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# Nodal Solutions of Elliptic Equations with Critical Sobolev Exponents

F. V. ATKINSON

*Department of Mathematics, University of Toronto, Toronto, Ontario, Canada M5S 1A1*

H. BREZIS

*Département de Mathématiques, Université Paris VI, Paris, France*

AND

L. A. PELETIER

*Mathematical Institute, Leiden University, Leiden, The Netherlands*

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## 1. INTRODUCTION

We consider the eigenvalue problem

$$
\text{(I)}\begin{cases} -\Delta u=\lambda u+|u|^{p-1}u & \text{in } B \qquad (1.1)\\ u\neq 0 & \text{in } B \qquad (1.2)\\ ui=0 & \text{on } \partial B, \qquad (1.3)\end{cases}
$$

where $B$ denotes the unit ball in $\mathbf{R}^N$ $(N\geqslant 3)$,

$$p=\frac{N+2}{N-2},$$

and $\lambda$ is a positive real number; for $\lambda\leqslant 0$ Problem (I) is known to have no solutions.

After detailed studies of the existence and properties of positive solutions of Problem (I) [BN, AP1] interest has recently grown in solutions which change sign. We shall call such solutions "nodal solutions." In this paper we shall discuss the existence of such solutions $u$ of (I) which possess *radial*



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*symmetry* for different values of $\lambda$ and we study the values of $\lambda$ when $\|u\|_\infty$ becomes unbounded.

Let $\mu_1$ be the first eigenvalue of $-\Delta$ on $B$ with Dirichlet boundary conditions. Then we recall from [BN] that if $N \geqslant 4$ there exists a positive solution of (I), which must necessarily be radial, for *every* $\lambda \in (0, \mu_1)$, whilst for $N=3$ there only exists a positive solution if $\lambda \in (\mu_1/4, \mu_1)$.

With nodal solutions a similar phenomenon occurs, although at a different value of $N$. Let $\mu_n$ denote the eigenvalue of $-\Delta$ on $B$ with Dirichlet boundary conditions, which corresponds to a radial eigenfunction with $n-1$ zeros. If $N \geqslant 7$, then for every $\lambda \in (0, \mu_n)$ Problem (I) has a solution with $n-1$ zeros [CSS, S], whilst if $N=4$, 5, or 6, there exists a number $\lambda^* > 0$ such that (I) has no radial nodal solution if $\lambda \in (0, \lambda^*)$ [ABP]. The only paper we know of which deals with nonradial solutions is [FJ]. Here it is proved that if $N \geqslant 4$, then for every $\lambda > 0$ there exist infinitely many solutions of (1.1)–(1.3). In view of Theorem A, these solutions cannot always be radial.

In the context of radially symmetric solutions it is not necessary to restrict the dimension $N$ to integer values, and it is natural to ask for the precise value of $N$—if any—at which the above transition occurs. For positive solutions it is well known to be $N=4$. For nodal solutions it was recently shown to be $N=6$ [ABP, AP2].

In this paper we shall focus on the behaviour of radial solutions of (I) which change sign when $4 \leqslant N \leqslant 6$. For a discussion of such solutions when $N > 6$ we refer to [AP2]. We begin by proving the following nonexistence theorem.

THEOREM A. *Suppose $4 \leqslant N \leqslant 6$. Then there exists a constant $\lambda^* > 0$ such that Problem* (I) *has no radial solutions which change sign if $\lambda \in (0, \lambda^*)$.*

We then turn to the asymptotic behaviour of the values $\lambda_n$ of $\lambda$ which correspond to solutions $u_n$ with $n-1$ zeros, as $\|u_n\|_\infty \to \infty$. For $N=3$ it was shown in [AP1] that

$$\lambda_n \to (n - \tfrac{1}{2})^2 \pi^2 \qquad \text{as} \quad \|u_n\|_\infty \to \infty.$$

For $4 \leqslant N \leqslant 6$ we shall prove the following theorem.

THEOREM B. (a) *Suppose $4 \leqslant N < 6$. Then for every $n \geqslant 2$,*

$$\lambda_n \to \mu_{n-1} \qquad \textit{as} \quad \|u_n\|_\infty \to \infty.$$

(b) *Suppose $N=6$. Then for every $n \geqslant 2$,*

$$\lambda_n \to \mu^*_{n-1} \qquad \textit{as} \quad \|u_n\|_\infty \to \infty,$$

*where $\mu^*_{n-1} \in (0, \mu_{n-1})$.*

*Remark.* Just as the eigenvalues $\mu_n$ of $-\Delta$ on $B$ are related to the zeros $\rho_n$ of the Bessel function $J_\nu$, where $\nu=(N-2)/2$, through the expression $\mu_n=\rho_n^2$, so are the numbers $\mu_n^*$ in part (b) of Theorem B related to the zeros $\rho_n^*$ of the solution of the—nonlinear—problem

$$v''+\frac{N-1}{r}v'+v(1+|v|)=0, \qquad r>0,$$

$$v(0)=\tfrac{1}{2}, \qquad v'(0)=0$$

through $\mu_n^*=(\rho_n^*)^2$, $n=1, 2, \ldots$.

Thus if $4 \leqslant N<6$ then as $\|u\|_\infty$ increases from zero to infinity, the branch of solutions with $n-1$ zeros moves from $\mu_n$ to $\mu_{n-1}$ and so skips precisely one eigenvalue of the associated linear problem. For $N=6$ the branch of solutions moves beyond $\mu_{n-1}$, but stays away from zero, whilst if $N>6$, it moves all the way back to zero (see Fig. 1).

