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REMARKS ON SUBLINEAR ELLIPTIC EQUATIONS

HAIM BREZIS and LUC OSWALD
Département de Mathématiques, Université Paris VI, 4 Place Jussieu, 75230 Paris Cedex 05, France

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1. INTRODUCTION

Consider the problem

$$\left.\begin{aligned} &-\Delta u = f(x, u) && \text{on } \Omega, \\ &u \geqslant 0, \quad u \neq 0 && \text{on } \Omega, \\ &u = 0 && \text{on } \partial\Omega \end{aligned}\right\} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $f(x, u) : \Omega \times [0, \infty) \to \mathbb{R}$.

We make the following assumptions:

$$\left.\begin{aligned} &\text{for a.e. } x \in \Omega \text{ the function } u \mapsto f(x, u) \text{ is continuous on} \\ &[0, \infty) \text{ and the function } u \mapsto f(x, u)/u \text{ is decreasing on } (0, \infty) \end{aligned}\right\} \tag{2}$$

$$\text{for each } u \geqslant 0 \text{ the function } x \mapsto f(x, u) \text{ belongs to } L^\infty(\Omega); \tag{3}$$

$$\left.\begin{aligned} &\text{there is a constant } C > 0 \text{ such that} \\ &f(x, u) \leqslant C(u + 1) \text{ for a.e. } x \in \Omega, \quad \forall u \geqslant 0. \end{aligned}\right\} \tag{4}$$

Set

$$a_0(x) = \lim_{u \downarrow 0} f(x, u)/u$$

$$a_\infty(x) = \lim_{u \uparrow \infty} f(x, u)/u$$

so that $-\infty < a_0(x) \leqslant +\infty$ and $-\infty \leqslant a_\infty(x) < -\infty$. By a solution of (1) we mean a function $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfying (1). It follows from (2), (3), (4) that $f(x, u) \in L^\infty(\Omega)$; indeed we have

$$-|f(x, \|u\|_\infty)| \leqslant f(x, u(x)) \leqslant C(|u(x)| + 1).$$

Consequently a solution of (1) belongs to $W^{2,p}(\Omega)$ for every $p < \infty$.

Our main result is the following:

THEOREM 1. There is at most one solution of (1). Moreover, a solution of (1) exists if and only if

$$\lambda_1(-\Delta - a_0(x)) < 0 \tag{5}$$

and

$$\lambda_1(-\Delta - a_\infty(x)) > 0. \tag{6}$$

Here $\lambda_1(-\Delta - a(x))$ denotes the first eigenvalue of $-\Delta - a(x)$ with zero Dirichlet condition. Since $a(x)$ may take the values $\pm\infty$ the precise meaning of (5) and (6) will be explained in Section 3. In the special case where $f(x, u) = f(u)$ is independent of x, then (5)–(6) is equivalent to

$$a_\infty < \lambda_1(-\Delta) < a_0.$$

Theorem 1 is closely related to a number of earlier results. We refer in particular to Krasnoselskii [12, theorems 7.14, 7.15], Keller and Cohen [11], Cohen and Laetsch [6], Keller [10], Simpson and Cohen [14], Laetsch [13], Amann [1, 2], Hess [9], DeFigueiredo [7], Berestycki [5], and Smoller and Wasserman [15].

The main novelties in our approach are the following:

(a) Our proof of uniqueness involves a simple "energy" device which is reminiscent of the device used in the theory of monotone operators—in contrast with all the previous proofs based on a comparison argument and on the maximum principle.

(b) Our proof of existence relies on a minimization technique while the earlier works used most often a sub-super-solution method. In addition, we point out that the functional to be minimized, namely

$$E(u) = \tfrac{1}{2}\int |\nabla u|^2 - \int F(x, u) \qquad \text{where} \qquad F(x, u) = \int_0^u f(x, s)\, \mathrm{d}s$$

is *convex* with respect to the variable $\rho = u^2$. This fact is based on an observation of Benguria [3] (see also [4]).

