi d\sigma = \int_\Omega D(v) \cdot \nabla\xi.$$

Inserting this in (1) we obtain

$$\begin{aligned} L(u) &= \frac{1}{4\pi} \sup_{\substack{\xi:\Omega\to\mathbb{R} \\ \|\nabla\xi\|_\infty \leq 1}} \int_\Omega [D(u) \cdot \nabla\xi - D(v) \cdot \nabla\xi] \\ &\leq \frac{1}{4\pi} \int_\Omega |D(u)| + |D(v)| dx \leq \frac{1}{8\pi}(E(u) + E(v)). \end{aligned}$$

Next, note that

$$\begin{aligned} \int |\nabla(u - v)|^2 dx &\geq -2\|\nabla u\|_2 \|\nabla v\|_2 + E(v) \geq (1 - \delta)E(v) - \frac{1}{\delta}E(u) \\ &\geq (1 - \delta)(8\pi L(u) - E(u)) - \frac{1}{\delta}E(u). \end{aligned}$$

We derive (15) by choosing $\varepsilon = \delta^2/2$.

There is a related notion of minimal connection defined as follows. Given $u \in H^1(\Omega; S^2)$ set

$$\tilde{L}(u) = \sup_{\substack{\xi:\Omega\to\mathbb{R} \\ \|\nabla\xi\|_\infty \leq 1,\ \xi=0 \text{ on } \partial\Omega}} \int D(u) \cdot \nabla\xi.$$

Note that $\tilde{L}(u)$ makes sense for any u in $H^1(\Omega; S^2)$—even if $\deg(u|_{\partial\Omega}) \neq 0$, or even if $\deg(u|_{\partial\Omega})$ is not well defined. In the case where u is smooth on $\overline{\Omega}$, except at a finite number of point singularities in Ω, then $\tilde{L}(u)$ coincides with the length of a minimal connection, "allowing connections to $\partial\Omega$." More precisely, one adds (an unspecified number of) artificial singularities on the boundary in such a way that the new configuration has the same number of positive and negative singularities. Then one looks for the minimal length connection in the usual Euclidean sense. $\tilde{L}(u)$

corresponds to the infimum of these quantities when varying the positions and the number of artificial boundary points (this is a result in [BCL]).

Warning: If $\deg(u|_{\partial\Omega}) = 0$, $\tilde{L}(u) \leq L(u)$ with strict inequality in general.

There are variants of the previous results for $\tilde{L}$.

Theorem $\tilde{1}$. *We have*

$$|\tilde{L}(u) - \tilde{L}(v)| \leq C\|\nabla u - \nabla v\|_2(\|\nabla u\|_2 + \|\nabla v\|_2), \quad \forall\, u, v \text{ in } H^1(\Omega; S^2).$$

Theorem $\tilde{2}$. *Given u in $H^1(\Omega; S^2)$ there is a sequence of maps v_n in $C^\infty(\overline{\Omega}; S^2)$ such that:*

$$v_n \rightharpoonup u \quad \textit{weakly in } H^1$$

and

$$\overline{\lim}_{n\to+\infty} \int_\Omega |\nabla v_n|^2 \leq \int_\Omega |\nabla u|^2 + 8\pi\tilde{L}(u).$$

Corollary $\tilde{1}$. *Given u in H^1, we have*

$$\inf_{v\in C^\infty(\overline{\Omega};S^2)} \int |\nabla(u-v)|^2 \leq 8\pi\tilde{L}(u).$$

The proof of Theorem $\tilde{1}$ is exactly the same as the one of Theorem 1 (in all integrations by parts boundary integrals vanish because $\xi = 0$ on $\partial\Omega$). The proof of Theorem $\tilde{2}$ follows the same idea as the one of Theorem 2 except that one should now use Theorem 2bis of [Be1] instead of Theorem 2 of [Be1]. Note that here the v_n's do not coincide with u on $\partial\Omega$.

Remark 2. We do not know whether the inequality of Corollary $\tilde{1}$ is optimal in the sense of Remark 1 (if one uses the same u as in Remark 1, $\tilde{L}(u) = \varepsilon$).

In the proof of Theorem 3 we shall use the following lemma.