*Remark.* Nodal solutions for the related equation

$$-\Delta u=\lambda\,|u|^{q-1}\,u+|u|^{4/(N-2)}\,u \qquad \text{in} \quad B,$$

where $1<q<(N+2)/(N-2)$, have been studied by Jones [J]. He showed (i) if $q>4/(N-2)$, there exist nodal solutions for all $\lambda>0$, and (ii) if $q<4/(N-2)$, there exists a neighbourhood of $\lambda=0$ in which there are none. More recently in was shown [K] that the second result can be extended to $q=4/(N-2)$.

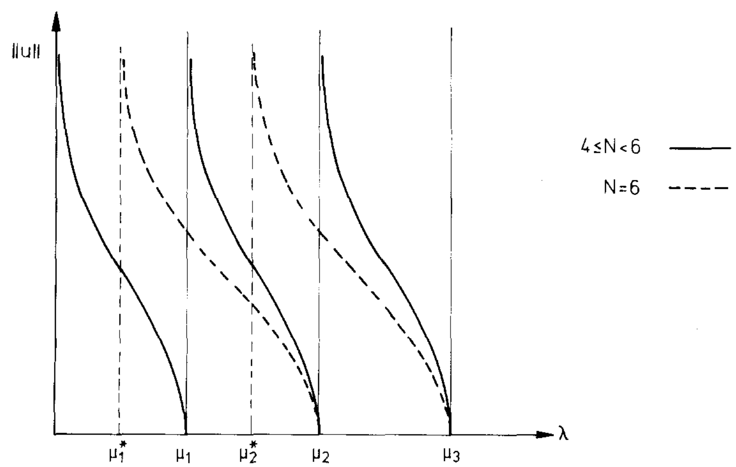


FIG. 1. Solution branches.

Let $u$ be a radial solution of (I). Then we can write $u = u(r)$, where $r = |x|$, and $u(r)$ is a solution of the two-point boundary value problem

$$u'' + \frac{N-1}{r} u' + \lambda u + |u|^{p-1} u = 0, \qquad 0 < r < 1 \tag{1.4}$$

$$u'(0) = 0, \qquad u(1) = 0, \tag{1.5}$$

in which primes denote differentiation with respect to $r$. By scaling $r$ and $u$ we can eliminate $\lambda$. Setting

$$\rho = \sqrt{\lambda} r, \qquad v(\rho) = \lambda^{-1/(p-1)} u(r) \tag{1.6}$$

we obtain

$$v'' + \frac{N-1}{\rho} v' + v + |v|^{p-1} v = 0, \qquad \rho > 0 \tag{1.7}$$

$$v'(0) = 0,$$

and, in addition, the boundary condition at $r = 1$,

$$v(R) = 0, \tag{1.8}$$

in which

$$R = \sqrt{\lambda}. \tag{1.9}$$

We study this problem by a shooting argument and thus for every fixed $\gamma \in \mathbf{R}$ we solve (1.7) together with the initial conditions

$$v(0) = \gamma, \qquad v'(0) = 0. \tag{1.10}$$

The problem (1.7), (1.10) has a unique solution $v(\rho, \gamma)$ which exists for all $\rho > 0$ and—as we shall see—has an infinite sequence of zeros,

$$0 < R_1(\gamma) < R_2(\gamma) < \cdots, \tag{1.11}$$

where $R_n(\gamma) \to \infty$ as $n \to \infty$. In view of (1.10) the eigenvalues $\lambda_n$ are related to the radii $R_n$ by

$$\lambda_n(\gamma) = R_n^2(\gamma), \qquad n = 1, 2, \ldots . \tag{1.12}$$

Thus we can study the properties of the eigenvalues $\lambda_n$ of (I) through an analysis of the zeros of the solutions $v(\rho, \gamma)$ of the initial value problem (1.7), (1.8).

Rather than studying (1.7), (1.8) directly we perform one more transformation, which eliminates the term in (1.7) involving the first order derivative. We set

$$t=\left(\frac{N-2}{\rho}\right)^{N-2}, \qquad y(t)=v(\rho). \tag{1.13}$$

Problem (1.7), (1.8) now becomes

$$y''+t^{-k}y(1+|y|^{p-1})=0, \qquad 0<t<\infty, \tag{1.14}$$

$$y(t)\to\gamma \qquad \text{as } t\to\infty, \tag{1.15}$$

where

$$k=2\,\frac{N-1}{N-2}, \qquad p=2k-3.$$

It is this problem that we shall study in the following sections. In Section 2 we establish some preliminary properties, in Section 3 we prove the non-existence of solutions in a neighbourhood of $\lambda=0$, and in Sections 4 and 5 we investigate the asymptotic behaviour of $\lambda_n$.

## 2. Preliminary Remarks

We consider the initial value problem

$$y''+t^{-k}f(y)=0, \qquad t<\infty \tag{2.1}$$

$$y(t)\to\gamma \qquad \text{as} \quad t\to\infty, \tag{2.2}$$

in which $k>2$ and

$$f(s)=s(1+|s|^{p-1}), \qquad p=2k-3. \tag{2.3}$$

It is well known that, because $k>2$, (2.1), (2.2) has for every $\gamma\in\mathbf{R}$ a unique solution, which we denote by $y(t,\gamma)$.