(c) In most earlier works it has not been noticed—or explicitly stated—that, under assumptions (2)–(4), there is indeed a simple necessary and sufficient condition for the existence of a solution of (1).

Our paper is organized as follows: 1. Introduction; 2. Uniqueness; 3. Condition (5)–(6) is necessary; 4. Condition (5)–(6) is sufficient.

2. UNIQUENESS

Here we use only assumptions (2) and (3). We start with the following lemma.

LEMMA 1. Assume (2), (3) and let u be a solution of (1). Then we have

$$u > 0 \qquad \text{on } \Omega \tag{7}$$

and

$$\frac{\partial u}{\partial n} < 0 \qquad \text{on } \partial\Omega \tag{8}$$

(n denotes the outward normal direction).

Proof. Since $u \leqslant \|u\|_\infty$ it follows that

$$\frac{f(x, u)}{u} \geqslant \frac{f(x, \|u\|_\infty)}{\|u\|_\infty}$$

and therefore

$$f(x,u) \geqslant -Mu \qquad \text{on } \Omega$$

for some constant $M \geqslant 0$. Hence u satisfies

$$-\Delta u + Mu \geqslant 0 \qquad \text{on } \Omega$$

and the conclusion follows from the strong maximum principle (see, e.g. Gilbarg and Trudinger [8]).

Proof of uniqueness. Suppose u_1 and u_2 are two solutions of (1). We write

$$-\frac{\Delta u_1}{u_1} + \frac{\Delta u_2}{u_2} = \frac{f(x,u_1)}{u_1} - \frac{f(x,u_2)}{u_2}, \tag{9}$$

multiply (9) through by $u_1^2 - u_2^2$ and integrate over Ω. Note that u_2^2/u_1 and u_1^2/u_2 belong to H_0^1 and

$$\nabla\left(\frac{u_2^2}{u_1}\right) = 2\frac{u_2}{u_1}\nabla u_2 - \frac{u_2^2}{u_1^2}\nabla u_1, \qquad \nabla\left(\frac{u_1^2}{u_2}\right) = 2\frac{u_1}{u_2}\nabla u_1 - \frac{u_1^2}{u_2^2}\nabla u_2.$$

(We use here the fact that u_1/u_2 and u_2/u_1 belong to L^∞, which is a consequence of lemma 1.) After some rearrangements we obtain the identity

$$\int\left(-\frac{\Delta u_1}{u_1} + \frac{\Delta u_2}{u_2}\right)(u_1^2 - u_2^2) = \int \left|\nabla u_1 - \frac{u_1}{u_2}\nabla u_2\right|^2 + \left|\nabla u_2 - \frac{u_2}{u_1}\nabla u_1\right|^2 \geqslant 0. \tag{10}$$

We deduce from (9) and (10) that

$$\int\left(\frac{f(x,u_1)}{u_1} - \frac{f(x,u_2)}{u_2}\right)(u_1^2 - u_2^2) \geqslant 0$$

and we conclude (using assumption (2)) that $u_1 = u_2$.

Remark 1. If instead of (2) we just assume that the function $u \mapsto f(x,u)/u$ is nonincreasing (for a.e. $x \in \Omega$), uniqueness may fail. However, we obtain

$$\frac{\nabla u_1}{u_1} = \frac{\nabla u_2}{u_2} \quad \text{and} \quad \frac{f(x,u_1)}{u_1} = \frac{f(x,u_2)}{u_2},$$

which implies in particular that u_1/u_2 is a constant. In many cases we can still conclude that $u_1 = u_2$.

