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Perturbations of Nonlinear Maximal Monotone Sets in Banach Space*

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Introduction

Let X be a real Banach space and let X^* be its dual. The value of $x^* \in X^*$ at $x \in X$ will be denoted by either (x^*, x) or (x, x^*). A subset of $X \times X^*$ is called *monotone* if for each pair $[x_i, x_i^*] \in A$, $i = 1, 2$, we have

$$(x_1^* - x_2^*, x_1 - x_2) \geqq 0 .$$

A monotone set is said to be *maximal monotone* if it is not properly contained in any other monotone set. Monotone sets are usually regarded as (graphs of) multivalued monotone mappings from X to X^*. Accordingly we shall use standard functional notation even when dealing with sets. Let A be a subset of $X \times X^*$. We define

$$A^{-1} = \{[y^*, x] : [x, y^*] \in A\}, \qquad Ax = \{z^* : [x, z^*] \in A\} ,$$
$$D(A) = \{x : Ax \neq \varnothing\}, \qquad R(A) = \bigcup_{x \in D(A)} Ax .$$

If α is real and B is a subset of $X \times X^*$ we also define

$$\alpha A = \{[x, \alpha y^*] : [x, y^*] \in A\}$$

and

$$A + B = \{[x, y^* + z^*] : [x, y^*] \in A \quad \text{and} \quad [x, z^*] \in B\} .$$

If C is a subset of X or X^* we define

$$|C| = \begin{cases} \inf\{\|z\| : z \in C\} & \text{if} \quad C \neq \varnothing , \\ +\infty & \text{if} \quad C = \varnothing . \end{cases}$$

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Let A and B be monotone sets in $X \times X^*$. $A + B$ is clearly monotone. However, if A and B are maximal monotone, then $A + B$ is not necessarily maximal monotone. $A + B$ may even be void; this happens when $D(A) \cap D(B) = \varnothing$. The main problem in the perturbation theory of maximal monotone sets is that of determining useful conditions under which the sum of two (or more) maximal monotone sets (or just monotone sets) is maximal monotone. Results in this direction have been established by Lescarret [9], Browder [4] and Rockafellar [15]. A common feature of most of these results is the assumption that at least one of the maximal monotone sets A or B has a domain with non-empty interior. In many applications however (see e.g. the examples in Section 3), the interior of the domains of the monotone sets, or operators, involved is void and the previous results cannot be applied directly.

The purpose of this paper is to obtain useful criteria for the case in which neither $D(A)$ nor $D(B)$ has a non-void interior. In Section 1 we present some preliminaries. We start with a renorming theorem for reflexive Banach spaces and continue with some convergence statements for monotone sets in Banach spaces. Section 2 contains the main results. It begins with a theorem, Theorem 2.1, which is the main tool used in proving all subsequent assertions. Theorem 2.1 is also used to obtain a simple proof of the main theorem of Rockafellar [15]. In Section 3 we give some examples of how the preceding results may be used in the study of certain nonlinear partial differential equations.

1. Monotone Sets in a Reflexive Banach Space

Let X be a real reflexive Banach space. It is known (see E. Asplund [1]) that if X is reflexive then there is an equivalent norm on X such that X is strictly convex under this norm and X^* is strictly convex under the dual norm. For our work we shall need a slightly stronger result:

THEOREM 1.1. *Let X be a reflexive Banach space with norm $\| \ \|$. For every $a > 1$ there exists an equivalent norm $\| \ \|_a$ on X such that*

(i) $\| \ \|_a$ *is a strictly convex norm and its dual norm* $\| \ \|_a^*$ *is also strictly convex,*

(ii) $a^{-1}\| \ \|_a \leqq \| \ \| \leqq a\| \ \|_a$ *and* $a^{-1}\| \ \|_a^* \leqq \| \ \|^* \leqq a\| \ \|_a^*$.

Proof: The proof of Theorem 1.1 follows, with minor changes, the arguments of E. Asplund in [1]. By a theorem of J. Lindenstrauss [10], there exists an equivalent norm $\| \ \|_X$ on X (respectively, $\| \ \|_{X^*}$ on X^*) which is strictly convex. (Note that $\| \ \|_X$ and $\| \ \|_{X^*}$ are not necessarily dual norms).

Let $\varepsilon > 0$ and define

$$f_0(x) = \tfrac{1}{2}(\|x\|^2 + \varepsilon\|x\|_X^2), \qquad x \in X,$$

$$g_0^*(y) = \tfrac{1}{2}(\|y\|^{*2} + \varepsilon\|y\|_{X^*}^2), \qquad y \in X^*.$$

