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ASYMPTOTICS FOR ELLIPTIC EQUATIONS INVOLVING CRITICAL GROWTH*

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Dedicated to Ennio De Giorgi on his sixtieth birthday

1. Introduction. Consider the problem

$$(1.1) \qquad (I) \qquad \begin{cases} -\Delta u - \lambda u = 3u^{5-\epsilon} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is the unit ball in $\mathbf{R}^3$, $\lambda \geq 0$ and $\epsilon \geq 0$. It is well known that if $\epsilon > 0$, Problem (I) has a solution u_ϵ for any $\lambda < \lambda_1 = \pi^2$. On the other hand, if $\epsilon = 0$ Problem (I) has a solution if and only if $\pi^2/4 < \lambda < \pi^2$ (See [BN]).

In this paper we return to the question, first studied in [AP2], of the asymptotic behaviour of u_ϵ as $\epsilon \to 0$. There it was shown that if $\lambda = 0$, any solution u_ϵ of Problem (I) has the following limiting behaviour:

$$(1.2) \qquad \lim_{\epsilon \to 0} \epsilon\, u_\epsilon^2(0) = \frac{32}{\pi}$$

* This work was done whilst the first author held the Kloosterman Chair of the Mathematical Institute of the University of Leiden.

and at any $x \in \Omega\backslash\{0\}$:

$$\lim_{\epsilon\to 0} \epsilon^{-1/2} u_\epsilon(x) = \frac{1}{4}\sqrt{\frac{\pi}{2}}\Big(\frac{1}{|x|} - 1\Big). \tag{1.3}$$

We return to the study of u_ϵ for two reasons. First we shall give a different proof of (1.2) and (1.3), which is mainly based on PDE methods and – we hope – will enable us in due course to handle non-spherical domains. The second reason is that the method of [AP2], which readily extends to the case $0 < \lambda < \pi^2/4$, cannot be applied when $\lambda = \pi^2/4$. Here, a new phenomenon occurs, which was first discovered by Budd [Bu]. A formal computation, based on the method of matched asymptotic expansions, suggests that

$$\lim_{\epsilon\to 0} \epsilon\, u_\epsilon^4(0) = 8\pi^2. \tag{1.4}$$

The method presented here does apply to the case $\lambda = \pi^2/4$ and provides a rigorous proof of (1.4).

As in [AP2], the method we use to prove (1.2) – (1.4) is based on estimating the different terms in the Pohozaev identity for Problem (I):

$$\frac{3\epsilon}{6-\epsilon}\int_\Omega u_\epsilon^{6-\epsilon} = J(u_\epsilon), \tag{1.5}$$

where

$$J(w) = \int_{\partial\Omega}(x,n)\Big(\frac{\partial w}{\partial n}\Big)^2 - 2\lambda\int_\Omega w^2.$$

Here n denotes the outward normal on $\partial\Omega$.

Writing $\mu = \big(u_\epsilon(0)\big)^{-2}$, we shall establish that

$$\mu^{-1/2} u_\epsilon \to 4\pi G_\lambda \quad \text{as} \quad \epsilon \to 0, \tag{1.6}$$

where G_λ is the Green's function of $-\Delta - \lambda$, i.e. G_λ solves

$$-\Delta G - \lambda G = \delta_0 \quad \text{in} \quad \Omega \tag{1.7}$$

$$G = 0 \quad \text{on} \quad \partial\Omega \tag{1.8}$$

in which δ_0 is the Dirac mass centered at the origin. By elliptic regularity theory (1.6) implies that

$$\frac{1}{\mu} J(u_\epsilon) \to J(4\pi G_\lambda). \tag{1.9}$$

On the other hand, we shall establish in Section 4 that

$$J(G_\lambda) = -g_\lambda(0)$$

so that

$$J(u_\epsilon) = -16\pi^2 g_\lambda(0)\mu + o(\mu) \quad \text{as} \quad \epsilon \to 0.$$

Here

$$g_\lambda(x) = G_\lambda(x) - \frac{1}{4\pi|x|}, \tag{1.10}$$

i.e. g_λ is the regular part of the Green's function.
Finally we check that

$$\int_\Omega u_\epsilon^{6-\epsilon} \to \frac{\pi^2}{4} \quad \text{as} \quad \epsilon \to 0 \tag{1.11}$$

and so, putting (1.9) and (1.11) into (1.5) we obtain that

$$\frac{\epsilon}{\mu} \to -\frac{32}{\pi} \cdot 4\pi g_\lambda(0) \quad \text{as} \quad \epsilon \to 0$$

or, since

$$g_\lambda(x) = \frac{1}{4\pi|x|}\Big\{\cos(\sqrt{\lambda}|x|) - \frac{\sin(\sqrt{\lambda}|x|)}{\tan\sqrt{\lambda}} - 1\Big\}, \tag{1.12}$$

this yields our first result:

Theorem 1. *Let u_ϵ be a solution of Problem (I).*
(a) If $0 \le \lambda \le \pi^2/4$, *then*

$$\lim_{\epsilon\to 0} \epsilon\, u_\epsilon^2(0) = \frac{32}{\pi}\frac{\sqrt{\lambda}}{\tan\sqrt{\lambda}}. \tag{1.13}$$

(b) If $0 \le \lambda < \pi^2/4$, *then at any* $x \ne 0$,

$$\lim_{\epsilon\to 0} \epsilon^{-1/2} u_\epsilon(x) = \Big(\frac{\pi^3}{2} \cdot \frac{\tan\sqrt{\lambda}}{\sqrt{\lambda}}\Big)^{1/2} G_\lambda(x),$$

where we define $\sqrt{\lambda}/\tan\sqrt{\lambda} = 1$ *if* $\lambda = 0$. If $\lambda = \pi^2/4$, the right hand side of (1.13) vanishes and all we can conclude is that

$$\epsilon\, u_\epsilon^2(0) = o(1) \quad \text{as} \quad \epsilon \to 0.$$

To obtain a precise estimate, we need a better global asymptotic approximation of u_ϵ than is given by (1.6). Hereby the following family of functions plays a central role

$$U_\mu(x) = \left(\frac{\mu}{\mu^2 + |x|^2}\right)^{1/2}; \tag{1.14}$$

it satisfies the equation

$$-\Delta u = 3u^5 \quad \text{in} \quad \mathbf{R}^3.$$

The function

$$\phi_\mu = U_\mu + 4\pi\mu^{1/2} g_\lambda$$

turns out to be the required approximation of u_ϵ. In Section 6 we shall establish that

(1.15)

$$J(u_\epsilon) = -16\pi^2\, g_\lambda(0)\mu + 4\pi^2\mu^2 + O(\epsilon\mu|\log\mu| + \mu^3|\log\mu|) \quad \text{as} \quad \epsilon \to 0$$

Putting (1.15), together with (1.11), in (1.5) again and using the fact that $g_\lambda(0) = 0$ when $\lambda = \pi^2/4$ we obtain our second result.

Theorem 2. *Let u_ϵ be a solution of Problem (I) in which $\lambda = \pi^2/4$. Then*

$$\lim_{\epsilon\to 0} \epsilon\, u_\epsilon^4(0) = 8\pi^2. \tag{a}$$

$$\lim_{\epsilon\to 0} \epsilon^{-1/4} u_\epsilon(x) = (8\pi^2)^{-1/4} \frac{\cos(\frac{\pi}{2}|x|)}{|x|}, \quad x \neq 0. \tag{b}$$

An important ingredient in the proof of Theorem 2, which is of some interest in its own right, is a Pohozaev–type identity for the Green's function. This identity, which is valid for arbitrary bounded domains Ω in $\mathbf{R}^3$ is derived in Section 4. Some other integral identities involving the Green's function are also established in this section. The proof of Theorem 2 is subsequently given in Sections 5 and 6.

In Section 7, we consider a different, but related problem:

$$(II) \qquad \begin{cases} -\Delta u - \left(\frac{\pi^2}{4} + \epsilon\right)u = 3u^5 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is again the unit ball in $\mathbf{R}^3$. It is known [BN] that if $0 < \epsilon < 3\pi^2/4$, Problem (II) has a solution u_ϵ, and that, as with Problem (I), $u_\epsilon(0) \to \infty$ as $\epsilon \to 0$.

With the machinery built for dealing with Problem (I), we can now readily establish the asymptotic properties of u_ϵ both at the origin, and away from the origin, as $\epsilon \to 0$.

Theorem 3. *Let u_ϵ be a solution of Problem (II). Then*

(a) $$\lim_{\epsilon \to 0} \epsilon\, u_\epsilon^2(0) = \frac{\pi^3}{2}.$$

(b) $$\lim_{\epsilon \to 0} \epsilon^{-1/2} u_\epsilon(x) = 4\sqrt{2} G_{\pi^2/4}(x), \quad x \neq 0.$$

Many of the arguments and results in this paper continue to hold when the domain Ω is not a ball. They furnish insight in the behaviour of solutions u_ϵ of Problems (I) and (II) as $\epsilon \to 0$ when Ω is a general domain in $\mathbf{R}^3$ or $\mathbf{R}^N (N > 2)$ and enable us to formulate a number of conjectures about the behaviour. This is done in Section 8.

2. Preliminary bounds. Since uniqueness for Problem (I) is not known, we shall always take u_ϵ to be *any* solution of Problem (I). Because $u_\epsilon > 0$, it is known to be radially symmetric and decreasing [GNN].

As a first observation, note that if $0 \leq \lambda \leq \pi^2/4$, $u_\epsilon(0) \to \infty$ as $\epsilon \to 0$. For suppose to the contrary that there exists a sequence $\{\epsilon_n\}$, $\epsilon_n \to 0$ as $n \to \infty$ such that $u_{\epsilon_n}(0)$ remains bounded as $n \to \infty$. Then u_{ϵ_n} remains bounded in $L^\infty(\Omega)$ and, in view of the elliptic regularity theory applied to (1.1), u_{ϵ_n} remains bounded in $C^1(\bar{\Omega})$. So we can extract a subsequence, still denoted by $\{u_{\epsilon_n}\}$, which converges uniformly to a limit v, which satisfies

$$\begin{cases} -\Delta v - \lambda v = 3v^5, \ v \geq 0 & \text{in } \Omega \\ \qquad\qquad v = 0 & \text{on } \partial\Omega. \end{cases}$$

This implies that $v = 0$ because $0 \leq \lambda \leq \pi^2/4$ [BN].

On the other hand, we assert that

$$\| u_\epsilon \|_\infty \geq \kappa > 0 \tag{2.1}$$

for some constant $\kappa > 0$ independent of ϵ, which contradicts the conclusion drawn above. To prove (2.1), we multiply (1.1) by the

principal eigenfunction ϕ_1 of $-\Delta$ (chosen positive) and integrate by parts. This leads to

$$(\lambda_1 - \lambda)\int_\Omega u_\epsilon \phi_1 = 3\int_\Omega u_\epsilon^{5-\epsilon}\phi_1$$
$$\leq 3 \parallel u_\epsilon \parallel_\infty^{4-\epsilon} \int_\Omega u_\epsilon \phi_1,$$

where λ_1 is the principal eigenvalue of $-\Delta$, and thus

$$\parallel u_\epsilon \parallel_\infty^{4-\epsilon} \geq \frac{1}{3}(\lambda_1 - \lambda) \geq \frac{\pi^2}{4},$$

because $\lambda_1 = \pi^2$ and $\lambda \leq \pi^2/4$.