3. CONDITION (5)–(6) IS NECESSARY

First we observe that

$$a_\infty(x) \leqslant f(x,1) \quad \text{and} \quad a_0(x) \geqslant f(x,1) \quad \text{for a.e. } x \in \Omega$$

and hence there is a constant $C \geqslant 0$ such that

$$a_\infty(x) \leqslant C \quad \text{and} \quad a_0(x) \geqslant -C \quad \text{for a.e. } x \in \Omega.$$

The precise meaning of $\lambda_1(-\Delta - a(x))$ is

$$\lambda_1(-\Delta - a(x)) = \operatorname*{Inf}_{\substack{\varphi \in H_0^1 \\ \|\varphi\|_2 = 1}} \left\{ \int |\nabla\varphi|^2 - \int_{[\varphi \neq 0]} a\varphi^2 \right\}.$$

Note that $\int_{[\varphi \neq 0]} a\varphi^2$ makes sense if $a(x)$ is any measurable function such that either $a(x) \leqslant C$ or $a(x) \geqslant -C$ a.e. on Ω. In the first case $\lambda_1(-\Delta - a(x)) \in (-\infty, +\infty]$ and in the second case $\lambda_1(-\Delta - a(x)) \in [-\infty, +\infty)$.

Proof of (5). By definition of $\lambda_1(-\Delta - a_0(x))$, and since $u > 0$ on Ω, we have

$$\lambda_1(-\Delta - a_0(x)) \leqslant \frac{\int |\nabla u|^2 - \int a_0 u^2}{\int u^2}.$$

On the other hand we have

$$\int |\nabla u|^2 = \int f(x, u)u < \int a_0(x)u^2$$

and (5) follows.

Proof of (6). Set

$$\tilde{a}(x) = \frac{f(x, \|u\|_\infty + 1)}{\|u\|_\infty + 1} \in L^\infty(\Omega)$$

and

$$\mu = \lambda_1(-\Delta - \tilde{a}(x)).$$

Let ψ denote the corresponding eigenfunction, that is,

$$\begin{cases} -\Delta\psi - \tilde{a}\psi = \mu\psi & \text{on } \Omega \\ \psi > 0 & \text{on } \Omega \\ \psi = 0 & \text{on } \partial\Omega. \end{cases}$$

Multiplying (1) through by ψ and integrating on Ω we find

$$\int u(\tilde{a}(x)\psi + \mu\psi) = \int f(x, u)\psi.$$

On the other hand we have $f(x, u) > \tilde{a}(x)u$ and thus we obtain $\mu \int u\psi > 0$; hence $\mu > 0$. Finally we claim that

$$\lambda_1(-\Delta - a_\infty) \geqslant \mu$$

(since $a_\infty(x) \leqslant \tilde{a}(x)$) and the conclusion follows.

4. EXISTENCE

We shall establish an existence result slightly stronger than announced in theorem 1. Instead of (2) we just assume that

$$\text{for a.e. } x \in \Omega \text{ the function } u \mapsto f(x, u) \text{ is continuous on } [0, \infty). \tag{11}$$

However, we also assume that

$$\text{for each } \delta > 0 \text{ there is a constant } C_\delta \geqslant 0 \text{ such that } f(x,u) \geqslant -C_\delta u \quad \forall u \in [0,\delta], \quad \text{a.e. } x \in \Omega. \tag{12}$$

(Note that (12) is a weaker assumption that (2) + (3).)

Set

$$a_0(x) = \liminf_{u \downarrow 0} \frac{f(x,u)}{u}$$

$$a_\infty(x) = \limsup_{u \uparrow \infty} \frac{f(x,u)}{u}.$$

Under assumptions (12) and (4) there is a constant C such that $a_0(x) \geqslant -C$ and $a_\infty(x) \leqslant C$.

THEOREM 2. Assume that (3), (4), (11), (12), (5) and (6) hold. Then there is a solution of (1).