We introduce two energy functionals,

$$E(t)=\frac{y'^2}{2}+t^{-k}F(y) \tag{2.4}$$

and

$$G(t)=\frac{t^k y'^2}{2}+F(y), \tag{2.5}$$

where

$$F(y) = \int_0^y f(s)\, ds.$$

If $y$ is a solution of (2.1), (2.2), we find upon differentiation that $E(t)$ is a *nonincreasing* and $G(t)$ a *nondecreasing* function of $t$. In particular we may conclude that

$$G(t) \leqslant F(\gamma) \qquad \text{for} \quad 0 < t < \infty$$

and thus that

$$|y(t, \gamma)| < |\gamma| \qquad \text{for} \quad 0 < t < \infty,$$

and that $y'(t, y)$ is uniformly bounded on $(0, \infty)$. Thus $y(t, \gamma)$ exists on the entire interval $(0, \infty)$.

LEMMA 1. (a) *Equation* (2.1) *is oscillatory near* $t = 0$;

(b) *The values of* $|y|$ *at the successive extrema, in the sense of increasing* $t$, *form an increasing sequence*;

(c) *The values of* $|y'|$ *at the successive zeros of* $y$, *in the sense of increasing* $t$, *form a decreasing sequence.*

*Proof.* (a) We write (2.1) as

$$y'' + t^{-2} a(t)\, y = 0, \tag{2.6}$$

where

$$a(t) = t^{2-k}(1 + |y(t)|^{2k-4}), \tag{2.7}$$

and compare it with the equation

$$z'' + t^{-2}(\tfrac{1}{4} + \varepsilon) z = 0 \tag{2.8}$$

in which $\varepsilon > 0$. Since for every $\varepsilon > 0$, (2.8) is oscillatory near zero [H, Theorem 7.1, p. 362] and, because $k > 2$, $a(t) > (\frac{1}{4} + \varepsilon)$ for $t$ small enough, it follows by the Sturm Comparison Theorem that (2.6) is oscillatory near $t = 0$.

Parts (b) and (c) follow immediately from the monotonicity properties of the energy functionals $E$ and $G$.

As a consequence we have

LEMMA 2. *If* $y(T)=0$, *then*

$$|y(t)| < |y'(T)|\,(T-t) \qquad \text{for} \quad 0<t<T. \tag{2.9}$$

*Proof.* Let

$$T^* = \inf\{t<T : |y|>0 \text{ on } (t, T)\}.$$

Then, by Eq. (2.1), $yy''<0$ on $(T^*, T)$ and (2.9) follows for $t \in [T^*, T)$.

Next, let $t<T^*$ and assume that $y(t) \neq 0$. Then, because (2.1) is oscillatory according to Lemma 1(a), there exists an interval $(t_1, t_2) \in (0, T)$ such that $t \in (t_1, t_2)$ and $|y|>0$ on $(t_1, t_2)$. By Lemma 1(b)

$$|y(t)| < \max_{(t_1, t_2)} |y| < \max_{(T^*, T)} |y| \leqslant |y'(T)|\,(T-\tau),$$

where $\tau$ is the point in $(T^*, T)$ at which $|y|$ reaches its maximum value. Since $t<T^*<\tau$, it follows that

$$|y(t)| < |y'(T)|\,(T-t).$$

Since $t$ was an arbitrary point in $(0, T^*)$ the proof is complete.

We shall denote the zeros of $y(t, y)$ by $T_n(\gamma)$, counting backwards, so as to be consistent with the numbering of the zeros $R_n(\gamma)$ of $v(\rho)$:

$$T_n(\gamma) = \left(\frac{N-2}{R_n(\gamma)}\right)^{N-2}. \tag{2.10}$$

Thus we have

$$\cdots < T_3(\gamma) < T_2(\gamma) < T_1(\gamma) < \infty.$$

A detailed analysis of the asymptotic behaviour of the largest zero $T_1(\gamma)$ and the slope $y'(T_1(\gamma), \gamma)$ as $\gamma \to \infty$ was made in [AP1]. Below we list in two lemmas those results which we shall need in the sequel. It will be convenient to introduce the number

$$k_1 = (k-1)^{1/(k-2)}.$$

LEMMA 3. (a) *Suppose* $k=3$. *Then*

$$T_1(\gamma) = 2\log\gamma[1+o(1)] \qquad \text{as} \quad \gamma \to \infty.$$

(b) *Suppose* $2<k<3$. *Then*

$$T_1(\gamma) = A(k)\,\gamma^{6-2k}[1+o(1)] \qquad \text{as} \quad \gamma \to \infty,$$

*where*

$$A(k) = k_1^{k-3} \frac{\Gamma((3-k)/(k-2))\, \Gamma((k-1)/(k-2))}{\Gamma(2/(k-2))}.$$

LEMMA 4. *For any* $k > 2$,

$$y'(T_1(\gamma), \gamma) = k_1 \gamma^{-1}[1 + o(1)] \qquad \textit{as} \quad \gamma \to \infty.$$

## 3. A NONEXISTENCE THEOREM

In this section we show that if $4 \leqslant N \leqslant 6$, then there exists a neighbourhood of $\lambda = 0$ in which (I) has no radial solutions with nodes. In the notation of the previous section this means that we need to show that if $2\frac{1}{2} \leqslant k \leqslant 3$, then

$$\sup\{T_2(\gamma) : \gamma \in (0, \infty)\} < \infty. \tag{3.1}$$

This implies, in view of (2.9) and (1.12), that

$$\lambda^* = \inf\{\lambda_2(\gamma) : \gamma \in (0, \infty)\} > 0.$$

Hence, since $\lambda_{n+1}(\gamma) > \lambda_n(\gamma)$ for every $n \geqslant 1$, it follows that

$$\lambda_n(\gamma) \geqslant \lambda^* > 0 \qquad \text{for} \quad \gamma \in (0, \infty) \text{ and } n \geqslant 1$$

and thus that there exist no nodal solutions for $0 < \lambda < \lambda^*$.