Whereas $u_\epsilon(0) \to \infty$ as $\epsilon \to 0$, $u_\epsilon(x) \to 0$ as $\epsilon \to 0$ at any point $x \neq 0$. This follows from the upper bound which we shall present next. Define the function

$$W_\mu(x) = \left(\frac{\mu}{\mu^2 + \alpha|x|^2}\right)^{1/2}, \tag{2.2}$$

where

$$\alpha = \mu^{\epsilon/2} + \frac{\lambda}{3}\mu^2. \tag{2.3}$$

It satisfies the equation

$$-\Delta u = \gamma^{-5} f(\gamma) u^5, \tag{2.4}$$

where $\gamma = \mu^{-1/2}$ and

$$f(s) = \lambda s + 3s^{5-\epsilon}. \tag{2.5}$$

Lemma 2.1. *Let u_ϵ be any solution of Problem (I). Then*

$$u_\epsilon(x) \leq W_\mu(x) \qquad \textit{for} \quad x \in \bar{\Omega},$$

where μ is given by

$$\mu = \{u_\epsilon(0)\}^{-2}. \tag{2.6}$$

Remark. In view of the observations made at the beginning of this section, (2.6) implies that $\mu \to 0$ as $\epsilon \to 0$. Thus, we shall always think of μ as a small quantity.

Remark. It follows from Lemma 2.1 that if $x \neq 0$,

$$u_\epsilon(x) = O(\mu^{1/2-\epsilon/2}) \quad \text{as} \quad \mu \to 0.$$

For the proof of Lemma 2.1 we refer to [AP1, Lemma 1(iii)].

The following elliptic estimates will be needed.
Consider the problem

$$(P) \qquad \begin{cases} -\Delta u - \lambda u = f & \text{in } \Omega \\ u = b & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^3$, with smooth boundary $\partial\Omega$ and $\lambda < \lambda_1$, λ_1 being the principal eigenvalue of $-\Delta$ in Ω.

Lemma 2.2. *Let u be a solution of Problem (P) in which $\lambda < \lambda_1$. Then*

$$\| u \|_{W^{1,q}(\Omega)} + \| \nabla u \|_{C^{0,\alpha}(\partial\Omega)} \leq$$
$$\leq C(\| f \|_{L^1(\Omega)} + \| f \|_{L^\infty(\omega)} + \| b \|_{C^{2,\alpha}(\partial\Omega)}),$$

for any $q < 3/2$, any $\alpha \in (0,1)$ and any neighbourhood ω of the boundary $\partial\Omega$ in Ω.

Remark. As a consequence of Lemma 2.2. we have

(i) $\| u \|_{L^2(\Omega)} + \| \nabla u \|_{L^2(\partial\Omega)} \leq$

(2.7) $$\leq C(\| f \|_{L^1(\Omega)} + \| f \|_{L^\infty(\omega)} + \| b \|_{C^{2,\alpha}(\partial\Omega)}).$$

(ii) If $\{f_n\}$ is a bounded sequence in $L^1(\Omega)$ and in $L^\infty(\omega)$, and $\{b_n\}$ is a bounded sequence in $C^{2,\alpha}(\partial\Omega)$, then the corresponding sequence of solutions $\{u_n\}$ has a compact closure in $L^2(\Omega)$, whilst the sequence $\{\nabla u_n\}$, restricted to $\partial\Omega$, has a compact closure in $L^2(\partial\Omega)$.

Proof of Lemma 2.2. First, in view of the classical Schauder estimates, we may always assume that $b = 0$. Next, we claim that for all $q < 3/2$,

$$\| u \|_{W^{1,q}(\Omega)} \leq C \| f \|_{L^1(\Omega)} .$$

This follows easily, by duality, from the fact that if v satisfies

$$\begin{cases} -\Delta v - \lambda v = f_0 + \sum_{i=1}^{3} \dfrac{\partial}{\partial x_i} f_i & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f_0, f_1, f_2 \in L^p(\Omega)$, then

$$\| v \|_{L^\infty(\Omega)} \leq C_p \sum_{i=0}^{3} \| f_i \|_{L^p(\Omega)}$$

for any $p > 3$ (see e.g. [GT], Theorem 8.15); in other words $(-\Delta - \lambda)^{-1}$ maps $W^{-1,p}(\Omega)$ into $L^\infty(\Omega)$ and, by duality, it also maps $L^1(\Omega)$ into $W_0^{1,p'}(\Omega)$ with $1/p + 1/p' = 1$.
Next, we claim that

$$\| \nabla u \|_{C^{0,\alpha}(\partial\Omega)} \leq C(\| f \|_{L^1(\Omega)} + \| f \|_{L^\infty(\omega)}).$$

Indeed let χ denote the characteristic function of ω and write $f = f_1 + f_2$ with $f_1 = f\chi$ and $f_2 = f(1-\chi)$. For $i = 1, 2$, let u_i be the solutions of the problems

$$\begin{aligned} -\Delta u_i - \lambda u_i &= f_i \quad \text{in } \Omega \\ u_i &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

so that $u = u_1 + u_2$.
By the L^p regularity theory (see e.g. [GT], Chapter 9) we have

$$\| u_1 \|_{W^{2,q}(\Omega)} \leq C \| f_1 \|_{L^\infty(\Omega)} = C \| f \|_{L^\infty(\omega)}$$

for any $q < \infty$, and consequently

$$\| u_1 \|_{C^{1,\alpha}(\bar{\Omega})} \leq C \| f \|_{L^\infty(\omega)}$$

for any $\alpha < 1$. On the other hand we have, as above,

$$\| u_2 \|_{W^{1,q}(\Omega)} \leq C \| f_1 \|_{L^1(\Omega)} \leq C \| f \|_{L^1(\Omega)}$$

and thus, by the Sobolev imbedding,

$$\| u_2 \|_{L^2(\Omega)} \leq C \| f \|_{L^1(\Omega)} .$$

Finally, we note that u_2 satisfies

$$\begin{cases} -\Delta u_2 - \lambda u_2 = 0 & \text{in } \omega \\ u_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

It follows from standard elliptic regularity that

$$\| u_2 \|_{C^{1,\alpha}(\overline{\omega'})} \leq C \| u_2 \|_{L^2(\omega)}$$

for any neighbourhood ω' of $\partial\Omega$, strictly smaller than ω. In particular we have

$$\| \nabla u_2 \|_{C^{0,\alpha}(\partial\Omega)} \leq C \| f \|_{L^1(\Omega)} .$$

3. A first estimate. We shall proceed in two steps. First we shall show that when $0 \leq \lambda \leq \pi^2/4$,

$$\epsilon \leq C\mu^{1-\epsilon/2}, \tag{3.1}$$

where C is some positive constant, and then we shall establish the desired limits.

To prove (3.1), note that (1.5) implies that

$$\frac{3\epsilon}{6-\epsilon} \int_\Omega u_\epsilon^{6-\epsilon} \leq \int_{\partial\Omega} (x,n) \Big(\frac{\partial u_\epsilon}{\partial n}\Big)^2 . \tag{3.2}$$

In the next two lemmas, we shall estimate the two integrals in (3.2).

Lemma 3.1. *Let u_ϵ be a solution of Problem (I) in which $\lambda < \pi^2$. Then*

$$\int_{\partial\Omega} (x,n) \Big(\frac{\partial u_\epsilon}{\partial n}\Big)^2 \leq C\mu^{1-\epsilon/2},$$

where C is some positive constant.

PROOF. According to Lemma 2.2 it is sufficient to estimate the right hand side of (1.1) in $L^1(\Omega)$ and $L^\infty(\omega)$, where ω is some neighbourhood of $\partial\Omega$ in Ω.
By Lemma 2.1

$$\begin{aligned} \int_\Omega u_\epsilon^{5-\epsilon} &\leq \int_\Omega W_\mu^{5-\epsilon} \\ &= 4\pi\mu^{(5-\epsilon)/2} \int_0^1 \frac{r^2 dr}{(\mu^2 + \alpha r^2)^{(5-\epsilon)/2}} . \end{aligned}$$

If we now set $r = \mu\alpha^{-1/2}s$, and observe that $\alpha \geq \mu^{\epsilon/2}$, we obtain

$$\int_\Omega u_\epsilon^{5-\epsilon} \leq 4\pi\mu^{1/2-\epsilon/4} \int_0^\infty \frac{s^2 ds}{(1+s^2)^{(5-\epsilon)/2}} \leq C\mu^{1/2-\epsilon/4} \tag{3.3}$$

for some positive constant C, because the integral is convergent. On the other hand, let ω be a neighbourhood of $\partial\Omega$ in Ω, which does not contain $x = 0$. Then, by Lemma 2.1, $u_\epsilon \leq C\mu^{1/2-\epsilon/4}$ in ω and hence

$$u_\epsilon^{5-\epsilon} \leq C\mu^{(5-\epsilon)(1/2-\epsilon/4)} \quad \text{in} \quad \omega. \tag{3.4}$$

Putting (3.3) and (3.4) into (2.7), the desired estimate follows.

Lemma 3.2. *We have*

$$\liminf_{\epsilon\to 0} \int_\Omega u_\epsilon^{6-\epsilon} \geq \frac{\pi^2}{4}.$$

PROOF. We first prove that

$$\liminf_{\epsilon\to 0} \int_\Omega u_\epsilon^{6-\epsilon} > 0. \tag{3.5}$$

Suppose, to the contrary, that there exists a subsequence $\{\epsilon_n\}, \epsilon_n \to 0$ as $n \to \infty$, but still denoted by ϵ, such that

$$\lim_{\epsilon\to 0} \int_\Omega u_\epsilon^{6-\epsilon} = 0. \tag{3.6}$$

Multiplying equation (1.1) by u_ϵ we deduce from (3.6), since $\lambda < \pi^2$, that

$$u_\epsilon \to 0 \quad \text{as} \quad \epsilon \to 0 \quad \text{in} \quad H^1(\Omega).$$

This implies by the Sobolev injection that

$$u_\epsilon \to 0 \quad \text{as} \quad \epsilon \to 0 \quad \text{in} \quad L^6(\Omega). \tag{3.7}$$

Next, we multiply (1.1) by u_ϵ^5 and integrate. This yields

$$\frac{5}{9}\int_\Omega |\nabla u_\epsilon^3|^2 = \lambda \int_\Omega u_\epsilon^6 + 3\int_\Omega u_\epsilon^{10-\epsilon},$$

and hence, using Sobolev's injection again, in view of (3.7),

$$\alpha \parallel u_\epsilon \parallel_{18}^6 \leq \int_\Omega u_\epsilon^{10-\epsilon} + o(1) \quad \text{as} \quad \epsilon \to 0, \tag{3.8}$$

where α is some positive number. However, using Hölder's inequality, we find that

$$\int_\Omega u_\epsilon^{10-\epsilon} \leq \parallel u_\epsilon \parallel_{18}^6 \parallel u_\epsilon \parallel_{6-3\epsilon/2}^{4-\epsilon} = o(1) \parallel u_\epsilon \parallel_{18}^6 . \tag{3.9}$$

Combining (3.8) and (3.9) we find that

$$\parallel u_\epsilon \parallel_{18} \to 0 \qquad \text{as} \quad \epsilon \to 0. \tag{3.10}$$

Going back to (1.1) and using standard elliptic regularity theory, we see that (3.10) implies that

$$\parallel u_\epsilon \parallel_{H^2(\Omega)} \to 0 \qquad \text{as} \quad \epsilon \to 0$$

and thus, since $N = 3$,

$$\parallel u_\epsilon \parallel_\infty \to 0 \qquad \text{as} \quad \epsilon \to 0.$$

This contradicts the observation made at the beginning of Section 2, that $u_\epsilon(0) \to \infty$ as $\epsilon \to 0$. Thus (3.5) is established .