Lemma 1. *Let u, p_1, p_2, p_3, be 4 vectors of $\mathbf{R}^3$. Set (in a given basis e_1, e_2, e_3)*

$$V = (u \cdot p_2 \wedge p_3, u \cdot p_3 \wedge p_1, u \cdot p_1 \wedge p_2).$$

Then, we have

$$|V| \leq \frac{1}{2}|u|(|p_1|^2 + |p_2|^2 + |p_3|^2). \tag{17}$$

Proof. We may always assume that $|u| = 1$. Let R be a rotation such that

$$R(u) = e_3 = (0, 0, 1).$$

Recall that

$$Ru \cdot Rp_i \wedge Rp_j = u \cdot p_i \wedge p_j, \quad \text{for } i, j \text{ in } \{1, 2, 3, \}.$$

Write in the basis e_1, e_2, e_3

$$Rp_1 = (a_1, b_1, c_1), Rp_2 = (a_2, b_2, c_2), Rp_3 = (a_3, b_3, c_3).$$

Thus

$$V = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) = a \wedge b$$

with

$$a = (a_1, a_2, a_3) \quad \text{and} \quad b = (b_1, b_2, b_3).$$

We have

$$|V| \leq |a|\,|b| \leq \frac{1}{2}(|a|^2 + |b|^2) \leq \frac{1}{2}(|Rp_1|^2 + |Rp_2|^2 + |Rp_3|^2)$$

which leads to (17).

Proof of Theorem 3. Since a supremum of sequentially lower semi-continuous functions is sequentially lower semi-continuous it suffices to check that for any fixed $\xi: \Omega \to \mathbf{R}$ with $\|\nabla\xi\|_\infty \leq 1$ the function

$$u \in H^1_\varphi \mapsto F_\xi(u) = \int_\Omega |\nabla u|^2 dx + 2\int_\Omega D(u) \cdot \nabla\xi \, dx$$

is sequentially lower semi-continuous for the weak H^1 topology (recall that the boundary integral $\int_{\partial\Omega} D(u) \cdot n\xi d\sigma$ depends only on φ).

Suppose that $u^n \rightharpoonup u$ weakly in H^1_φ and set $v^n = u^n - u$. We have, when $n \to +\infty$,

$$\int_\Omega |\nabla u^n|^2 = \int_\Omega |\nabla u|^2 + \int_\Omega |\nabla v^n|^2 + o(1).$$

Write

$$2\int D(u^n)\cdot\nabla\xi = A^n + B^n + C^n$$

where

$$A^n = 2\int u^n\cdot(u_y\wedge u_z\xi_x + u_z\wedge u_x\xi_y + u_x\wedge u_y\xi_z)$$

$$B^n = 2\int u^n\cdot(v_y^n\wedge u_z + u_y\wedge v_z^n)\xi_x + 2\int u^n\cdot(v_z^n\wedge u_x + u_z\wedge v_x^n)\xi_y$$

$$+\ 2\int u^n\cdot(v_x^n\wedge u_y + u_x\wedge v_y^n)\xi_z$$

$$C^n = 2\int u^n\cdot(v_y^n\wedge v_z^n\xi_x + v_z^n\wedge v_x^n\xi_y + v_x^n\wedge v_y^n\xi_z)$$

clearly $A^n \to 2\int D(u)\cdot\nabla\xi$ since $u^n \rightharpoonup u$ weak * in L^∞. On the other hand, $B^n \to 0$ since for example $v_y^n \rightharpoonup 0$ weakly in L^2 and $(u^n\wedge u_z)\xi_x$ converges strongly in L^2 to $(u\wedge u_z)\xi_x$ (by dominated convergence). Finally we claim that:

$$|C^n| \le \int |\nabla v^n|^2.$$

Indeed the integrand may be written as $V^n\cdot\nabla\xi$ where

$$V^n = (u^n\cdot v_y^n\wedge v_z^n, u^n\cdot v_z^n\wedge v_x^n, u^n\cdot v_x^n\wedge v_y^n),$$

and by Lemma 1, we have $|V^n| \le \frac{1}{2}|\nabla v^n|^2$. We conclude that

$$\underline{\lim}_{n\to\infty}\int|\nabla u^n|^2 + 2\int D(u^n)\cdot\nabla\xi \ge \int|\nabla u|^2 + 2\int D(u)\cdot\nabla\xi.$$

This completes the proof of Theorem 3.

2. Some Properties of Minimizers for the F Problem

We first prove Theorem 4, i.e., every minimizer for the F problem is weakly harmonic.

Let $u \in H^1_\varphi$ and let $\psi \in C_0^\infty(\Omega;\mathbf{R}^3)$. Set $u(\psi) = \frac{u+t\psi}{|u+t\psi|}$. Note that $|u+t\psi| = 1 + 0(t)$ so that for t small enough $u(t) \in H^1_\varphi$. The following lemma will be used in the proof of Theorem 4.

Lemma 2. *We have for $|t|$ small enough*

$$L(u(t)) = L(u).$$

Proof. The conclusion is obvious if u is in R^0_φ since $u(t)$ and u have the same singularities. In the general case, as in the proof of Theorem 2, there is a sequence $u_n \in R^0_\varphi$ such that $u_n \to u$ in H^1_φ. It is easy to see that $u_n(t) \to u(t)$ in H^1_φ for every $|t|$ small enough. The conclusion then follows from the continuity of L (Theorem 1).