LEMMA 5. *Suppose* $2\frac{1}{2} \leqslant k \leqslant 3$. *Then* (3.1) *holds.*

*Proof.* Since $\lambda_2 \to \mu_2$ as $\gamma \to 0$, it is sufficient to show that

$$\limsup_{\gamma \to \infty} T_2(\gamma) < \infty.$$

We use a Sturmian comparison argument, comparing the solution $y(t, \gamma)$ with the solmution $z(t) = \sqrt{t}$ of the equation

$$z'' + \frac{1}{4t^2} z = 0.$$

Recall that $y$ satisfies the equation

$$y'' + \frac{1}{t^2} a(t)\, y = 0$$

in which

$$a(t)=t^{2-k}(1+|y|^{2k-4}).$$

Hence, by the Sturm Comparison Theorem, $y$ cannot have a zero on any interval $[t_0, T_1)$ on which $a(t)\leqslant\frac{1}{4}$.

By Lemma 2, we have

$$|y(t,\gamma)|<T_1(\gamma)\,|y'(T_1(\gamma),\gamma)| \qquad \text{for} \quad 0<t<T_1(\gamma),$$

and so, by Lemma 4, there exists a constant $K>0$ such that

$$|y(t,\gamma)|<K\,\frac{T_1(\gamma)}{\gamma} \qquad \text{for} \quad 0<t<T_1(\gamma) \tag{3.2}$$

when $\gamma$ is large enough. Thus,

$$a(t)<t^{2-k}[1+\{K\gamma^{-1}T_1(\gamma)\}^{2k-4}] \tag{3.3}$$

for $\gamma$ large enough.

According to Lemma 3 we have when $k=3$

$$\gamma^{-1}T_1(\gamma)=O(\gamma^{-1}\log\gamma) \qquad \text{as} \quad \gamma\to\infty \tag{3.4}$$

and when $2<k<3$

$$\gamma^{-1}T_1(\gamma)=O(\gamma^{5-2k}) \qquad \text{as} \quad \gamma\to\infty. \tag{3.5}$$

Thus if $k\geqslant 2\frac{1}{2}$, then $\gamma^{-1}T_1(\gamma)$ is uniformly bounded for large values of $\gamma$ and so, by (3.3),

$$a(t)<\mathscr{C}t^{2-k} \qquad \text{for} \quad 0<t<T_1(\gamma), \tag{3.6}$$

where $\mathscr{C}$ is some positive constant. If we then choose $t_0=(4\mathscr{C})^{1/(k-2)}$, we conclude from (3.6) that $a(t)\leqslant\frac{1}{4}$ on $[t_0, T_1(\gamma))$ and therefore that

$$T_2(\gamma)<t_0 \qquad \text{for large } \gamma.$$

This completes the proof.

## 4. Asymptotic Analysis when $2\frac{1}{2}<k<3$ $(4\leqslant N<6)$

In this section and the next, we study the asmptotic behaviour of the zeros $T_n(\gamma)$ of the solution $y(t,\gamma)$ as $\gamma\to\infty$, and thus obtain asymptotic estimates for the eigenvalues $\lambda_n$ of Problem (I).

As we showed in the previous section, if $2\frac{1}{2} < k \leqslant 3$ then

$$y(t, \gamma) \to 0 \qquad \text{as} \quad \gamma \to \infty$$

uniformly in compact sets. Thus we may expect $y(t, \gamma)$ to converge to a solution of the linear equation associated with (2.1),

$$z'' + t^{-k} z = 0, \tag{4.1}$$

whereas if $k = 2\frac{1}{2}$ there is no reason to expect this.

We use the method of variation of parameters to substantiate our conjecture. Let $\alpha(t)$ and $\beta(t)$ be solutions of (4.1) so that

$$\alpha(t) \to 1 \qquad \text{and} \qquad \beta'(t) \to 1 \qquad \text{as} \quad t \to \infty. \tag{4.2}$$

Plainly, $\alpha(t)$ is uniquely determined, but $\beta(t)$ is not. However, this will not affect the final result. In any case we have

$$\alpha(t)\, \beta'(t) - \alpha'(t)\, \beta(t) \equiv 1. \tag{4.3}$$

Specifically we can take, with $\nu = 1/(k-2)$,

$$\alpha(t) = A_\nu \sqrt{t}\, J_\nu(2\nu t^{-1/2\nu})$$

and

$$\beta(t) = B_\nu \sqrt{t}\, Y_\nu(2\nu t^{-1/2\nu}),$$

where $A_\nu = \nu^{-\nu}\Gamma(\nu+1)$ if $\nu \in [1, 2)$, $B_\nu = -\nu^\nu \sin(\pi\nu)\, \Gamma(1-\nu)$ if $\nu \in (1, 2)$, and $B_1 = -\pi$. For further reference we note that

$$\alpha(t), \beta(t) = O(t^{k/4}) \qquad \text{as} \quad t \to 0. \tag{4.4}$$

We now introduce functions $a(t)$ and $b(t)$ such that

$$y = a\alpha + b\beta, \qquad y' = a\alpha' + b\beta'. \tag{4.5}$$

Such functions exist in view of (4.3). Solving for $a$ and $b$ we obtain

$$a = -y'\beta + y\beta', \qquad b = y'\alpha - y\alpha'. \tag{4.6}$$

At $t = T_1(\gamma)$, we have since $y(T_1) = 0$,

$$a(T_1) = -y'(T_1, \gamma)\, \beta(T_1)$$
$$b(T_1) = y'(T_1, \gamma)\, \alpha(T_1).$$