To complete the proof of Lemma 3.2, we multiply (1.1) by u_ϵ, use Sobolev's inequality and the observation that

$$\int_\Omega u_\epsilon^2 \leq \int_\Omega W_\mu^2 \leq 4\pi\mu^{1-\epsilon/2}.$$

This yields

$$S \parallel u_\epsilon \parallel_6^2 \leq 3 \int_\Omega u_\epsilon^{6-\epsilon} + o(1) \qquad \text{as} \quad \epsilon \to 0,$$

where S is the best Sobolev constant. On the other hand, by Hölder's inequality,

$$\parallel u_\epsilon \parallel_{6-\epsilon} \leq \parallel u_\epsilon \parallel_6 (1 + o(1)) \qquad \text{as} \quad \epsilon \to 0.$$

So

$$\left(S+o(1)\right) \| u_\epsilon \|_{6-\epsilon}^2 \leq 3 \| u_\epsilon \|_{6-\epsilon}^{6-\epsilon} + o(1) \quad \text{as} \quad \epsilon \to 0. \tag{3.11}$$

Set

$$L = \liminf_{\epsilon \to 0} \int_\Omega u_\epsilon^{6-\epsilon}.$$

We have shown that $L > 0$. If $L = \infty$, the lemma is proved, so suppose that $L < \infty$. Then, if we let $\epsilon \to 0$ through a sequence chosen so that $\| u_\epsilon \|_{6-\epsilon}^{6-\epsilon} \to L$, we deduce from (3.11) that

$$SL^{1/3} \leq 3L$$

or

$$L \geq (\frac{1}{3}S)^{3/2} = \frac{\pi^2}{4},$$

(see for instance [T]). This finishes the proof.

If we use the Lemmas 3.1 and 3.2 in (3.2), we obtain the following first estimate.

Lemma 3.3. *Suppose* $0 \leq \lambda \leq \pi^2/4$. *Then there exists a constant* $C > 0$ *such that*

$$\epsilon \leq C\mu^{1-\epsilon/2}$$

for ϵ *sufficiently small.*

Corollary 3.4. *Suppose* $0 \leq \lambda \leq \pi^2/4$. *Then*

$$|\mu^\epsilon - 1| = O(\mu|\log \mu|) \qquad \text{as} \quad \epsilon \to 0.$$

We now return to the Pohozaev identity

$$\frac{3\epsilon}{6-\epsilon} \int_\Omega u_\epsilon^{6-\epsilon} = \int_{\partial\Omega} (x,n)\left(\frac{\partial u_\epsilon}{\partial n}\right)^2 - 2\lambda \int_\Omega u_\epsilon^2 \tag{3.12}$$

for a precise estimate. With Lemma 3.2 it is easy to establish the limit of the first integral.

Lemma 3.5. *We have*

$$\lim_{\epsilon\to 0}\int_\Omega u_\epsilon^{6-\epsilon} = \frac{\pi^2}{4}.$$

PROOF. By Lemma 2.1

$$\begin{aligned}\int_\Omega u_\epsilon^{6-\epsilon} &\le \int_\Omega W_\mu^{6-\epsilon}\\ &\le 4\pi\mu^{-\epsilon/4}\int_0^\infty \frac{s^2 ds}{(1+s^2)^{3-(\epsilon/2)}}\\ &\to \frac{\pi^2}{4} \qquad \text{as} \quad \epsilon\to 0\end{aligned}$$

by Corollary 3.4. Remembering Lemma 3.2, this completes the proof.

To estimate the right hand side of (3.12), we investigate the behaviour of u_ϵ as $\epsilon \to 0$.

Lemma 3.6. *We have*

$$\mu^{1/2}u_\epsilon(\mu x) \to U_1(x) = \frac{1}{(1+|x|^2)^{1/2}} \quad \textit{as} \quad \epsilon\to 0$$

uniformly on $\mathbf{R}^3$, *where* u_ϵ *is extended by* 0 *outside* $B_{1/\mu}$.

PROOF. We use a scaling argument, and define the family of functions

$$v_\epsilon(x) = \mu^{1/2}u_\epsilon(\mu x). \tag{3.13}$$

They satisfy

$$\begin{aligned}-\Delta v_\epsilon - \lambda\mu^2 v_\epsilon &= 3\mu^{\epsilon/2}v_\epsilon^{5-\epsilon} \quad \text{on} \quad B_{1/\mu}\\ v_\epsilon(0) &= 1,\end{aligned} \tag{3.14}$$

and, in addition, by Lemma 2.1,

$$v_\epsilon(x) \le \mu^{1/2}W_\mu(\mu x) \le \frac{1}{(1+\mu^{\epsilon/2}|x|^2)^{1/2}} \qquad \text{on} \quad B_{1/\mu}.$$

Thus, in view of Corollary 3.4, the family $\{v_\epsilon\}$ is bounded. The elliptic regularity theory implies that $\{v_\epsilon\}$ is equicontinuous on every compact subset of $B_{1/\mu}$. Hence, by Arzéla–Ascoli there exists a sequence, also denoted by v_ϵ, which converges to some function V uniformly on compact sets, but since $v_\epsilon(x) \to 0$ as $x \to \infty$, uniformly with respect to ϵ, also on $\mathbf{R}^3$. Taking the limit in (3.13), using Corollary 3.4 again, we conclude that V satisfies

$$\begin{aligned} -\Delta V &= 3V^5 \qquad \text{in} \quad \mathbf{R}^3 \\ V(0) &= 1. \end{aligned}$$

In addition, V depends only on $|x|$. Thus $V = U_1$. Since V is determined uniquely it follows that the entire family v_ϵ converges to U_1 as $\epsilon \to 0$.

We use this limit to estimate the right hand side of (1.1).

Lemma 3.7. *We have*

$$\lim_{\epsilon \to 0} \mu^{-1/2} \int_\Omega u_\epsilon^{5-\epsilon} = \frac{4\pi}{3}.$$

PROOF. Defining v_ϵ as in (3.13), we can write the integral as

$$\begin{aligned} \mu^{-1/2} \int_{B_1} u_\epsilon^{5-\epsilon}(x)dx &= \mu^{-3+\epsilon/2} \int_{B_1} v_\epsilon^{5-\epsilon}\Big(\frac{x}{\mu}\Big)dx \\ &= \mu^{\epsilon/2} \int_{B_{1/\mu}} v_\epsilon^{5-\epsilon}(y)dy. \\ &\to \int_{\mathbf{R}^3} U_1^5(y)dy \qquad \text{as} \quad \epsilon \to 0 \\ &= \frac{4\pi}{3}. \end{aligned}$$

(The last integral can easily be computed by recalling that $-\Delta U_1 = 3U_1^5$ on $\mathbf{R}^3$).

Remark. Because at any $x \neq 0$,

$$\mu^{-1/2} u_\epsilon^{5-\epsilon}(x) \leq \frac{C}{|x|^5}\mu^2 \to 0 \qquad \text{as} \quad \epsilon \to 0,$$

Lemma 3.7 implies that

$$\mu^{-1/2}u_\epsilon^{5-\epsilon} \to \frac{4\pi}{3}\delta_0 \qquad \text{as} \quad \epsilon \to 0, \tag{3.15}$$

where δ_0 is the Dirac mass centered at the origin.

Now, define the function

$$w_\epsilon = \mu^{-1/2}u_\epsilon.$$

It satisfies

$$\begin{cases} -\Delta w_\epsilon - \lambda w_\epsilon = 3\mu^{-1/2}u_\epsilon^{5-\epsilon} & \text{in } \Omega \\ \qquad\qquad w_\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

By Lemma 2.2 and the subsequent Remark, it follows from (3.15) that

$$w_\epsilon \to 4\pi G_\lambda \quad \text{as} \quad \epsilon \to 0 \quad \text{in} \quad L^2(\Omega), \tag{3.16}$$

and also uniformly away from the origin. Here G_λ is the Green's function defined by (1.7) and (1.8). In addition, restricted to $\partial\Omega$ we have

$$\nabla w_\epsilon \to 4\pi\nabla G_\lambda \qquad \text{as} \quad \epsilon \to 0 \quad \text{in} \quad L^2(\partial\Omega).$$

Thus, writing the right hand side of (3.12) again as $J(u_\epsilon)$, we conclude that

$$\frac{1}{\mu}J(u_\epsilon) \to 16\pi^2 J(G_\lambda) \quad \text{as} \quad \epsilon \to 0.$$

We now use a result, which will be proved in the next section:

Lemma 3.8. *Let G_λ be the Green's function defined by (1.7) and (1.8). Then for any domain $\Omega \subset \mathbf{R}^3$ with smooth boundary $\partial\Omega$,*

$$\int_{\partial\Omega}(x,n)\Big(\frac{\partial G_\lambda}{\partial n}\Big)^2 - 2\lambda\int_\Omega G_\lambda^2 = -g_\lambda(0),$$

where

$$g_\lambda(x) = G_\lambda(x) - \frac{1}{4\pi|x|}.$$

Thus, we find that the right hand side of (3.12) behaves asymptotically as

$$J(u_\epsilon) = 16\pi^2 g_\lambda(0)\mu[1+o(1)] \quad \text{as} \quad \epsilon \to 0. \tag{3.17}$$

Remembering the behaviour of the integral on the left hand side of (3.12), given in Lemma 3.5, we arrive at the final result

$$\lim_{\epsilon\to 0} \frac{\epsilon}{\mu} = -\frac{32}{\pi} \cdot 4\pi g_\lambda(0). \tag{3.18}$$

This completes part (a) of the proof of the first part of Theorem 1. The proof of part (b) follows immediately from (3.16) and (3.18) and the observation that

$$-4\pi g_\lambda(0) = \frac{\sqrt{\lambda}}{\tan\sqrt{\lambda}} > 0 \qquad \text{for} \quad \lambda < \pi^2/4.$$

4. A Pohozaev-type identity for the Green's function. Throughout this section, Ω is an *arbitrary* bounded domain in $\mathbf{R}^N$ $(N > 2)$ with smooth boundary $\partial\Omega$, and λ_1 the first eigenvalue of $-\Delta$ in Ω with Dirichlet boundary conditions.

Let $G_\lambda(x,y)$ be the Green's function of $-\Delta - \lambda$ in Ω, i.e. for any $y \in \Omega$, $G_\lambda(\cdot, y)$ satisfies

$$-\Delta G_\lambda - \lambda G_\lambda = \delta_y \qquad \text{in} \quad \Omega \tag{4.1}$$

$$G_\lambda = 0 \qquad \text{on} \quad \partial\Omega, \tag{4.2}$$

where δ_y denotes the Dirac mass concentrated at $x = y$, and let $g_\lambda(x,y)$ denote the regular part of G_λ,

$$g_\lambda(x,y) = G_\lambda(x,y) - \frac{1}{(N-2)\sigma_N |x-y|^{N-2}}, \tag{4.3}$$

where σ_N denotes the area of the unit sphere in $\mathbf{R}^N$.

Remark. When $\lambda \neq 0$, then if $N = 3$, $g_\lambda(\cdot, y) \in C(\Omega)$, but if $N \geq 4$, $g_\lambda(x,y)$ has no finite limit as $x \to y$. Of course, if $\lambda = 0$, $g_0(\cdot, y)$ is a smooth function for any $N > 2$.

Conforming with the notation of the previous sections, we shall write $G_\lambda(x) = G_\lambda(x,0)$ and $g_\lambda(x) = g_\lambda(x,0)$ if $0 \in \Omega$.