Proof of Theorem 4. Let u be a minimizer for F in H^1_φ. We have

$$E(u) + 8\pi L(u) \leq E(u(t)) + 8\pi L(u(t)) = E(u(t)) + 8\pi L(u)$$

(by Lemma 2). It follows that $\frac{d}{dt}E(u(t))|_{t=0} = 0$. This implies, by a standard computation, that u satisfies

$$\int \nabla u \cdot \nabla \psi = \int (u \cdot \psi)|\nabla u|^2$$

which says that u is weakly harmonic.

Proof of Theorem 5. First if $\mu_\varphi < \overline{\mu}_\varphi$ it is clear that (P_φ) holds since every smooth map has an energy which is at least $\overline{\mu}_\varphi$.

Conversely, we assume that (P_φ) holds.

Claim: *Let u be any minimizer for μ_φ. Then u cannot be a minimizer for F.*

We postpone the proof of the claim and complete the proof of Theorem 5. Suppose, by contradiction, that $\mu_\varphi = \overline{\mu}_\varphi$. By definition of $\overline{\mu}_\varphi$ there is a sequence $u_n \in H^1_\varphi \cap C^0$ such that $E(u_n) \to \overline{\mu}_\varphi$. We may also assume that $u_n \rightharpoonup u$ weakly in H^1 and thus u is a minimizer for F, by Corollary 2. Since $\overline{\mu}_\varphi = \mu_\varphi$, u is also a minimizer for E in H^1_φ. This contradicts the claim.

Proof of the claim. Let $\eta(t)|_{t\in[-1,+1]}$ be a smooth family of diffeomorphisms from Ω into itself, satisfying $\eta(0) = \mathrm{Id}_\Omega$ and $\eta(t) = \mathrm{Id}|_{\partial\Omega}$ on $\partial\Omega$ for all $t \in [-1,+1]$. Set $u(t) = u \circ \eta(t)$. Since u is a minimizer for E we have:

$$\frac{d}{dt}E(u(t))|_{t=0} = 0.$$

(Any weakly harmonic map in H^1_φ satisfying this condition for every $\eta(t)$ is called a stationary map.)

Suppose by contradiction that u is also a minimizer for F. Then we have $\frac{d}{dt}F(u(t))|_{t=0} = 0$, and therefore

$$\frac{d}{dt}L(u(t))|_{t=0} = 0. \tag{18}$$

By assumption (P_φ), u has a finite nonempty set of singularities. Let x_0 be one of the singularities of u. In a minimal connection x_0 is connected to some other singularity x_1 (both have degree ± 1). Let e be the unit vector tangent to x_0 to a geodesic curve connecting x_0 to x_1, pointing towards x_1. Let $X \in C^\infty(\mathbf{R}^3, \mathbf{R}^3)$ be any vector-field with support in a small neighbourhood of x_0, containing no other singularity of u other than x_0, and such that $X(x_0) = e$. Let $\eta(t)$ be the corresponding flow; the map $u(t)$ has the same singularities as u except that x_0 is replaced by $\eta(t)(x_0)$. Therefore $L(u(t)) = L(u) - t + o(t)$ as $t \to 0$. This contradicts (18).

3. The Existence of Infinitely Many (Weakly) Harmonic Maps

The Section is devoted to the proof of Theorem 6. Here we assume that φ is any map in $H^1(\partial\Omega; S^2) \cap C^0(\partial\Omega; S^2)$. Note that the definition of $L(u,v)$ given by (11) makes sense for any $u, v \in H^1_\varphi$ and for any φ (even if $\deg(\varphi; \partial\Omega) \neq 0$).

The proof relies on the following Lemmas:

Lemma 3. *Let u, v in H^1_φ be smooth except at a finite number of singularities, counted according to their multiplicity. Let (p_i) denote the positive singularities of u together with the negative singularities of v. Let (n_i) denote the negative singularities of u together with the positive singularities of v. Then $L(u,v)$ is the minimal connection associated to (p_i, n_i).*

Lemma 4. *Given v in H^1_φ set*

$$G(u) = E(u) + 8\pi L(u,v).$$

Then the conclusions of Theorem 3 and 4 hold, i.e., G is sequentially lower semi-continuous on H^1_φ and every minimizer of G is weakly harmonic.

Proof of Lemma 3. Integrating by parts in (11), we find

$$\begin{aligned} L(u,v) &= \frac{1}{4\pi} \sup_{\substack{\xi:\Omega\to\mathbf{R} \\ \|\nabla\xi\|_\infty \le 1}} \int_\Omega (\operatorname{div} D(u) - \operatorname{div} D(v))\xi \\ &= \sup_{\substack{\xi:\Omega\to\mathbf{R} \\ \|\nabla\xi\|_\infty \le 1}} \Sigma\xi(p_i) - \Sigma\xi(n_i), \end{aligned}$$

which is the length of the desired minimal connection.