Because $T_1(\gamma) \to \infty$ as $\gamma \to \infty$, we conclude from (4.2) and Lemmas 3 and 4 that

$$a(T_1(\gamma)) = -Q\omega(\gamma)[1 + o(1)] \qquad \text{as} \quad \gamma \to \infty, \tag{4.7}$$

where

$$\omega(\gamma)=\log\gamma/\gamma \text{ if } k=3, \qquad \text{and} \qquad \omega(\gamma)=\gamma^{5-2k} \text{ if } k\in(2,3), \tag{4.8}$$

and

$$Q=4 \text{ if } k=3, \qquad \text{and} \qquad Q=k_1A(k) \text{ if } k\in(2,3). \tag{4.9}$$

Similarly we obtain for $b$ that

$$b(T_1(\gamma))=\frac{k_1}{\gamma}\,[1+o(1)] \qquad \text{as} \quad \gamma\to\infty. \tag{4.10}$$

LEMMA 6. *Let* $2<k\leqslant 3$. *Then*

$$a(t)=-Q\omega(\gamma)[1+o(1)] \qquad \text{as} \quad \gamma\to\infty$$
$$b(t)=o(\omega(\gamma)) \qquad \text{as} \quad \gamma\to\infty$$

*uniformly on* $(\delta, T_1(\gamma))$ *for any* $\delta>0$.

*Proof.* Differentiating (4.6) we obtain

$$a'=t^{-k}\,|y|^{2k-4}\,y\beta$$

and

$$b'=-t^{-k}\,|y|^{2k-4}\,y\alpha$$

and hence, upon integration over $(t, T_1(\gamma))$,

$$a(t)=a(T_1)-\int_t^{T_1} s^{-k}\,|y|^{2k-4}\,y\beta\,ds, \tag{4.11}$$

$$b(t)=b(T_1)+\int_t^{T_1} s^{-k}\,|y|^{2k-4}\,y\alpha\,ds. \tag{4.12}$$

In view of the asymptotic behaviour of $a(T_1)$ and $b(T_1)$ given in (4.7) and (4.10) we need to show that the integrals are $o(\omega(\gamma))$ as $\gamma\to\infty$. Note that

$$|\alpha(t)|\leqslant 1 \qquad \text{and} \qquad |\beta(t)|\leqslant C\max\{t^{k/4}, t\} \qquad \text{on} \quad (0,\infty),$$

where $C$ denotes some generic positive constant. Therefore, when $t\geqslant\delta$,

$$\left|\int_t^{T_1} s^{-k}\,|y|^{2k-4}\,y\alpha\,ds\right|\leqslant C\{\omega(\gamma)\}^{2k-3}\int_\delta^\infty s^{-k}\,ds=C_1(\delta)\{\omega(\gamma)\}^{2k-3}.$$

Similarly,

$$\left|\int_t^{T_1} s^{-k}\,|y|^{2k-4}\,y\beta\,ds\right| \leqslant C\{\omega(\gamma)\}^{2k-3}\left(\int_\delta^1+\int_1^\infty\right)s^{-k}\,|\beta(s)|\,ds$$
$$\leqslant C\{\omega(\gamma)\}^{2k-3}\left(\int_\delta^1 s^{-3k/4}\,ds+\int_1^\infty s^{1-k}\,ds\right)$$
$$=C_2(\delta)\{\omega(\gamma)\}^{2k-3}.$$

Thus, since $2k-3>2$ in the range of values of $k$ we consider, both integrals are indeed $o(\omega(\gamma))$ as $\gamma\to\infty$, and the proof is complete.

Returning to $y$ we conclude from (4.5) and Lemma 6 that

$$\frac{y(t,\gamma)}{\omega(\gamma)}=-Q\alpha(t)+o(1) \qquad \text{as} \quad \gamma\to\infty \tag{4.13a}$$

and

$$\frac{y'(t,\gamma)}{\omega(\gamma)}=-Q\alpha'(t)+o(1) \qquad \text{as} \quad \gamma\to\infty \tag{4.13b}$$

uniformly on compact subsets of $(0,\infty)$.

Let

$$\tau_1>\tau_2>\tau_3>\cdots$$

be the zeros of $\alpha(t)$, $\tau_1$ being the first one, so that $\alpha(t)>0$ on $(\tau_1,\infty)$.

THEOREM 1. *Suppose $2\frac{1}{2}<k\leqslant 3$. Then for $n\geqslant 2$,*

$$T_n(\gamma)\to\tau_{n-1} \qquad \textit{as} \quad \gamma\to\infty.$$

*Proof.* It is clear from (4.13) that the zeros of $y(t,\gamma)$ converge to those of $\alpha(t)$, as $\gamma\to\infty$; what remains to be established is that $T_2(\gamma)\to\tau_1$ as $\gamma\to\infty$.