Theorem 4.1. *Suppose $N = 3$, and let $G_\lambda(\cdot, y)$ be defined by (4.1), (4.2) and $g_\lambda(\cdot, y)$ by (4.3). Then*

$$\int_{\partial\Omega} (x-y,n)\Big(\frac{\partial G_\lambda}{\partial n}(x,y)\Big)^2 - 2\lambda \int_\Omega G_\lambda^2(x,y) = -g_\lambda(y,y).$$

In the proof of Theorem 4.1 we shall use the following Lemma about the linear problem

$$-\Delta u - \lambda u = f(x) \quad \text{in} \quad \Omega \tag{4.4}$$

$$u = 0 \quad \text{on} \quad \partial\Omega. \tag{4.5}$$

Lemma 4.2. *Let u be a solution of (4.4), (4.5) in which $\lambda < \lambda_1$ and $f \in L^\infty(\Omega)$. Then*

$$\int_{\partial\Omega} (x,n)\Big(\frac{\partial u}{\partial n}\Big)^2 - 2\lambda \int_\Omega u^2 = -\int_\Omega f\{(N-2)u + 2(x,\nabla u)\}. \tag{4.6}$$

PROOF. Observe that since $f \in L^\infty$, $u \in W^{2,p}$ for any $p > 1$, whence the integrals in (4.6) are all well defined.
We first multiply (4.4) by u and integrate over Ω. This yields, in view of the boundary condition,

$$\int_\Omega |\nabla u|^2 - \lambda \int_\Omega u^2 = \int_\Omega fu. \tag{4.7}$$

Next, we multiply (4.4) by $(x, \nabla u)$ and integrate over Ω to obtain

$$\begin{aligned} &-\int_{\partial\Omega} (x,n)\Big(\frac{\partial u}{\partial n}\Big)^2 + \int_\Omega (\nabla u, \nabla(x,\nabla u)) - \lambda \int_\Omega u(x,\nabla u) \\ &= \int_\Omega f(x,\nabla u), \end{aligned}$$

which we can write as

$$\begin{aligned} -\int_{\partial\Omega} (x,n)\Big(\frac{\partial u}{\partial n}\Big)^2 + \int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega (x, \nabla(|\nabla u|^2)) + \\ -\frac{1}{2}\lambda \int_\Omega (x, \nabla u^2) = \int_\Omega f(x,\nabla u) \end{aligned}$$

or

$$-\frac{1}{2}\int_{\partial\Omega}(x,n)\Big(\frac{\partial u}{\partial n}\Big)^2+\Big(1-\frac{N}{2}\Big)\int_\Omega|\nabla u|^2+ \\ +\lambda\frac{N}{2}\int_\Omega u^2=\int_\Omega f(x,\nabla u). \tag{4.8}$$

If we now use (4.7) to eliminate the second integral on the left hand side in (4.8) we end up with the required identity.

Proof of Theorem 4.1. Without loss of generality we set $y=0$. Define the family of functions

$$\delta_\rho=\frac{1}{|B_\rho|}\chi_{B_\rho}\quad \rho>0$$

with $|B_\rho|=\frac{4}{3}\pi\rho^3$. Plainly the functions δ_ρ converge weakly to δ_0 as $\rho\to 0$.
Let v_ρ be the solution of the equation

$$-\Delta v=\delta_\rho \qquad \text{in} \quad \mathbf{R}^3$$

such that $v_\rho(x)\to 0$ as $|x|\to\infty$. It is readily verified that

$$v_p(x)=\begin{cases}-\dfrac{r^2}{8\pi\rho^3}+\dfrac{3}{8\pi\rho} & \text{if } 0<r<\rho\\ \dfrac{1}{4\pi r} & \text{if } \rho\le r<\infty\,.\end{cases}$$

Let u_ρ be the solution of the problem

$$\begin{cases}-\Delta u-\lambda u=\delta_\rho & \text{in } \Omega\\ \qquad\qquad u=0 & \text{on } \partial\Omega\end{cases}$$

so that by Lemma 4.2,

$$\int_{\partial\Omega}(x,n)\Big(\frac{\partial u_\rho}{\partial n}\Big)^2-2\lambda\int_\Omega u_\rho^2=-\int_\Omega\delta_\rho\{u_\rho+2(x,\nabla u_\rho)\}. \tag{4.9}$$

Now let $\rho\to 0$. Then by Lemma 2.2 and the subsequent Remark, $u_\rho\to G_\lambda$ and the *left hand side* of (4.9) converges to

$$\int_{\partial\Omega}(x,n)\Big(\frac{\partial G_\lambda}{\partial n}\Big)^2-2\lambda\int_\Omega G_\lambda^2. \tag{4.10}$$

To evaluate the limit of the *right hand side* we write it as

$$(4.11)\quad -\int_\Omega \delta_\rho\{(u_\rho - v_\rho) + 2(x, \nabla(u_\rho - v_\rho))\} - \int_\Omega \delta_\rho\{v_\rho + 2(x, \nabla v_\rho)\}.$$

It follows by direct computation that

$$(4.12)\qquad \int_\Omega \delta_\rho\{v_\rho + 2(x, \nabla v_\rho)\} = 0.$$

On the other hand,

$$\begin{cases} -\Delta(u_\rho - v_\rho) - \lambda(u_\rho - v_\rho) = \lambda v_\rho & \text{in } \Omega \\ u_\rho - v_\rho = -v_\rho & \text{on } \partial\Omega \end{cases}$$

and hence, since $v_\rho \to 1/4\pi r$ as $\rho \to 0$ in $L^2(\Omega)$, $u_\rho - v_\rho \to g_\lambda$ as $\rho \to 0$ in $H^2(\Omega)$, and hence uniformly in $\bar{\Omega}$, where g_λ is the solution of the problem

$$\begin{aligned} -\Delta g - \lambda g &= \frac{\lambda}{4\pi r} && \text{in } \Omega \\ g &= -\frac{1}{4\pi r} && \text{on } \partial\Omega, \end{aligned}$$

i.e. g_λ is the regular part of the Green's function. Therefore

$$(4.13)\qquad \int_\Omega \delta_\rho(u_\rho - v_\rho) \to g_\lambda(0) \quad \text{as} \quad \rho \to 0.$$

Finally,

$$(4.14)\qquad \left|\int_\Omega \delta_\rho(x, \nabla(u_\rho - v_\rho))\right| \le \frac{3}{4\pi\rho^2}\int_{B_\rho} |\nabla(u_\rho - v_\rho)|.$$

But $\nabla(u_\rho - v_\rho)$ is bounded in $H^1(\Omega)$, and hence in $L^6(\Omega)$. So, by Hölder's inequality,

$$\int_{B_\rho} |\nabla(u_\rho - v_\rho)| \le C\|\nabla(u_\rho - v_\rho)\|_6 \rho^{5/2}$$

for some positive constant C, which implies by (4.14) that

$$(4.15)\qquad \left|\int_\Omega \delta_\rho(x, \nabla(u_\rho - v_\rho))\right| = O(\rho^{1/2}) \qquad \text{as} \quad \rho \to 0.$$

So, (4.12), (4.13) and (4.15) together imply that the right hand side of (4.9), given by (4.11), tends to

$$-g_\lambda(0) \tag{4.16}$$

as $\rho \to 0$. Thus we conclude that letting $\rho \to 0$ in (4.9) yields the desired identity.

In the next two theorems we establish Pohozaev–type identities for the Green's function G_λ, when $\lambda = 0$, in arbitrary domains in $\mathbf{R}^N (N > 2)$. The second of these will be used in Section 8. For convenience we write $G(x,y) = G_0(x,y)$ and $g(x,y) = g_0(x,y)$.

Theorem 4.3. *We have, for every $y \in \Omega$,*

$$\int_{\partial\Omega} (x-y,n)\Big(\frac{\partial G}{\partial n}(x,y)\Big)^2 dx = -(N-2)g(y,y),$$

where $n = n(x)$ denotes the outward normal to $\partial\Omega$ at x.

PROOF. As in the proof of Theorem 4.1 we may assume that $y = 0$ and consider the family of functions

$$\delta_\rho = \frac{1}{|B_\rho|}\chi_{B_\rho}$$

with $|B_\rho| = \sigma_N \rho^N/N$. Let v_ρ be the solution of the equation

$$-\Delta v = \delta_\rho \quad \text{in} \quad \mathbf{R}^N$$

such that $v_\rho(x) \to 0$ as $|x| \to \infty$. It is readily verified that

$$v_\rho(x) = \begin{cases} -\dfrac{r^2}{2\sigma_N\rho^N} + \dfrac{N}{2\sigma_N(N-2)\rho^{N-2}} & \text{if } 0 < r < \rho \\ \dfrac{1}{(N-2)\sigma_N r^{N-2}} & \text{if } \rho \le r < \infty. \end{cases}$$

Let u_ρ be the solution of the problem

$$\begin{cases} -\Delta u = \delta_\rho & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

so that, by Lemma 4.2

$$\int_{\partial\Omega}(x,n)\Big(\frac{\partial u_\rho}{\partial n}\Big)^2 = -\int_\Omega \delta_\rho\{(N-2)u_\rho + 2(x,\nabla u_\rho)\}.$$

Now let $\rho \to 0$. Clearly $u_\rho \to G$ in $C^k(\omega)$ for any integer k and any neighbourhood ω of $\partial\Omega$ which does not contain 0, and thus the left hand side converges to

$$\int_{\partial\Omega}(x,n)\Big(\frac{\partial G}{\partial n}\Big)^2.$$

To evaluate the right hand side we write it as

$$\begin{aligned}&-\int_\Omega \delta_\rho\{(N-2)(u_\rho - v_\rho) + 2(x,\nabla(u_\rho - v_\rho))\}\\ &-\int_\Omega \delta_\rho\{(N-2)v_\rho + 2(x,\nabla v_\rho)\}.\end{aligned}$$

It follows by direct computation that

$$\int_\Omega \delta_\rho\{(N-2)v_\rho + 2(x,\nabla v_\rho)\} = 0.$$

On the other hand,

$$\begin{aligned}-\Delta(u_\rho - v_\rho) &= 0 \quad \text{in } \Omega\\ u_\rho - v_\rho &= -v_\rho \quad \text{on } \partial\Omega\end{aligned}$$

and hence, since $v_\rho(x) \to 1/\{(N-2)\sigma_N r^{N-2}\}$ as $\rho \to 0, u_\rho - v_\rho \to g$ as $\rho \to 0$ in $C^k(\overline{\Omega})$ for any integer k. Therefore

$$\int_\Omega \delta_\rho(u_\rho - v_\rho) \to g(0,0)$$

and

$$\int_\Omega \delta_\rho(x,\nabla(u_\rho - v_\rho)) \to 0$$

as $\rho \to 0$.

Theorem 4.4. *We have for every $y \in \Omega$,*

$$\int_{\partial\Omega} \Big(\frac{\partial G}{\partial n}(x,y)\Big)^2 n(x)dx = -\nabla\phi(y),$$

where $\phi(y) = g(y,y)$ and $n(x)$ denotes the outward normal to $\partial\Omega$ at x.