Proof. Consider the functional

$$E(u) = \tfrac{1}{2}\int |\nabla u|^2 - \int F(x,u), \qquad u \in H_0^1(\Omega)$$

where $F(x,u) = \int_0^u f(x,t)\,dt$ and $f(x,u)$ is extended to be $f(x,0)$ for $u \leqslant 0$. Note that $E(u) \in (-\infty, +\infty]$ is well-defined since $F(x,u) \leqslant C(\frac{1}{2}u^2 + |u|)$ $\forall u \in \mathbb{R}$. We claim that:

$$E \text{ is coercive on } H_0^1, \text{ that is, } \lim_{\|u\|_{H_0^1} \to \infty} E(u) = \infty; \tag{13}$$

$$E \text{ is l.s.c. for the weak } H_0^1 \text{ topology}; \tag{14}$$

$$\text{there is some } \phi \in H_0^1 \text{ such that } E(\phi) < 0. \tag{15}$$

Proof of (13). Assume, by contradiction, that there is some sequence (u_n) in H_0^1 such that

$$\|u_n\|_{H_0^1} \to \infty \qquad \text{and} \qquad E(u_n) \leqslant C.$$

We have

$$\tfrac{1}{2}\int |\nabla u_n|^2 \leqslant \int F(x,u_n) + C \tag{16}$$

and consequently we obtain

$$\tfrac{1}{2}\int |\nabla u_n|^2 \leqslant C\int (u_n^2 + 1). \tag{17}$$

Set

$$t_n = \|u_n\|_2 \qquad \text{and} \qquad v_n = u_n/t_n.$$

It follows from (17) that

$$t_n \to \infty, \qquad \|v_n\|_2 = 1 \quad \text{and} \quad \|v_n\|_{H_0^1} \leqslant C.$$

We may therefore assume that

$$v_n \rightharpoonup v \text{ weakly in } H_0^1,\ v_n \to v \text{ strongly in } L^2 \text{ and a.e. with } \|v\|_2 = 1.$$

We claim that

$$\limsup_{n\to\infty} \int \frac{F(x, t_n v_n)}{t_n^2} \leqslant \tfrac{1}{2}\int_{[v>0]} a_\infty v^2. \tag{18}$$

Indeed we write

$$\int F(x, t_n v_n) = \int_{[v>0]} F(x, t_n v_n^+) + \int_{[v\leqslant 0]} F(x, t_n v_n^+) + \int_{[v_n\leqslant 0]} F(x, t_n v_n). \tag{19}$$

We estimate the second integral by

$$\int_{[v\leqslant 0]} F(x, t_n v_n^+) \leqslant C\int_{[v\leqslant 0]} [t_n^2 (v_n^+)^2 + 1]$$

and we deduce that

$$\int_{[v\leqslant 0]} \frac{F(x, t_n v_n^+)}{t_n^2} \leqslant o(1) \qquad \text{as } n\to\infty \tag{20}$$

since $v_n \to v$ in L^2.

We estimate the third integral by

$$\int_{[v_n\leqslant 0]} F(x, t_n v_n) \leqslant C\int t_n |v_n|$$

and thus we obtain

$$\int_{[v_n\leqslant 0]} \frac{F(x, t_n v_n)}{t_n^2} \leqslant o(1) \qquad \text{as } n\to\infty. \tag{21}$$

We now turn to the first integral. We note that

$$\limsup_{u\to+\infty} \frac{F(x, u)}{u^2} \leqslant \tfrac{1}{2} a_\infty(x) \qquad \text{for a.e. } x\in\Omega$$

and therefore

$$\limsup_{n\to+\infty} \frac{F(x, t_n v_n^+(x))}{t_n^2} \leqslant \tfrac{1}{2} a_\infty(x) v^2(x) \qquad \text{a.e. on } [v>0]. \tag{22}$$

On the other hand we have

$$\frac{F(x, t_n v_n^+)}{t_n^2} \leqslant C\left[(v_n^+)^2 + \frac{1}{t_n^2}\right]$$

and since $v_n \to v$ in L^2 we may find a fixed function $h \in L^1$ such that (for some subsequence)

$$\frac{F(x, t_n v_n^+)}{t_n^2} \leqslant h \qquad \text{a.e. on } \Omega,\ \forall n. \tag{23}$$