Suppose to the contrary that $T_2(\gamma)\to\tau_l$ as $\gamma\to\infty$ for some $l>1$. In view of (4.13), $y(t,\gamma)$ has a zero $T^*(\gamma)$ which converges to $\tau_1$ as $\gamma\to\infty$. Because $\tau_1>\tau_l$ by assumption, it follows that $T^*(\gamma)>T_2(\gamma)$ for $\gamma$ large enough. Hence, $T^*(\gamma)=T_1(\gamma)$. However, $T_1(\gamma)\to\infty$ as $\gamma\to\infty$ whence we have a contradiction. This proves the theorem.

We finally return to the original variables $r$, $u$, and $\lambda$. Thus we set $k=2(N-1)/(N-2)$. Following the transformations made in Section 2 backwards, we find that the functions

$$\phi_l(x)=\alpha(\tau_l\,|x|^{2-N}), \qquad l=1,2,\ldots$$

satisfy

$$-\Delta\phi_l = \mu_l \phi_l \quad \text{in } B$$
$$\phi_l = 0 \quad \text{on } \partial B,$$

where,

$$\mu_l = (N-2)^2 \tau_l^{-2/(N-2)}.$$

However, by (1.12) and (2.10),

$$\lambda_n(\gamma) = (N-2)^2 \{T_n(\gamma)\}^{-2/(N-2)}. \tag{4.14}$$

Thus we conclude from Lemma 6 that for $n \geqslant 2$

$$\lambda_n(\gamma) \to \mu_{n-1} \quad \text{as} \quad \gamma \to \infty.$$

This completes the proof of part (a) of Theorem B.

## 5. Asymptotic Analysis when $k = 2\frac{1}{2}$ $(N = 6)$

When $k = 2\frac{1}{2}$, Eq. (2.1) becomes

$$y'' + t^{-5/2} y(1 + |y|) = 0 \tag{5.1}$$

and, according to Lemma 3, the asymptotic behaviour of $T_1(\gamma)$ and $y'(T_1(\gamma), \gamma)$ as $\gamma \to \infty$ is given by

$$T_1(\gamma) = \frac{2}{9}\gamma[1 + o(1)] \quad \text{as} \quad \gamma \to \infty, \tag{5.2}$$

$$y'(T_1(\gamma), \gamma) = \frac{9}{4\gamma}[1 + o(1)] \quad \text{as} \quad \gamma \to \infty. \tag{5.3}$$

Thus we can conclude from Lemma 2 that

$$|y(t, \gamma)| \leqslant \tfrac{1}{2}[1 + o(1)] \quad \text{as} \quad \gamma \to \infty, \tag{5.4}$$

i.e., $y(t, \gamma)$ is uniformly bounded on $[0, T_1(\gamma)]$.

To estimate the asymptotic behaviour of the zeros $T_n(\gamma)$ of $y(t, \gamma)$, we proceed in two steps. First we determine the location $(t_0, y_0)$ of the largest zero of $y'(t, \gamma)$, i.e.,

$$t_0(\gamma) = \inf\{t \in (0, \infty) : y' > 0 \text{ on } (t, \infty)\}$$

and

$$y_0(\gamma) = y(t_0(\gamma), \gamma).$$

Having done so, we approximate $y(t, \gamma)$ for $t < t_0$.

About $(t_0, y_0)$ we prove the following asymptotic estimates.

THEOREM 2. *Suppose* $k = 2\frac{1}{2}$. *Then*

$$\text{(a)} \quad y_0(\gamma) = -\tfrac{1}{2}[1 + o(1)] \qquad \textit{as} \quad \gamma \to \infty;$$
$$\text{(b)} \quad t_0(\gamma) = (\tfrac{2}{9}\gamma)^{2/3}\,[1 + o(1)] \qquad \textit{as} \quad \gamma \to \infty.$$

Before turning to the proof of Theorem 2, we establish a few preliminary lemmas. It will be convenient to use the abbreviations

$$\kappa(\gamma) = y'(T_1(\gamma), \gamma) \qquad \text{and} \qquad \sigma(\gamma) = \kappa(\gamma)\, T_1(\gamma).$$

LEMMA 7. *We have*

$$t_0^{3/2} < \tfrac{2}{3}(1 + \sigma) T_1. \tag{5.5}$$

*Proof.* Integration of (5.1) over $(t_0, T_1)$ yields

$$\kappa = \int_{t_0}^{T_1} s^{-5/2}\, |y(s)|\, (1 + |y(s)|)\, ds. \tag{5.6}$$

Hence, because $|y(t)| < \sigma$ on $(0, T_1)$ by Lemma 2, it follows that

$$\kappa < \sigma(1 + \sigma) \int_{t_0}^{T_1} s^{-5/2}\, ds,$$

and thus that

$$1 < \tfrac{2}{3} T_1 (1 + \sigma) t_0^{-3/2}.$$

The desired bound is now immediate.