PROOF. Without loss of generality we may assume that $y = 0$. We use the same notation as in the proof of Theorem 4.1. Multiplying the equation

$$-\Delta u_\rho = \delta_\rho \qquad \text{in} \quad \Omega$$

through by $\partial u_\rho/\partial x_i$ and integrating over Ω we obtain

$$-\frac{1}{2}\int_{\partial\Omega}\Big(\frac{\partial u_\rho}{\partial n}\Big)^2(n,e_i) = \int_\Omega \delta_\rho \frac{\partial u_\rho}{\partial x_i}dx, \tag{4.17}$$

where e_i denotes the unit vector along the x_i-axis. But

$$\int_\Omega \delta_\rho \frac{\partial u_\rho}{\partial x_i}dx = \int_\Omega \delta_\rho \frac{\partial}{\partial x_i}(u_\rho - v_\rho)dx + \int_\Omega \delta_\rho \frac{\partial v_\rho}{\partial x_i}dx$$

and, since the last integral is zero by symmetry, we have

$$\int_\Omega \delta_\rho \frac{\partial u_\rho}{\partial x_i}dx = \int_\Omega \delta_\rho \frac{\partial}{\partial x_i}(u_\rho - v_\rho)dx. \tag{4.18}$$

On the other hand $(u_\rho - v_\rho)(x) \to g(x,0)$ in $C^k(\overline{\Omega})$ for any integer k, as $\rho \to 0$ and thus from (4.18) we deduce that

$$\lim_{\rho\to 0}\int_\Omega \delta_\rho \frac{\partial u_\rho}{\partial x_i} = \frac{\partial}{\partial x_i} g(x,0)\Big|_{x=0}.$$

Passing to the limit in (4.17) we are led to

$$-\frac{1}{2}\int_\Omega \Big(\frac{\partial G}{\partial n}(x,0)\Big)^2(n,e_i) = \frac{\partial}{\partial x_i} g(x,0).$$

Finally, we differentiate the relation $\phi(x) = g(x,x)$ and we find

$$\frac{\partial\phi}{\partial x_i}(0) = \frac{\partial g}{\partial x_i}(0,0) + \frac{\partial g}{\partial y_i}(0,0) = 2\frac{\partial g}{\partial x_i}(0,0)$$

by the symmetry of the function $g(x, y)$.

5. A good approximation for u_ϵ. To study the properties of u_ϵ as $\epsilon \to 0$ when $\lambda = \pi^2/4$, we need a better estimate for $J(u_\epsilon)$ than is given in (3.17). Thus it is necessary to establish with greater precision the behaviour of $u_\epsilon(x)$ as $\epsilon \to 0$. Writing as before

$$\mu = (u_\epsilon(0))^{-2}, \tag{5.1}$$

We shall prove in this section that in all of Ω, u_ϵ can be very well approximated – in some adequate norms – by the function

$$\phi_\mu = U_\mu + 4\pi\sqrt{\mu} g_\lambda, \tag{5.2}$$

where g_λ is the regular part of the Green's function defined in (1.10) and $0 \le \lambda \le \pi^2/4$. Consider the "error term" η defined by

$$\eta = u_\epsilon - \phi_\mu. \tag{5.3}$$

Then an easy calculation shows that

$$-\Delta\eta - \lambda\eta = f \quad \text{in} \quad \Omega \tag{5.4}$$

$$\eta = b \quad \text{on} \quad \partial\Omega, \tag{5.5}$$

where

$$f = \lambda\Big(U_\mu - \frac{\sqrt{\mu}}{|x|}\Big) + 3\Big(u_\epsilon^{5-\epsilon} - U_\mu^5\Big), \tag{5.6}$$

$$b = -(U_\mu + 4\pi\sqrt{\mu} g_\lambda). \tag{5.7}$$

Recall, moreover, from Section 3 that

$$\epsilon = O(\mu) \quad \text{as} \quad \mu \to 0. \tag{5.8}$$

Our main estimate for η will be the following.

Proposition 5.1. *Let η be defined by (5.3). Then*

$$\|\eta\|_{L^2(\Omega)} + \|\nabla\eta\|_{L^2(\partial\Omega)} \le C(\mu^{5/2}|\log\mu| + \epsilon\mu^{1/2}|\log\mu|),$$

where $\nabla\eta$ denotes the full gradient of η, and not just the tangential gradient.

The proof of Proposition 5.1 is based on Lemma 2.2 which requires estimates for f in $L^1(\Omega)$ and $L^\infty(\omega)$, where ω is a suitable neighbourhood of $\partial\Omega$ and for b in $C^{2,\alpha}(\partial\Omega)$. We shall now derive these bounds in succession.

Lemma 5.2. *Let $\alpha \in (0,1)$. Then*

$$\|b\|_{C^{2,\alpha}(\partial\Omega)} = O(\mu^{5/2}) \quad as \quad \mu \to 0.$$

PROOF. On $\partial\Omega$, $G_\lambda = 0$, and so

$$b(x) = -\sqrt{\mu}\Big(\frac{1}{\sqrt{\mu^2 + |x|^2}} - \frac{1}{|x|}\Big).$$

Since $|x|$ is bounded away from zero on $\partial\Omega$, it readily follows that

$$\|b\|_{C^k(\partial\Omega)} = O(\mu^{5/2}) \quad \text{as} \quad \mu \to 0$$

for any $k \geq 0$.

Lemma 5.3. *Let ω be a neighbourhood of $\partial\Omega$ in Ω, which does not contain the origin. Then*

$$\|f\|_{L^\infty(\omega)} = O(\mu^{5/2}) \quad as \quad \mu \to 0.$$

PROOF. As in the proof of Lemma 5.2, one proves that the first term in (5.6) is $O(\mu^{5/2})$ as $\mu \to 0$. As to the second term, plainly

$$\|U_\mu\|_{L^\infty(\omega)} \leq C\sqrt{\mu},$$

where C is some generic constant, and, since by Lemma 2.1 $u_\epsilon \leq W_\mu$, we have

$$\|u_\epsilon\|_{L^\infty(\omega)} \leq |W_\mu|_{L^\infty(\omega)} \leq C\sqrt{\mu}$$

Thus

$$\|U_\mu^5\|_{L^\infty(\omega)} \leq C\mu^{5/2}$$

and

$$\|u_\epsilon^5\|_{L^\infty(\omega)} \leq C\mu^{5/2}\mu^{-\epsilon/2} \leq C\mu^{5/2}$$

in view of (5.8).

To estimate f in $L^1(\Omega)$, we split it into three pieces:

$$f = f_1 + f_2 + f_3,$$

where

$$f_1 = \lambda\Big(U_\mu - \frac{\sqrt{\mu}}{|x|}\Big), \tag{5.9}$$

$$f_2 = 3(u_\epsilon^{5-\epsilon} - W_\mu^{5-\epsilon}), \tag{5.10}$$

$$f_3 = 3(W_\mu^{5-\epsilon} - U_\mu^5). \tag{5.11}$$

Lemma 5.4. *Let f_1 be given by (5.9). Then*

$$\|f_1\|_{L^1(\Omega)} = O(\mu^{5/2}|\log\mu|) \quad as \quad \mu \to 0.$$

PROOF. By the expression for U_μ, we have

$$\int_\Omega |f_1| = 4\pi\lambda\sqrt{\mu}\int_0^1 \Big|\frac{1}{r} - \frac{1}{\sqrt{\mu^2+r^2}}\Big| r^2 dr,$$

or, if we set $r = \mu s$,

$$\begin{aligned}\int_\Omega |f_1| &= 4\pi\lambda\mu^{5/2}\int_0^{1/\mu}\Big|\frac{1}{s} - \frac{1}{\sqrt{1+s^2}}\Big| s^2 ds \\ &\le C\mu^{5/2}\int_0^{1/\mu}\frac{s\,ds}{1+s^2} \\ &< C\mu^{5/2}|\log\mu| \qquad \text{as} \quad \mu \to 0.\end{aligned}$$

The splitting of the second term in the function f into f_2 and f_3 was inspired by the fact that $u_\epsilon \le W_\mu$ in Ω. Thus $f_2 \le 0$ and to estimate it we merely need a lower bound for u_ϵ. This bound is given in the following Lemma. It is very similar to a bound given in [AP3], but for completeness, we give the proof in the Appendix.

Lemma 5.5. *Let u_ϵ be any solution of (1.1) in which $0 \le \lambda < \pi^2$. Then, for $x \in \overline{\Omega}$,*

$$u_\epsilon(x) \ge W_\mu(x)\Big(1 - \frac{\lambda}{2}|x|^2\Big) - \frac{1}{\sqrt{\mu}}[\{\sqrt{\mu}W_\mu(x)\}^{-\epsilon} - 1].$$

Lemma 5.6. *Let f_2 be given by (5.10). Then for μ small*

$$\|f_2\|_{L^1(\Omega)} \le C(\epsilon\mu^{1/2}|\log\mu| + \mu^{5/2}|\log\mu|),$$

where C is some positive constant.

PROOF. By Lemmas 2.1 and 5.5

$$\begin{aligned} |f_2| &\le CW_\mu^{4-\epsilon}(W_\mu - u_\epsilon) \\ &\le C\mu^{-1/2}W_\mu^{4-\epsilon}[\{\sqrt{\mu}W_\mu\}^{-\epsilon} - 1] + C|x|^2W_\mu^{5-\epsilon}, \end{aligned} \tag{5.12}$$

where, like throughout this proof, C is some positive generic constant.
Observe that

$$\{\sqrt{\mu}W_\mu(x)\}^{-\epsilon} = \Big(\frac{\mu^2 + \alpha|x|^2}{\mu^2}\Big)^{\epsilon/2},$$

where $\alpha = \mu^{\epsilon/2} + \frac{1}{3}\lambda\mu^2$. Hence, because $|x| \le 1$,

$$\begin{aligned} \{\sqrt{\mu}W_\mu(x)\}^{-\epsilon} - 1 &\le (2\mu^{-2})^{\epsilon/2} - 1 \\ &= C\epsilon|\log\mu| \end{aligned} \tag{5.13}$$

in view of (5.8). Also, note that since $\alpha \ge \mu^{\epsilon/2}$ and $\mu^{\epsilon/2} < 1$,

$$W_\mu(x) \le \Big(\frac{\mu}{\mu^2 + \mu^{\epsilon/2}|x|^2}\Big)^{1/2} < \mu^{-\epsilon/4}U_\mu(x). \tag{5.14}$$

Thus, (5.13) and (5.14), as well as (5.8) enable us to simplify (5.12) to the following bound:

$$\begin{aligned} |f_2| &\le C\epsilon\mu^{-1/2}|\log\mu|U_\mu^{4-\epsilon} + C|x|^2U_\mu^{5-\epsilon} \\ &\le C\epsilon\mu^{-1/2}|\log\mu|U_\mu^4 + C|x|^2U_\mu^5 \end{aligned} \tag{5.15}$$

since $U_\mu^{-\epsilon} \le C$. However

$$\int_\Omega U_\mu^4 = 4\pi\mu^2 \int_0^1 \frac{r^2 dr}{(\mu^2 + r^2)^2}$$
$$= 4\pi\mu \int_0^{1/\mu} \frac{s^2 ds}{(1+s^2)^2},$$

where we have set $r = \mu s$. Because the last integral is convergent if we let μ tend to zero, we may conclude that

$$\int_\Omega U_\mu^4 \le C\mu. \tag{5.16}$$

Similarly, we obtain

$$\int_\Omega U_\mu^5 |x|^2 = 4\pi\mu^{5/2} \int_0^{1/\mu} \frac{s^4 ds}{(1+s^2)^{5/2}}$$
$$\le C\mu^{5/2} |\log \mu|. \tag{5.17}$$

Thus, integrating (5.15) over Ω, and using (5.16) and (5.17), we end up with the desired bound

$$\int_\Omega |f_2| \le C(\epsilon\mu^{1/2} \mid \log\mu \mid + \mu^{5/2} \mid \log\mu \mid).$$

Lemma 5.7. *Let f_3 be given by (5.11). Then for μ small*

$$\|f_3\|_{L^1(\Omega)} \le C(\epsilon\mu^{1/2}|\log\mu| + \mu^{5/2}),$$

where C is some positive constant.