Choose ε small enough so that $f_0(x) \leqq \frac{1}{2}a^2 \|x\|^2$ and $g_0^*(y) \leqq \frac{1}{2}a^2 \|y\|^{*2}$, where g_0 is the conjugate function of g_0^*. Clearly there exists a $C > 0$ such that $g_0 \leqq f_0 \leqq (1 + C)g_0$. We consider the iteration procedure of [1]:

$$f_{n+1}(x) = \tfrac{1}{2}(f_n(x) + g_n(x)) ,$$
$$g_{n+1}(x) = \inf_{y \in X} \tfrac{1}{2}(f_n(x + y) - g_n(x - y)) ,$$

and obtain $h(x) = \lim f_n(x) = \lim g_n(x)$. Since $g_0(x) \leqq h(x) \leqq f_0(x)$, we have $(1/2a^2) \| x\|^2 \leqq h(x) \leqq \frac{1}{2}a^2\|x\|^2$. Let $\| \quad \|_a = \sqrt{2h(x)}$. By a theorem in [1], $\| \quad \|_a$ and $\| \quad \|_a^*$ are strictly convex norms (since f_0 and g_0^* are strictly convex) and they clearly satisfy (ii).

We shall henceforth assume that X is a real reflexive Banach space. The norms $\| \quad \|$ in X and $\| \quad \|^*$ in X^* will always be dual norms and if there is no danger of confusion we shall omit the star from the norm $\| \quad \|^*$ in X^* and denote both the norm in X and its dual norm in X^* by $\| \quad \|$.

Let F be the *duality map* of X, i.e., the subset of $X \times X^*$ defined by

$$F = \{[x, x^*] : x \in X, x^* \in X^* \text{ and } (x, x^*) = \|x^2\| = \|x^*\|^2\} .$$

If X and X^* are strictly convex, then F is a single-valued function defined on all of X. We shall write $x^* = F(x)$ if $[x, x^*] \in F$. In this case, it is easy to check that F is one-to-one and onto. Moreover, the map F is strictly monotone (i.e., $(F(x) - F(y), x - y) > 0$ for $x \neq y$), hemicontinuous (i.e., the mapping $t \mapsto (F(x + ty), z)$ is continuous in t for $x, y, z \in X$), maps bounded sets into bounded sets and is coercive (i.e., $(F(x), x)/\|x\| \to \infty$ as $\|x\| \to \infty$). The duality map F clearly depends on the norm in X. If $\| \quad \|_a$ is an equivalent norm on X, then we denote the duality map corresponding to $\| \quad \|_a$ by F_a.

Let X and X^* be strictly convex and let A be a monotone set in $X \times X^*$; then it is well known (see e.g. Browder [4]) that A is maximal monotone if and only if $R(F + A) = X^*$. The following lemma is related to this result and will be needed later.

LEMMA 1.1. *Let $\{ \| \quad \|_a\}$ be a family of strictly convex equivalent norms on X such that their dual norms are also strictly convex. Let F_a be the duality map corresponding to the norm $\| \quad \|_a$. Let A be a monotone set in $X \times X^*$. If for every $f^* \in X^*$ and $u \in X$ there exists an a for which the equation*

$$v^* + F_a(v) = f^* + F_a(u) , \qquad [v, v^*] \in A , \tag{1.1}$$

has a solution $v \in X$, then A is maximal monotone.

Proof: Let $u \in X$, $f^* \in X^*$ be such that

$$(v^* - f^*, v - u) \geqq 0 \quad \text{for} \quad \text{every} \quad [v, v^*] \in A .$$

Solving equation (1.1) yields $(F_a(v) - F_a(u),\ v - u) \leqq 0$. But F_a is strictly monotone, and therefore $v = u$. Using (1.1) again, we obtain $v^* = f^*$, i.e., $[u, f^*] \in A$ and A is maximal monotone.

Our next lemma is analogous to Lemma 1.2 in Brezis-Stampacchia [2].

LEMMA 1.2. *Let B be a maximal monotone set in $X \times X^*$. If $[u_n, v_n^*] \in B$, $u_n \rightharpoonup u$ (weak convergence is denoted by $\rightharpoonup$), $v_n^* \rightharpoonup v^*$ and either*

$$\overline{\lim_{n,m\to\infty}}\ (u_n - u_m, v_n^* - v_m^*) \leqq 0 , \tag{1.2}$$

or

$$\overline{\lim_{n\to\infty}}\ (u_n - u, v_n^* - v^*) \leqq 0 , \tag{1.3}$$

then $[u, v^*] \in B$ and $(u_n, v_n^*) \to (u, v^*)$.

Proof: We prove the lemma with the condition (1.2). From the monotonicity of B it follows that

$$\lim_{n,m\to\infty} (u_n - u_m, v_n^* - v_m^*) = 0 . \tag{1.4}$$

Let $\{n_i\}$ be a subsequence of $\{n\}$ such that $(u_{n_i}, v_{n_i}^*) \to L$. From (1.4) we obtain

$$\begin{aligned} 0 &= \lim_{n_i\to\infty} \left[\lim_{n_k\to\infty} (u_{n_i} - u_{n_k}, v_{n_i}^* - v_{n_k}^*) \right] \\ &= \lim_{n_i\to\infty} [(u_{n_i}, v_{n_i}^*) - (u_{n_i}, v^*) - (u, v_{n_i}^*) + L] \\ &= 2L - 2(u, v^*) . \end{aligned}$$

Hence $L = (u, v^*)$ and therefore (since L is unique) $(u_n, v_n^*) \to (u, v^*)$. This implies that $(x - u, y^* - v^*) \geqq 0$ for every $[x, y^*] \in B$, and $[u, v^*] \in B$ now follows from the maximality of B. The proof of the lemma with the condition (1.3) is similar.