COROLLARY 1. *We have*

$$\frac{t_0(\gamma)}{T_1(\gamma)} \to 0 \qquad \textit{as} \quad \gamma \to \infty.$$

LEMMA 8. *We have*

$$\lim_{\gamma \to \infty} \frac{|y_0(\gamma)|}{\sigma(\gamma)} = 1.$$

*Proof.* By Lemma 2, we have

$$|y_0| = |y(t_0)| \leqslant |y'(T_1)|\ (T_1 - t_0) < \sigma,$$

and so

$$\limsup_{\gamma \to \infty} \frac{|y_0(\gamma)|}{\sigma(\gamma)} \leqslant 1.$$

Thus, it suffices to prove that

$$\liminf_{\gamma \to \infty} \frac{|\gamma_0(\gamma)|}{\sigma(\gamma)} \geqslant 1. \tag{5.7}$$

Integrating (5.1) twice, we obtain for $t < T_1$ that

$$|y(t)| = \kappa(T_1 - t) - \int_t^{T_1} (s-t) s^{-5/2}\, |y(s)|\ (1 + |y(s)|)\, ds. \tag{5.8}$$

Hence, remembering that $|y(t)| < \sigma$ if $t < T_1$ we conclude that

$$|y(t)| > \kappa(T_1 - t) - \sigma(1+\sigma) \int_t^{T_1} s^{-3/2}\, ds$$

or

$$|y(t)| > \kappa(T_1 - t) - 2\sigma(1+\sigma) t^{-1/2}.$$

By Lemma 7, $t_0 < cT_1^{2/3}$ for some $c > 0$ and so $|y_0| > |y(cT_1^{2/3})|$. Therefore

$$|y_0| > \kappa(T_1 - cT_1^{2/3}) - 2\sigma(1+\sigma)(cT_1^{2/3})^{-1/2},$$

and hence

$$|y_0|/\sigma > 1 + O(T_1^{-1/2}) \qquad \text{as} \quad \gamma \to \infty.$$

Remembering that $T_1(\gamma) \to \infty$ as $\gamma \to \infty$, (5.7) follows.

LEMMA 9. *We have*

$$\liminf_{\gamma \to \infty} \frac{t_0^{3/2}(\gamma)}{T_1(\gamma)} \geqslant \frac{2}{3} \liminf_{\gamma \to \infty} (1 + |y_0(\gamma)|).$$

*Proof.* By the strict convexity of $y$ on $(t_0, T_1)$,

$$|y(t)| > \frac{|y_0|}{T_1 - t_0}\,(T_1 - t) \qquad \text{on} \quad (t_0, T_1)$$

and so, by (5.6),

$$\kappa > \frac{|y_0|}{T_1 - t_0} \int_{t_0}^{T_1} (T_1 - s) s^{-5/2} \left(1 + \frac{|y_0|}{T_1 - t_0} (T_1 - s)\right) ds. \tag{5.9}$$

Introducing the variable $u = s/T_1$, and setting $\tau = t_0/T_1$, we can write (5.9) as

$$\kappa \geqslant \frac{|y_0|}{1-\tau} \tau_1^{-3/2} \int_\tau^1 (1-u) u^{-5/2} \left(1 + \frac{|y_0|}{1-\tau} (1-u)\right) du$$

$$= \frac{2}{3} |y_0| (1 + |y_0|) t_0^{-3/2} [1 + o(1)] \qquad \text{as} \quad \gamma \to \infty,$$

because $\tau \to 0$ as $\gamma \to \infty$ by Corollary 1. Thus

$$\frac{t_0^{3/2}}{T_1} \geqslant \frac{2}{3} (1 + |y_0|) \frac{|y_0|}{\sigma} [1 + o(1)] \qquad \text{as} \quad \gamma \to \infty,$$

which, together with Lemma 8, yields the desired lower bound.

COROLLARY 2. *We have*

$$\liminf_{\gamma \to \infty} \frac{t_0^{3/2}(\gamma)}{T_1(\gamma)} \geqslant \frac{2}{3}.$$

We can now readily complete the proof of Theorem 2 by means of the estimates (5.2) and (5.3) for $T_1(\gamma)$ and $y'(T_1(\gamma))$.

*Proof of Theorem* 2. (a) Since

$$\sigma(\gamma) = T_1 \, y'(T_1(\gamma)),$$

it follows from (5.2) and (5.3) that $\lim_{\gamma \to \infty} \sigma(\gamma) = \frac{1}{2}$. Hence, by Lemma 8,

$$\lim_{\gamma \to \infty} |y_0(\gamma)| = \tfrac{1}{2}.$$

Because $y_0(\gamma) < 0$, the desired limit follows.

(b) By Lemma 7 we have

$$\limsup_{\gamma \to \infty} \frac{t_0^{3/2}(\gamma)}{T_1(\gamma)} \leqslant 1,$$

and by Lemma 9 and Theorem 2(a) we have

$$\liminf_{\gamma \to \infty} \frac{t_0^{3/2}(\gamma)}{T_1(\gamma)} \geqslant 1.$$

Thus

$$\lim_{\gamma \to \infty} \frac{t_0^{3/2}(\gamma)}{T_1(\gamma)} = 1.$$

The proof is completed by means of (5.2).

Having shown in Theorem 2 that the first local minimum of $y(t, \gamma)$ (coming from $t = \infty$) moves to $t = \infty$ as $\gamma \to \infty$, and that its value $y_0(\gamma)$ tends to $-\frac{1}{2}$, one expects that the solution $y(t, \gamma)$ converges to the solution $Y(t)$ of the problem

$$Y'' + t^{-5/2} Y(1 + |Y|) = 0, \qquad t > 0 \tag{5.10}$$

$$Y(t) \to -\tfrac{1}{2} \quad \text{as} \quad t \to \infty \tag{5.11}$$

when $\gamma \to \infty$. In Theorem 3 we show that this is indeed so.