PROOF. We split f_3 as follows:

$$f_3 = 3(W_\mu^{5-\epsilon} - U_\mu^{5-\epsilon}) + 3(U_\mu^{5-\epsilon} - U_\mu^5)$$
$$= f_{31} + f_{32}$$

in an obvious notation. By the Mean Value Theorem

$$|f_{31}| \le C\mu^{(5-\epsilon)/2} \frac{|\mu^{\epsilon/2} - 1| + \lambda\mu^2}{(\mu^2 + \mu^{\epsilon/2}|x|^2)^{(7-\epsilon)/2}} \cdot |x|^2,$$

and hence, because $\epsilon = O(\mu)$ by (5.8),

$$|f_{31}| \leq C\mu^{5/2}(\epsilon|\log\mu| + \mu^2)\frac{|x|^2}{(\mu^2+|x|^2)^{7/2}}.$$

This yields upon integration over Ω,

$$\int_\Omega |f_{31}| \leq C(\epsilon\mu^{1/2}|\log\mu| + \mu^{5/2}). \tag{5.18}$$

As to the second term,

$$\begin{aligned} |f_{32}| &= 3U_\mu^5|U_\mu^{-\epsilon} - 1| \\ &= 3U_\mu^5\Big|\Big(\frac{\mu^2+|x|^2}{\mu}\Big)^{\epsilon/2} - 1\Big| \\ &\leq C\epsilon|\log\mu|U_\mu^5, \end{aligned}$$

and so

$$\int_\Omega |f_{32}| \leq C\epsilon\mu^{1/2}|\log\mu|. \tag{5.19}$$

Together (5.18) and (5.19) yield the required estimate .

From Lemmas 5.4, 5.6 and 5.7 we finally conclude that for μ small

$$\int_\Omega |f| \leq C(\epsilon\mu^{1/2}|\log\mu| + \mu^{5/2}|\log\mu|),$$

C being some positive constant. This estimate, together with earlier estimates for f near $\partial\Omega$ and b on $\partial\Omega$ given in Lemmas 5.3 and 5.2 suffice to prove Proposition 5.1 with the help of Lemma 2.2.

The following two bounds for ϕ_μ will prove useful later.

Lemma 5.8. *Let ϕ_μ be given by (5.2). Then*

$$(a) \qquad \int_\Omega \phi_\mu^2 \leq C\mu,$$

$$(b) \qquad \int_{\partial\Omega} |\nabla\phi_\mu|^2 \leq C\mu,$$

where C is some positive constant.

PROOF. Both estimates follow immediately from the definition of U_μ and the fact that λ does not depend on μ.

6. The main results. We now return again to Pohozaev's identity for Problem (I):

$$\frac{3\epsilon}{6-\epsilon}\int_\Omega u_\epsilon^{6-\epsilon} = \int_{\partial\Omega}(x,n)\Big(\frac{\partial u_\epsilon}{\partial n}\Big)^2 - 2\lambda\int_\Omega u_\epsilon^2, \tag{6.1}$$

and use the approximation

$$u_\epsilon = \phi_\mu + \eta, \tag{6.2}$$

where

$$\phi_\mu = U_\mu + 4\pi\sqrt{\mu}g_\lambda, \tag{6.3}$$

which was discussed in the previous section to obtain an expansion for the right hand side $J(u_\epsilon)$ of (6.1).

Theorem 6.1. *We have the expansion, as $\epsilon \to 0$*

$$J(u_\epsilon) = -16\phi^2 g_\lambda(0)\mu + 4\pi^2\lambda\mu^2 + O(\epsilon\mu|\log\mu| + \mu^3|\log\mu|).$$

The proof of Theorem 6.1 is given in two lemmas.

Lemma 6.2. *Let ϕ_μ be defined by (6.3). Then*

$$J(u_\epsilon) = J(\phi_\mu) + R_\epsilon,$$

where

$$|R_\epsilon| \le C(\epsilon\mu|\log\mu| + \mu^3|\log\mu|),$$

in which C is some positive constant.

PROOF. By (6.2),

$$\begin{aligned} R_\epsilon &= J(\phi_\mu + \eta) - J(\phi_\mu) \\ &= \int_{\partial\Omega}(x,n)\left\{2\frac{\partial\phi_\mu}{\partial n}.\frac{\partial\eta}{\partial\eta} + \Big(\frac{\partial\eta}{\partial\eta}\Big)^2\right\} - 2\lambda\int_\Omega(2\phi_\mu\eta + \eta^2) \end{aligned}$$

and hence, by Cauchy– Schwarz's inequality and Lemma 5.8

$$|R_\epsilon| \le C\{\mu^{1/2}(\|\nabla\eta\|_{L^2(\partial\Omega)} + \|\eta\|_{L^2(\Omega)}) \\ +\|\nabla\eta\|^2_{L^2(\partial\Omega)} + \|\eta\|^2_{L^2(\Omega)}\}.$$

If we now use Proposition 5.1 to estimate η, we arrive at the required bound.

The function ϕ_μ is explicitly given in terms of the parameter μ, and closely related to the Green's function G_λ. Indeed

$$\begin{aligned}\phi_\mu(x) &= 4\pi\sqrt{\mu}\Big(\frac{1}{4\pi\sqrt{\mu^2+|x|^2}} + g_\lambda(x)\Big)\\ &\simeq 4\pi\sqrt{\mu}\Big(\frac{1}{4\pi|x|} + g_\lambda(x)\Big)\\ &= 4\pi\sqrt{\mu}G_\lambda(x).\end{aligned}$$

In the next Lemma we shall exploit this relationship, and our knowledge of $J(G_\lambda)$ from Theorem 4.1, to determine the first two terms of the asymptotic expansion of $J(\phi_\mu)$ as $\mu \to 0$.

Lemma 6.3. *Let ϕ_μ be given by (6.3). Then, as $\mu \to 0$,*

$$J(\phi_\mu) = -16\pi^2 g_\lambda(0)\mu + 4\lambda\pi^2\mu^2 + O(\mu^3|\log\mu|).$$

PROOF. Recall that

$$\phi_\mu = 4\pi\sqrt{\mu}G_\lambda + \Big(U_\mu - \frac{\sqrt{\mu}}{|x|}\Big) \tag{6.4}$$

Hence,

$$\frac{\partial\phi_\mu}{\partial n} = 4\pi\sqrt{\mu}\frac{\partial G_\lambda}{\partial n} + \sqrt{\mu}\frac{\partial}{\partial n}\Big(\frac{1}{\sqrt{\mu^2+|x|^2}} - \frac{1}{|x|}\Big),$$

and therefore, since $\Omega = B_1$,

$$\frac{\partial\phi_\mu}{\partial n}\Big|_{\partial\Omega} = 4\pi\sqrt{\mu}\frac{\partial G_\lambda}{\partial n}\Big|_{\partial\Omega} + O(\mu^{5/2}) \quad \text{as} \quad \mu \to 0.$$

Thus

(6.5)
$$\int_{\partial\Omega}(x,n)\Big(\frac{\partial\phi_\mu}{\partial n}\Big)^2 = 16\pi^2\mu\int_{\partial\Omega}(x,n)\Big(\frac{\partial G_\lambda}{\partial n}\Big)^2 + O(\mu^3) \quad \text{as} \quad \mu\to 0.$$

Next, by (6.3) we can write the second integral in $J(\phi_\mu)$ as

$$\begin{aligned}\int_\Omega \phi_\mu^2 &= \int_\Omega \frac{\mu}{\mu^2+|x|^2} + 8\pi\mu\int_\Omega \frac{g_\lambda}{\sqrt{\mu^2+|x|^2}} + 16\pi^2\mu\int_\Omega g_\lambda^2\\ &= 16\pi^2\mu\int_\Omega G_\lambda^2 + X_1 + X_2,\end{aligned}$$

where

$$X_1 = 8\pi\mu\int_\Omega\Big(\frac{1}{\sqrt{\mu^2+|x|^2}} - \frac{1}{|x|}\Big)g_\lambda$$

and

$$X_2 = \mu\int_\Omega\Big(\frac{1}{\mu^2+|x|^2} - \frac{1}{|x|^2}\Big).$$

Plainly, because $g_\lambda \in L^\infty(\Omega)$,

$$\begin{aligned}|X_1| &\le C\mu\int_0^1\Big|\frac{1}{\sqrt{\mu^2+r^2}} - \frac{1}{r}\Big|r^2dr\\ &\le C\mu^3|\log\mu|.\end{aligned}$$

On the other hand

$$X_2 = -4\pi\mu^3\int_0^1\frac{dr}{\mu^2+r^2} = -2\pi^2\mu^2 + O(\mu^3).$$

Thus

$$\int_\Omega \phi_\mu^2 = 16\pi^2\mu\int_\Omega G_\lambda^2 - 2\pi^2\mu^2 + O(\mu^3|\log\mu|) \quad \text{as} \quad \mu\to 0. \tag{6.6}$$

Putting (6.5) and (6.6) together, we obtain

$$J(\phi_\mu) = 16\pi^2\mu J(G_\lambda) + 4\pi^2\lambda\mu^2 + O(\mu^3|\log\mu|) \quad \text{as} \quad \mu\to 0.$$

Since $J(G_\lambda) = -g_\lambda(0)$ according to Theorem 4.1, the proof is complete.

Plainly, Lemma's 6.2 and 6.3 yield Theorem 6.1.

We are now ready to prove Theorem 2. We have $\lambda = \pi^2/4$, and so $g_\lambda(0) = 0$. This implies, according to Theorem 6.1, that $J(u_\epsilon) = O(\mu^2)$ as $\epsilon \to 0$. Therefore, we divide (6.1) by μ^2 and let ϵ tend to zero. Remembering the limit of the integral on the left hand side of (6.1), given in Lemma 3.5, we obtain

$$\frac{\pi^2}{8} \lim_{\epsilon \to 0} \frac{\epsilon}{\mu^2} = \pi^4, \tag{6.7}$$

or, since $\mu = (u_\epsilon(0))^{-2}$,

$$\lim_{\epsilon \to 0} \epsilon u_\epsilon^4(0) = 8\pi^2$$

which is the content of Part (a).
As to Part (b), we recall from (3.16) that

$$\mu^{-1/2} u_\epsilon(x) \to 4\pi G_{\pi^2/4}(x) \quad \text{as} \quad \epsilon \to 0$$

in $L^2(\Omega)$, and pointwise for $x \neq 0$. If we now use (6.7) to eliminate μ, we find the limit

$$\epsilon^{-1/4} u_\epsilon(x) \to 2^{5/4} \sqrt{\pi} G_{\pi^2/4}(x) \quad \text{as} \quad \epsilon \to 0$$

in $L^2(\Omega)$, which is the content of Part (b).

7. A related problem. Consider the problem

$$(II) \qquad \begin{cases} -\Delta u - \left(\dfrac{\pi^2}{4} + \epsilon\right) u = 3u^5 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is the unit ball in $\mathbf{R}^3$ and ϵ a small positive number. As in Problem (I), Problem (II) has a solution u_ϵ if $\epsilon > 0$ (and small enough), but it has no solution if $\epsilon = 0$ [BN]. It is again our objective to study the behaviour of u_ϵ as $\epsilon \to 0$, and specifically, to prove Theorem 3.