From (22), (23) and Fatou's lemma we obtain

$$\limsup_{n\to\infty} \int_{[v>0]} \frac{F(x, t_n v_n^+)}{t_n^2} \leqslant \tfrac{1}{2} \int_{[v>0]} a_\infty v^2. \tag{24}$$

Combining (19), (20), (21) and (24) we see that (18) holds. Passing to the limit in (16) we find (using (18))

$$\tfrac{1}{2} \int |\nabla v|^2 \leqslant \tfrac{1}{2} \int_{[v>0]} a_\infty v^2. \tag{25}$$

Finally we have (by definition of α)

$$\int |\nabla v^+|^2 - \int_{[v>0]} a_\infty v^2 \geqslant \alpha \|v^+\|_2^2 \tag{26}$$

where $\alpha = \lambda_1(-\Delta - a^\infty(x)) > 0$.

Combining (25) and (26) we deduce that $v^+ = 0$ and going back to (25) we obtain $v = 0$—a contradiction since $\|v\|_2 = 1$.

Proof of (14). Suppose $u_n \rightharpoonup u$ weakly in H_0^1. Since $F(x, u_n) \leqslant C(u_n^2 + 1)$ we may apply Fatou's lemma and conclude that

$$\limsup_{n\to\infty} \int F(x, u_n) \leqslant \int F(x, u).$$

Proof of (15). We fix any $\phi \in H_0^1$ satisfying

$$\int |\nabla \phi|^2 - \int_{[\phi \neq 0]} a_0 \phi^2 < 0$$

(such a ϕ exists by assumption (5)).

We may always assume that $\phi > 0$ and that $\phi \in L^\infty$ (otherwise we replace ϕ by $|\phi|$ and truncate ϕ). We note that

$$\liminf_{u\downarrow 0} \frac{F(x, u)}{u^2} \geqslant \tfrac{1}{2} a_0(x)$$

and thus

$$\liminf_{\varepsilon\downarrow 0} \frac{F(x, \varepsilon\phi(x))}{\varepsilon^2} \geqslant \tfrac{1}{2} a_0(x)\phi^2(x) \qquad \text{a.e. on } [\phi \neq 0].$$

On the other hand we deduce from (12) that

$$\frac{F(x, \varepsilon\phi)}{\varepsilon^2} \geqslant -C\phi^2 \geqslant -C.$$

We may therefore apply Fatou's lemma and conclude that

$$\liminf_{\varepsilon\downarrow 0} \int_{[\phi\neq 0]} \frac{F(x, \varepsilon\phi)}{\varepsilon^2} \geqslant \tfrac{1}{2} \int_{[\phi\neq 0]} a_0 \phi^2;$$

thus we have

$$\liminf_{\varepsilon \downarrow 0} \int \frac{F(x, \varepsilon\varphi)}{\varepsilon^2} \geqslant \tfrac{1}{2} \int_{[\varphi \neq 0]} a_0 \varphi^2.$$

Hence we obtain

$$\tfrac{1}{2} \int |\nabla \varphi|^2 - \int \frac{F(x, \varepsilon\varphi)}{\varepsilon^2} < 0$$

for $\varepsilon > 0$ small enough.

Proof of theorem 2 *concluded.* Using (13), (14) and (15) we see that $\operatorname{Inf}_{u \in H_0^1} E(u)$ is achieved by some $u \not\equiv 0$. We may always assume that $u \geqslant 0$—otherwise we replace u by u^- and use the fact that $F(x, u) \leqslant F(x, u^+)$ (which holds since $F(x, u) = f(x, 0)u \leqslant 0$ for $u \leqslant 0$). If we knew in addition that $u \in L^\infty$ we would easily conclude that u is a solution of (1). We claim that indeed we may also assume that

$$u \in L^\infty. \tag{27}$$

For this purpose we introduce a truncated problem. We set, for each integer $k > 0$

$$\begin{cases} f^k(x, u) = \operatorname{Max}\{f(x, u), -ku\} & \text{if } u \geqslant 0 \\ f^k(x, u) = f^k(x, 0) = f(x, 0) & \text{if } u \leqslant 0 \end{cases}$$

and

$$a_0^k(x) = \liminf_{u \downarrow 0} \frac{f^k(x, u)}{u}, \qquad a_\infty^k(x) = \limsup_{u \uparrow \infty} \frac{f^k(x, u)}{u}.$$