THEOREM 3. *Suppose $k = 2\frac{1}{2}$. Then for every $t > 0$,*

$$\lim_{\gamma \to \infty} (t, \gamma) = Y(t).$$

*Proof.* We integrate (5.1) and (5.10) twice over $(t, t^0)$. This yields the integral equations

$$y(t) = y_0 - \int_t^{t_0} (s - t) s^{-5/2} f(y(s))\, ds \tag{5.12}$$

and

$$Y(t) = Y(t_0) - Y'(t_0)(t_0 - t) - \int_t^{t_0} (s - t) s^{-5/2} f(Y(s))\, ds, \tag{5.13}$$

where now

$$f(z) = z(1 + |z|).$$

For convenience we have dropped the reference to $\gamma$. If we now write

$$w(t) = |y(t) - Y(t)|,$$

subtract (5.13) from (5.12), and take absolute values we obtain

$$w(t) \leqslant A + B \int_t^{t_0} s^{-3/2} w(s)\, ds, \tag{5.14}$$

where

$$A = y_0 - Y(t_0) - t_0 Y_0'(t_0),$$
$$B = \max\{f'(z) : |z| \leqslant \tfrac{1}{2}\} = 2.$$

By Gronwall's inequality, (5.14) implies that

$$w(t) \leqslant A e^{2B/\sqrt{t}}, \qquad t > 0. \tag{5.15}$$

By Theorem 2,

$$y_0(\gamma) - Y(t_0(\gamma)) \to 0 \qquad \text{as} \quad \gamma \to \infty \tag{5.16}$$

and from (5.10), (5.11) we deduce that for $t_0$ sufficiently large,

$$0 < t_0 Y'(t_0) = t_0 \int_{t_0}^{\infty} s^{-5/2} f(Y(s))\, ds$$
$$< \frac{3}{4} t_0 \int_{t_0}^{\infty} s^{-5/2}\, ds$$
$$= \frac{1}{2} t_0^{-1/2}.$$

Hence, using Theorem 2 again we conclude that

$$t_0(\gamma)\, Y'(t_0(\gamma)) \to 0 \qquad \text{as} \quad \gamma \to \infty. \tag{5.17}$$

Together, (5.16) and (5.17) imply that $A(\gamma) \to 0$ as $\gamma \to \infty$, and thus, by (5.15), that for every $t > 0$,

$$y(t, \gamma) - Y(t) \to 0 \qquad \text{as} \quad \gamma \to \infty.$$

Let us denote the zeros of $Y(t)$ by $\tau_n^*$, and number them so that

$$\cdots < \tau_n^* < \cdots < \tau_2^* < \tau_1^* < \infty.$$

Because $Y'(\tau_n^*) \neq 0$ for every $n \geqslant 1$, the following theorem follows readily from Theorem 3.

THEOREM 4. *Suppose $k = 2\frac{1}{2}$. Then for every $n \geqslant 2$,*

$$T_n(\gamma) \to \tau_{n-1}^* \qquad \textit{as} \quad \gamma \to \infty.$$

For the proof we refer to the proof of Theorem 2.

About the zeros $\tau_n^*$, we have the following comparison lemma.

LEMMA 10. *Let $\{\tau_n\}$ be the zeros of the solution of the problem*

$$\alpha'' + t^{-5/2}\alpha = 0, \qquad 0 < t < \infty,$$
$$\alpha(t) \to 1 \qquad \text{as} \quad t \to \infty.$$

*Then*

$$\tau_n^* > \tau_n \qquad \text{for every} \quad n \geq 1.$$

*Proof.* We first prove the lemma for $n = 1$. Suppose to the contrary that $y > 0$ on $(\tau_1, \infty)$. Then

$$0 = \int_{\tau_1}^{\infty} \alpha\{y'' + t^{-5/2}y(1 + |y|)\}\, dt$$
$$= \alpha'(\tau_1)\, y(\tau_1) + \int_{\tau_1}^{\infty} t^{-5/2}\alpha y\, |y|\, dt.$$

Because the first term on the right side is nonnegative, and the second term is positive, we have a contradiction. Therefore $\tau_1^* > \tau_1$.

Next, suppose that for some $n \geq 2$, $\tau_n^* \leq \tau_n$. Then there exists an index $m \in \{1, ..., n-1\}$ such that $y(t)$ has one sign on $(\tau_{m+1}, \tau_m)$. Because $1 + |y| > 1$ on $(\tau_{m+1}, \tau_m)$ this is impossible by the Sturm Comparison Principle. It follows that $\tau_n^* > \tau_n$ for every $n \geq 1$.

As in the previous section, we find, upon returning to the original variables, that

$$\mu_l^* = (N-2)^2\, (\tau_l^*)^{-2/(N-2)} = (\rho_l^*)^2, \qquad l = 1, 2, ..., \tag{5.18}$$

where $\rho_l^*$ is the $l$th zero of the solution of the problem

$$v'' + \frac{N-1}{\rho}\, v' + v(1 + |v|) = 0, \qquad \rho > 0$$
$$v(0) = \tfrac{1}{2}, \qquad v'(0) = 0.$$

Comparing (4.14) and (5.18), we find that Lemma 10 implies that $\mu_l^* < \mu_l$, for all $l = 1, 2, ...$. This completes the proof of the last line of Theorem B.

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