It is readily shown, as in Section 2, that $u_\epsilon(0) \to \infty$ as $\epsilon \to 0$, and from [AP1] one has the upper bound

$$u_\epsilon(x) \le W_\mu(x) = \Big(\frac{\mu}{\mu^2 + \alpha|x|^2}\Big)^{1/2} \quad \text{in} \quad \overline{\Omega}, \tag{7.1}$$

where

$$\mu = (u_\epsilon(0))^{-2} \tag{7.2}$$

and

$$\alpha = 1 + \frac{1}{3}\Big(\frac{\pi^2}{4} + \epsilon\Big)\mu^2. \tag{7.3}$$

One also has from [AP3] the lower bound

$$u_\epsilon(x) \ge W_\mu(x)\Big\{1 - \frac{1}{2}\Big(\frac{\pi^2}{4} + \epsilon\Big)|x|^2\Big\}. \tag{7.4}$$

The two bounds (7.1) and (7.4) relate $u_\epsilon(x)$ via the parameter μ to its central value $u_\epsilon(0)$. To relate μ to ϵ we use Pohozaev's identity again. For Problem (II) it becomes

$$J(u_\epsilon) = \int_{\partial\Omega} (x,n)\Big(\frac{\partial u_\epsilon}{\partial n}\Big)^2 - 2\Big(\frac{\pi^2}{4} + \epsilon\Big)\int_\Omega u_\epsilon^2 = 0. \tag{7.5}$$

We now estimate the two terms in $J(u_\epsilon)$ by finding a good approximation for u_ϵ. As before we use for this purpose the function

$$\phi = U_\mu + 4\pi\sqrt{\mu}\, g_{(\pi^2/4)+\epsilon},$$

where U_μ has been defined by (1.14) and g_λ in (1.10). To see that ϕ is indeed a good approximation, we note that the remainder term

$$\eta = u_\epsilon - \phi,$$

is a solution of the problem

$$-\Delta\eta - \Big(\frac{\pi^2}{4} + \epsilon\Big)\eta = f \tag{7.6}$$

$$\eta = b, \tag{7.7}$$

where

$$f = \left(\frac{\pi^2}{4} + \epsilon\right)\left(U_\mu - \frac{\sqrt{\mu}}{|x|}\right) + 3\left(u_\epsilon^5 - U_\mu^5\right)$$
$$b = -(U_\mu + 4\pi\sqrt{\mu}g_{(\pi^2/4)+\epsilon}).$$

and hence that it can be estimated by

$$\|\eta\|_{L^2(\Omega)} + \|\nabla\eta\|_{L^2(\partial\Omega)} \leq C\mu^{5/2}|\log\mu|. \tag{7.8}$$

The proof of (7.8) is very similar to that of Proposition 5.1 (actually it is a little simpler) and we therefore omit it.

It follows from (7.8) that $J(u_\epsilon)$ can be expressed as

$$J(u_\epsilon) = J(\phi) + R_\epsilon, \tag{7.9}$$

where

$$R_\epsilon = O(\mu^3|\log\mu|) \quad \text{as} \quad \epsilon \to 0. \tag{7.10}$$

For the proof of (7.9) and (7.10) we refer to Lemma 6.2.

If we now use Lemma 6.3 to evaluate $J(\phi)$, setting $\lambda = (\pi^2/4) + \epsilon$, we conclude from (7.9) that

$$-16\pi^2 g_{(\pi^2/4)+\epsilon}(0)\mu + 4\pi^2\left(\frac{\pi^2}{4} + \epsilon\right)\mu^2 = O(\mu^3|\log\mu|) \quad \text{as} \quad \epsilon \to 0$$

or, if we divide by $-16\pi^2\mu$,

$$g_{(\pi^2/4)+\epsilon}(0) = \frac{1}{4}\left(\frac{\pi^2}{4} + \epsilon\right)\mu + O(\mu^2|\log\mu|) \quad \text{as} \quad \epsilon \to 0. \tag{7.11}$$

However, $g_\lambda(0)$ is given by

$$g_\lambda(0) = -\frac{1}{4\pi}\sqrt{\lambda}\,\text{cotan}\,\sqrt{\lambda},$$

and so, if $\lambda = (\pi^2/4) + \epsilon$,

$$g_{(\pi^2/4)+\epsilon}(0) = \frac{\epsilon}{8\pi} + O(\epsilon^2) \quad \text{as} \quad \epsilon \to 0. \tag{7.12}$$

Inserting (7.12) into (7.11) we conclude first that $\mu = O(\epsilon)$ as $\epsilon \to 0$ and subsequently that

$$\frac{\epsilon}{\mu} = \frac{\pi^3}{2} + O(\epsilon|\log\epsilon|) \quad \text{as} \quad \epsilon \to 0. \tag{7.13}$$

As to the limiting behaviour of u_ϵ as $\epsilon \to 0$, we recall that by (7.8),

$$\mu^{-1/2} u_\epsilon \to 4\pi G_{\pi^2/4} \quad \text{as} \quad \epsilon \to 0$$

in $L^2(\Omega)$, and pointwise away from the origin. Hence, by (7.13)

$$\epsilon^{-1/2} u_\epsilon(x) \to 4\sqrt{2/\pi} G_{\pi^2/4} \quad \text{as} \quad \epsilon \to 0.$$

This completes the proof of Theorem 3.

8. Conjectures for general domains. We shall now formulate various conjectures for general domains in $\mathbf{R}^N$. They are motivated partly by the results in the previous sections and partly by some recent results of [R1,2]. We shall also present evidence in support of these conjectures.

CONJECTURE 1. Let $\Omega \subset \mathbf{R}^N, N \geq 3$, be a bounded domain with smooth boundary. Let u_ϵ be a solution of

$$-\Delta u_\epsilon = N(N-2) u_\epsilon^{p-\epsilon} \quad \text{in} \quad \Omega \tag{8.1}$$

$$u_\epsilon > 0 \quad \text{in} \quad \Omega \tag{8.2}$$

$$u_\epsilon = 0 \quad \text{on} \quad \partial\Omega, \tag{8.3}$$

where $p = (N+2)/(N-2)$.
We denote again by $G(x,y) = G_0(x,y)$ the Green's function of $-\Delta$ and by $g(x,y) = g_0(x,y)$ its regular part i.e.

$$g(x,y) = G(x,y) - \frac{1}{(N-2)\sigma_N |x-y|^{N-2}},$$

where σ_N is the area of the unit sphere in $\mathbf{R}^N$:

$$\sigma_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}.$$

Recall that $g(x,y)$ is smooth on $\Omega \times \Omega$.

Conjecture 1. *Assume u_ϵ is a minimizing sequence for the Sobolev inequality, i.e.*

$$\frac{\int_\Omega |\nabla u_\epsilon|^2}{\|u_\epsilon\|_{p+1}^2} = S_N + o(1), \tag{8.4}$$

where S_N is the best Sobolev constant in $\mathbf{R}^N$:

$$S_N = \pi N(N-2)\left\{\frac{\Gamma(N/2)}{\Gamma(N)}\right\}^{2/N}.$$

Then

$$\lim_{\epsilon\to 0} \epsilon\|u_\epsilon\|_{L^\infty}^2 = 2\sigma_N^2\left[\frac{N(N-2)}{S_N}\right]^{N/2}|g|,$$

where g is a critical value of the function $\phi(x) = g(x,x)$ i.e. $g = \phi(x_0)$ for some point $x_0 \in \Omega$ such that $\nabla\phi(x_0) = 0$.

Evidence. Recall that Pohozaev's identity says that if u is a solution of the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

then,

$$\left(1-\frac{N}{2}\right)\int_\Omega uf(u) + N\int_\Omega F(u) = \frac{1}{2}\int_{\partial\Omega}(x-y,n)\left(\frac{\partial u}{\partial n}\right)^2, \tag{8.5}$$

where $F(u) = \int_0^u f(t)dt$ and y is any point in $\mathbf{R}^N$. Applying this identity to (8.1) – (8.3) we obtain

$$\frac{N(N-2)^3}{2N-\epsilon(N-2)}\epsilon\int_\Omega u_\epsilon^{p+1-\epsilon} = \int_{\partial\Omega}(x-y,n)\left(\frac{\partial u_\epsilon}{\partial n}\right)^2,$$

or

$$\frac{1}{2}(N-2)^3\epsilon\|u_\epsilon\|_{p+1}^{p+1} \simeq \int_{\partial\Omega}(x-y,n)\left(\frac{\partial u_\epsilon}{\partial n}\right)^2. \tag{8.6}$$

If we multiply (8.1) by u_ϵ, integrate over Ω and use (8.4) we obtain

$$\|u_\epsilon\|_{p+1}^{p-1} \simeq \frac{S_N}{N(N-2)},$$

and therefore

$$\|u_\epsilon\|_{p+1}^{p+1} \simeq \left[\frac{S_N}{N(N-2)}\right]^{N/2}.$$

In view of (8.6) we thus find that

$$\int_{\partial\Omega}(x-y,n)\left(\frac{\partial u_\epsilon}{\partial n}\right)^2 \simeq \frac{1}{2}(N-2)^3\left[\frac{S_N}{N(N-2)}\right]^{N/2}\epsilon. \tag{8.7}$$

On the other hand we know that u_ϵ "concentrates" around some point x_0 (see [S], [L]) and near x_0 we have

$$u_\epsilon(x) \simeq \frac{\mu^{(N-2)/2}}{(\mu^2+|x-x_0|^2)^{(N-2)/2}} \tag{8.8}$$

for some appropriate $\mu = \mu_\epsilon$ which tends to 0 as $\epsilon \to 0$. Then, we have

$$\|u_\epsilon\|_{L^\infty(\Omega)} \simeq \mu^{-(N-2)/2}. \tag{8.9}$$

From (8.8) we deduce that, near $x = x_0$,

$$\begin{aligned} N(N-2)u_\epsilon^{p-\epsilon} &\simeq N(N-2)\frac{\mu^{(N+2)/2}}{(\mu^2+|x-x_0|^2)^{(N+2)/2}} \\ &= \mu^{(N-2)/2}\frac{N(N-2)\mu^2}{(\mu^2+|x-x_0|^2)^{(N+2)/2}} \\ &\simeq \mu^{(N-2)/2}K_N\delta_{x_0}, \end{aligned} \tag{8.10}$$

where

$$K_N = N(N-2)\int_{\mathbf{R}^N}\frac{dx}{(1+|x|^2)^{(N+2)/2}} = N(N-2)\int_{\mathbf{R}^N}U^p(x)dx$$

and

$$U(x) = \frac{1}{(1+|x|^2)^{(N-2)/2}}.$$

Since U satisfies the equation $-\Delta U = N(N-2)U^p$ we see that

$$N(N-2)\int_{\mathbf{R}^N} U^p dx = \int_{\mathbf{R}^N}(-\Delta U) = -\lim_{r\to\infty} r^{N-1}\sigma_N U'(r)$$
$$= (N-2)\sigma_N,$$

and therefore

$$K_N = (N-2)\sigma_N. \tag{8.11}$$

Going back to (8.1) – (8.3) we conclude that, globally on Ω,

$$u_\epsilon(x) \simeq \mu^{(N-2)/2} K_N G(x, x_0) \tag{8.12}$$

and therefore

(8.13)
$$\int_{\partial\Omega}(x-y,n)\Big(\frac{\partial u_\epsilon}{\partial n}\Big)^2 dx \simeq \int_{\partial\Omega}(x-y_0,n)\mu^{N-2}K_N^2\Big(\frac{\partial G}{\partial n}(x,x_0)\Big)^2 dx.$$

Recall that (see Theorem 4.3)

$$\int_{\partial\Omega}(x-x_0,n)\Big(\frac{\partial G}{\partial n}(x,x_0)\Big)^2 dx = -(N-2)g(x_0,x_0).$$

Hence, if we put $y = x_0$ in (8.13) we obtain using (8.11)

$$\int_{\partial\Omega}(x-x_0,n)\Big(\frac{\partial u_\epsilon}{\partial n}\Big)^2 dx = -\mu^{N-2}\sigma_N^2(N-2)^3 g(x_0,x_0). \tag{8.14}$$

Putting (8.7) and (8.14) together we are led to

$$\frac{1}{2}(N-2)^3\left[\frac{S_N}{N(N-2)}\right]^{N/2}\epsilon \simeq -\mu^{(N-2)}\sigma_N^2(N-2)^3 g(x_0,x_0).$$

Consequently, using (8.9), we conjecture that

$$\|u_\epsilon\|^2_{L^\infty(\Omega)} \simeq \mu^{-(N-2)} \simeq \frac{2}{\epsilon}\sigma_N^2\left[\frac{N(N-2)}{S_N}\right]|g(x_0,x_0)|$$

since $g(x,y) < 0$ (by the maximum principle).