Proof of Lemma 4. In order to prove that G is sequentially lower semi-continuous one uses exactly the same argument as in the proof of

Theorem 3. The fact that every minimizer of G is weakly harmonic is proved using the same method as in Theorem 4; it suffices to observe that:

$$|L(u_1, v) - L(u_2, v)| \leq L(u_1, u_2) \leq C\|\nabla(u_1 - u_2)\|_2(\|\nabla u\|_2 + \|\nabla v\|_2).$$

Proof of Theorem 6 in the case where $\deg(\varphi; \partial\Omega) = 0$ and (P_φ) holds. Set for $\lambda \in (0, 1]$ and $u \in H^1_\varphi$

$$F_\lambda(u) = \int_\Omega |\nabla u|^2 + 8\pi\lambda L(u).$$

The conclusions of Theorem 3, 4 and 5 are valid for the functional F_λ (the proofs are unchanged). For each $\lambda \in (0, 1]$ one may minimize F_λ on H^1_φ. It is reasonable to conjecture that for every $\lambda < \lambda_0$ sufficiently small these minimizers are distinct (thus providing a "branch" of harmonic maps near a minimizer for E) and that they are smooth except on a set of low dimension (possibly isolated points). In fact, one can prove that, for some sequence $\lambda_n \to 0$ the minimizers of F_{λ_n} are distinct. Indeed, set

$$A_1 = \min\{E(u); u \text{ is a minimizer for } F_1 \text{ in } H^1_\varphi\}.$$

It is clear, by Theorem 3, that this minimum is achieved by some u_1 which is weakly harmonic. We now construct a second weakly harmonic map u_2. By Theorem 5, $A_1 > \mu_\varphi$. Let v_1 be a minimizer for μ_φ in H^1_φ. We claim that

$$L(v_1) \leq \frac{1}{8\pi}(\mu_\varphi + \overline{\mu}_\varphi).$$

Indeed $E(v_1) + E(w) \geq 8\pi L(v_1)\ \forall w \in H^1_\varphi \cap C^0$ (see (16)) and then the conclusion follows by taking the infimum over w.

Let $0 < \lambda_2 < 1$ be small enough so that

$$\mu_\varphi + \lambda_2(\mu_\varphi + \overline{\mu}_\varphi) < A_1.$$

By construction

$$\begin{aligned} F_{\lambda_2}(v_1) &= \int |\nabla v_1|^2 + 8\pi\lambda_2 L(v_1) = \mu_\varphi + 8\pi\lambda_2 L(v_1) \\ &\leq \mu_\varphi + \lambda_2(\mu_\varphi + \overline{\mu}_\varphi) < A_1. \end{aligned}$$

Let u_2 be some minimizer for F_{λ_2} on H^1_φ. Since $F_{\lambda_2}(u_2) \leq F_{\lambda_2}(v_1) < A_1$ it follows that $E(u_2) < A_1$ and hence u_2 is not a minimizer for F_1. Also

u_2 is not a minimizer for E (since Theorem 5 holds for F_{λ_2}). Hence u_2 is weakly harmonic and $u_2 \neq u_1$.

We now construct a third weakly harmonic map u_3. Let

$$A_2 = \min\{E(u); u \text{ is a minimizer for } F_{\lambda_2} \text{ on } H^1_\varphi\}.$$

By Theorem 5 (applied to F_{λ_2} instead of F) we have $A_2 > \mu_\varphi$. Let $\lambda_3 < \lambda_2$ be small enough so that

$$\mu_\varphi + \lambda_3(\mu_\varphi + \overline{\mu}_\varphi) < A_2.$$

We have (with the same v_1 as above)

$$F_{\lambda_3}(v_1) < A_2.$$

Let u_3 be some minimizer for F_{λ_3} on H^1_φ. Note that $E(u_3) < A_2 < A_1$. Thus $u_3 \neq u_2$ and $u_3 \neq u_1$ is a third weakly harmonic map. Iterating this construction we find infinitely many weakly harmonic maps.

Proof of Theorem 6 in case $\deg(\varphi; \partial\Omega) \neq 0$.

Case 1: There are infinitely many minimizers for E on H^1_φ. This case is trivial.

Case 2: There are finitely many minimizers $w_1, w_2, \ldots, w_k$. Every map in H^1_φ must have a singularity. We follow the same argument as above except that we replace $L(u)$ by $L(u, v)$ where $v \in R^\infty_\varphi$ is such that $L(w_i; v) \neq 0$, $i = 1, 2, \ldots, k$.

Remark 3. The existence of infinitely many minimizing harmonic maps for some special boundary data φ (axially symmetric) has been recently established by Hardt–Kinderlehrer–Lin [HKL].

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