Assumptions (3), (4), (11) and (12) hold for $f^k(x, u)$. Assumption (5) holds for a^k since

$$\lambda_1(-\Delta - a_0^k(x)) \leqslant \lambda_1(-\Delta - a_0(x)) < 0$$

because $f \leqslant f^k$ and thus $a_0 \leqslant a_0^k$. Moreover, assumption (6) holds for a^k provided k is large enough. Indeed, it is easy to check that $\lambda_1(-\Delta - a_\infty^k(x)) \uparrow \lambda_1(-\Delta - a_\infty(x))$ since $a_\infty^k \downarrow a_\infty$ as $k \uparrow \infty$.

Set

$$E_k(u) = \tfrac{1}{2} \int |\nabla u|^2 - \int F^k(x, u), \qquad u \in H_0^1.$$

It follows from the previous argument that $\operatorname{Inf}_{u \in H_0^1} E_k(u)$ is achieved by some u_k. Moreover, u_k satisfies

$$\begin{cases} -\Delta u_k = f^k(x, u_k) & \text{on } \Omega \\ u_k \geqslant 0, \quad u_k \not\equiv 0 & \text{on } \Omega \\ u_k = 0 & \text{on } \partial\Omega \end{cases}$$

(note that E_k is of class C^1 since $|f^k(x, u)| \leqslant C_k(|u| + 1)$).

A standard bootstrap argument shows that $u_k \in L^\infty$.
Set

$$v = \mathrm{Min}\{u, u_k\}.$$

We claim that

$$E(v) \leqslant E(u), \tag{28}$$

and this will conclude the proof of (27).

Indeed, we have

$$\tfrac{1}{2}\int |\nabla u_k|^2 - \int F^k(x, u_k) \leqslant \tfrac{1}{2}\int |\nabla \phi|^2 - \int F^k(x, \phi) \qquad \forall \phi \in H_0^1. \tag{29}$$

In (29) we choose $\phi = \mathrm{Max}\{u, u_k\}$ and we find

$$\tfrac{1}{2}\int_{[u_k<u]} |\nabla u_k|^2 - \int_{[u_k<u]} F^k(x, u_k) \leqslant \tfrac{1}{2}\int_{[u_k<u]} |\nabla u|^2 - \int_{[u_k<u]} F^k(x, u). \tag{30}$$

On the other hand we have

$$E(v) - E(u) = \int_{[u_k<u]} \{\tfrac{1}{2}|\nabla u_k|^2 - \tfrac{1}{2}|\nabla u|^2 - F(x, u_k) + F(x, u)\}$$

and using (30) we obtain

$$F^k(x, u_k) - F^k(x, u) - F(x, u_k) + F(x, u) = \int_{u_k}^{u} [f(x, t) - f^k(x, t)]\, \mathrm{d}t \leqslant 0$$

on the set $[u_k < u]$. Thus (28) is proved.

Remark 2. We assume again that (2) holds. Then the functional E is convex with respect to the variable $\rho = u^2$. More precisely, the functional $\rho \mapsto E(\sqrt{\rho})$ defined on the convex set

$$K = \{\rho \in L^1; \rho \geqslant 0 \text{ a.e. and } \sqrt{\rho} \in H_0^1\}$$

is *convex*. Indeed, it is known (and easy to prove) that the functional $\rho \mapsto \int |\nabla \sqrt{\rho}|^2$ is convex (see [3] and also [4]) while the function $\rho \mapsto -F(x, \sqrt{\rho})$ is convex since its derivative

$$-\tfrac{1}{2}\frac{f(x, \sqrt{\rho})}{\sqrt{\rho}}$$

is increasing.

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