Finally, we claim that the point of concentration x_0 is a critical point of the function $\phi(x) = g(x,x)$. First, note that in Pohozaev's identity (8.5) the point y is arbitrary and thus we have

$$\int_{\partial\Omega} \left(\frac{\partial u_\epsilon}{\partial n}\right)^2 n dx = 0. \tag{8.15}$$

From (8.12) we deduce that

$$\int_{\partial\Omega} \left(\frac{\partial G}{\partial n}(x, x_0)\right)^2 n(x) dx = 0. \tag{8.16}$$

To complete the argument it now suffices to apply Theorem 4.4.

Remark. Conjecture 1 is consistent with the results obtained in [AP2] when Ω is a ball.

CONJECTURE 2. Let $\Omega \subset \mathbf{R}^N, N \geq 4$, be a bounded domain with smooth boundary. Let u_ϵ be a solution of

$$\begin{cases} -\Delta u_\epsilon = N(N-2)u_\epsilon^p + \epsilon u_\epsilon & \text{in } \Omega, \\ u_\epsilon > 0 & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where $p = (N+2)/(N-2)$.

Conjecture 2. *Assume $\{u_\epsilon\}$ is a minimizing sequence for the Sobolev inequality. Then*

$$\begin{aligned} \lim_{\epsilon\to 0} \epsilon \|u_\epsilon\|_{L^\infty(\Omega)}^{2(N-4)/N-2)} &= \frac{(N-2)^3\sigma_N}{2a_N}|g| \quad && \text{if } N > 4 \\ \lim_{\epsilon\to 0} \epsilon \log \|u_\epsilon\|_{L^\infty(\Omega)} &= 4\sigma_4|g| && \text{if } N = 4, \end{aligned}$$

where g is a critical value of the function $\phi(x) = g(x,x)$ and

$$a_N = \int_0^\infty \frac{r^{N-1}dr}{(1+r^2)^{N-2}} = 2\frac{N-1}{N-4}\frac{\{\Gamma(N/2)\}^2}{\Gamma(N)}.$$

Evidence. From Pohozaev's identity we obtain

$$\epsilon \int_\Omega u_\epsilon^2 = \frac{1}{2}\int_{\partial\Omega} (x-y, n)\left(\frac{\partial u_\epsilon}{\partial n}\right)^2 \tag{8.17}$$

for any point $y \in \mathbf{R}^N$. As above, we have (8.8) at the point of concentration x_0 and thus

$$\int_\Omega u_\epsilon^2 \simeq \mu^2 \int_{\mathbf{R}^N} \frac{dx}{(1+|x|^2)^{N-2}} = \mu^2 \sigma_N a_N \quad \text{if} \quad N > 4 \tag{8.18}$$

and

$$\int_\Omega u_\epsilon^2 \simeq \mu^2 |\log \mu| \sigma_4 \qquad \text{if} \quad N = 4. \tag{8.19}$$

On the other hand, we have as in Conjecture 1,

$$\begin{aligned} \int_{\partial\Omega} (x - x_0, n)\Big(\frac{\partial u_\epsilon}{\partial n}\Big)^2 &\simeq \int_{\partial\Omega} (x - x_0, n)\mu^{N-2} K_N^2 \Big(\frac{\partial G}{\partial n}(x, x_0)\Big)^2 dx \\ &= -\mu^{N-2}(N-2)^3 \sigma_N^2 g(x_0, x_0). \end{aligned} \tag{8.20}$$

Putting together (8.17) with $y = x_0$, (8.18), (8.19) and (8.20) we are led to Conjecture 2. The argument for showing that x_0 as a critical point of ϕ is the same as in Conjecture 1. Part of this programme has been made rigorous by O. Rey [R2].

Remark. Conjecture 2 is consistent with the results for the ball in [AP3].

CONJECTURE 3. Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with smooth boundary. Let u_ϵ be the solution of

$$\begin{cases} -\Delta u_\epsilon = 3u_\epsilon^{5-\epsilon} + \lambda u_\epsilon & \text{in } \Omega, \\ u_\epsilon > 0 & \text{in } \Omega, \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

As before, we denote by $G_\lambda(x,y)$ the Green's function of $-\Delta - \lambda$ on Ω and by $g_\lambda(x,y)$ its regular part. It is not difficult to see that $\phi_\lambda(x) = g_\lambda(x,x)$ is smooth on Ω

Conjecture 3. *(i) Assume $\phi_\lambda(x) \le 0$ on Ω and $\{u_\epsilon\}$ is a minimizing sequence for the Sobolev inequality. Then*

$$\lim_{\epsilon \to 0} \epsilon \|u_\epsilon\|^2_{L^\infty(\Omega)} = 128|g_\lambda|,$$

where g_λ is a critical value of the function ϕ_λ, i.e $g_\lambda = \phi_\lambda(x_0)$ for some point $x_0 \in \Omega$ such that $\nabla\phi_\lambda(x_0) = 0$.
(ii) If $\phi_\lambda(x_0) = 0$, then

$$\lim_{\epsilon \to 0} \epsilon \|u_\epsilon\|^4_{L^\infty(\Omega)} = 32\lambda.$$

Remark. This would be consistent with Theorems 1 and 2.

Appendix.

Let u_ϵ be a positive radial solution of

$$(A.1) \qquad -\Delta u - \lambda u = 3u^{5-\epsilon} \quad \text{in} \quad \Omega$$

$$(A.2) \qquad u(0) = \mu^{-1/2}.$$

Then, according to [AP1], we have the *upper bound*

$$(A.3) \qquad u_\epsilon(x) \leq W_\mu(x),$$

where

$$(A.4) \qquad W_\mu(x) = \left(\frac{\mu}{\mu^2 + \alpha|x|^2}\right)^{1/2}$$

and

$$(A.5) \qquad \alpha = \mu^{\epsilon/2} + \frac{1}{3}\lambda\mu^2.$$

In addition we have the *lower bound*:

Lemma 5.5. *We have*

$$(A.6) \qquad u_\epsilon(x) \geq W_\mu(x)\left(1 - \frac{\lambda}{2}|x|^2\right) - \mu^{-1/2}[\{\mu^{1/2}W_\mu(x)\}^{-\epsilon} - 1].$$

PROOF. We follow [AP3], exploiting the radial symmetry of u_ϵ. Setting

$$t = \frac{1}{|x|} \quad \text{and} \quad y(t) = u_\epsilon(x),$$

we transform (A.1), (A.2) to

$$(A.7) \qquad y'' + t^{-4}f(y) = 0$$

$$(A.8) \qquad y(t) \to \gamma \quad \text{as} \quad t \to \infty,$$

where f is given by

$$f(s) = \lambda s + 3s^{5-\epsilon}$$

and $\gamma = \mu^{-1/2}$. Rephrasing (A.3) we have

$$(A.9) \qquad y(t) \le z(t) \qquad \text{for} \quad 1 \le t < \infty,$$

where $z(t) = W_\mu(x)$, and is given explicitly by

$$z(t) = \gamma t\Big(t^2 + \frac{1}{3}\frac{f(\gamma)}{\gamma}\Big)^{-1/2}.$$

Note that z is a solution of the problem

$$(A.10) \qquad z'' + t^{-4}\gamma^{-5}f(\gamma)z^5 = 0$$

$$(A.11) \qquad z(t) \to \gamma \quad \text{as} \quad t \to \infty.$$

We now integrate the differential equation (A.7) for y twice. That yields, if we use (A.8),

$$\begin{aligned} y(t) &= \gamma - \int_t^\infty (s-t)s^{-4}f(y(s))ds \\ &\ge \gamma - \int_t^\infty (s-t)s^{-4}f(z(s))ds \\ &= \gamma - \lambda I_1 - 3I_2, \end{aligned} \qquad (A.12)$$

where

$$I_1 = \int_t^\infty (s-t)s^{-4}z(s)ds$$

$$I_2 = \int_t^\infty (s-t)s^{-4}z^{5-\epsilon}(s)ds.$$

By the concavity and positivity of z, we have

$$\frac{z(s)}{s} < \frac{z(t)}{t} \quad \text{if} \quad s > t,$$

and hence we can estimate I_1 by

$$(A.13) \qquad I_1 < \frac{z(t)}{t}\int_t^\infty (s-t)s^{-3}ds = \frac{z(t)}{2t^2}.$$

To estimate I_2, we note that since z is increasing,

$$z^{-\epsilon}(s) < z^{-\epsilon}(t) \quad \text{if} \quad s > t$$

and so

$$I_2 < z^{-\epsilon}(t) \int_t^\infty (s-t)s^{-4}z^5(s)ds. \tag{A.14}$$

If we integrate the differential equation (A.10) for z twice and use (A.11), we obtain

$$z(t) = \gamma - \gamma^{-5}f(\gamma) \int_t^\infty (s-t)s^{-4}z^5(s)ds.$$

Using this relation to eliminate the integral from (A.14) we arrive at the estimate

$$I_2 < z^{-\epsilon}(t)\frac{\gamma - z(t)}{\gamma^{-5}f(\gamma)}.$$

This yields, remembering the definition of f,

$$I_2 < \frac{1}{3}\{\gamma - z(t)\}\Big(\frac{\gamma}{z(t)}\Big)^\epsilon. \tag{A.15}$$

Putting the estimates (A.13) and (A.15) for I_1 and I_2 into (A.12) finally yields the lower bound

$$\begin{aligned} y(t) &> z(t)\Big(1 - \frac{\lambda}{2t^2}\Big) - \{\gamma - z(t)\}\Big[\Big(\frac{\gamma}{z(t)}\Big)^\epsilon - 1\Big] \\ &< z(t)\Big(1 - \frac{\lambda}{2t^2}\Big) - \gamma\Big[\Big(\frac{\gamma}{2(t)}\Big)^\epsilon - 1\Big] \end{aligned}$$

because $\gamma > z(t)$. This reads, in terms of the original variables,

$$u_\epsilon(x) \geq W_\mu(x)\Big(1 - \frac{\lambda}{2}|x|^2\Big) - \mu^{-1/2}[\{\mu^{1/2}W_\mu(x)\}^{-\epsilon} - 1],$$

i.e. the lower bound we set out to prove.

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