Let X and X^* be reflexive and strictly convex and let B be a maximal monotone set in $X \times X^*$. As a consequence of a theorem of Browder [4] the equations

$$F(x_\lambda - x) + \lambda x_\lambda^* = 0 , \qquad [x_\lambda, x_\lambda^*] \in B , \tag{1.5}$$

have a unique solution $[x_\lambda, x_\lambda^*]$ for every $x \in X$ and $\lambda > 0$. We define

$$x_\lambda = J_\lambda x, \qquad x \in X, \tag{1.6}$$

$$x_\lambda^* = B_\lambda x, \qquad x \in X, \tag{1.7}$$

for every $x \in X$. We collect some elementary properties of J_λ and B_λ in the following lemma.

LEMMA 1.3. *Let X and X^* be strictly convex.*

(a) *B_λ is a (single-valued) monotone mapping of all of X into X^*.*

(b) *B_λ and J_λ map bounded sets into bounded sets.*

(c) *B_λ (respectively J_λ) is continuous from X with the strong topology to X^* (respectively X) with the weak topology.*

(d) *For every $x \in D(B)$, $\|B_\lambda x\| \leqq |Bx|$ and, for every $x \in \overline{\text{conv}\,(D(B))}$, $\lim_{\lambda\to 0} J_\lambda x = x$.*

(e) *If $\lambda_n \to 0$, $x_n \rightharpoonup x$, $B_{\lambda_n} x_n \rightharpoonup y^*$ and*

$$\overline{\lim_{n,m\to\infty}}\,(x_n - x_m, B_{\lambda_n}x_n - B_{\lambda_m}x_m) \leqq 0,$$

then $[x, y^*] \in B$ and

$$\lim_{n,m\to\infty}(x_n - x_m, B_{\lambda_n}x_n - B_{\lambda_m}x_m) = 0.$$

Proof: (a) B_λ is clearly defined on all of X and is single-valued. Moreover,

$$\begin{aligned}(B_\lambda u - B_\lambda v, u - v) &= (B_\lambda u - B_\lambda v, J_\lambda u - J_\lambda v)\\ &\quad + (B_\lambda u - B_\lambda v, (u - J_\lambda u) - (v - J_\lambda v))\\ &= (B_\lambda u - B_\lambda v, J_\lambda u - J_\lambda v)\\ &\quad + \frac{1}{\lambda}(F(J_\lambda u - u) - F(J_\lambda v - v), (J_\lambda u - u) - (J_\lambda v - v)\\ &\geqq 0,\end{aligned}$$

and hence B_λ is monotone.

(b) Let $[u, v^*] \in B$. Multiplying (1.5) by $J_\lambda x - u$ yields

$$(F(J_\lambda x - x), J_\lambda x - u) \leqq \lambda(v^*, u - J_\lambda x)$$

which implies that J_λ maps bounded sets into bounded sets and since F maps bounded sets into bounded sets also B_λ has this property.

(c) Let $x_n \to x_0$ in X. Let $u_n = J_\lambda x_n$, $v_n^* = B_\lambda x_n$; then u_n and v_n^* are bounded (by (b)). We have $F(u_n - x_n) + \lambda v_n^* = 0$, and therefore

$$(F(u_n - x_n) - F(u_m - x_m), (u_n - x_n) - (u_m - x_m)) + \lambda(v_n^* - v_m^*, u_n - u_m)$$
$$= (F(u_n - x_n) - F(u_m - x_m), x_m - x_n) .$$

Since the right-hand side tends to zero as $n,m \to \infty$ and the two terms on the left-hand side are non-negative we have

$$\lim_{n,m\to\infty} (v_n^* - v_m^*, u_n - u_m) = 0$$

and

$$\lim_{n,m\to\infty} (F(u_n - x_n) - F(u_m - x_m), (u_n - x_n) - (u_m - x_m)) = 0 .$$

Let $\{n_k\}$ be a subsequence of $\{n\}$ such that $u_{n_k} \rightharpoonup u$, $v_{nk}^* \rightharpoonup v^*$ and $F(u_{n_k} - x_{n_k}) \rightharpoonup \eta^*$. Then $[u, v^*] \in B$ and $F(u - x_0) + \lambda v^* = 0$, by Lemma 1.2. Consequently, $u = J_\lambda u_0$ and $v^* = B_\lambda x_0$ and therefore (since the limits are unique) $J_\lambda x_n \rightharpoonup J_\lambda x_0$ and $B_\lambda x_n \rightharpoonup B_\lambda x_0$ and the proof of (c) is complete.

(d) Let $[x, x^*] \in B$ and $F(x_\lambda - x) + \lambda x_\lambda^* = 0$; then

$$0 \leqq (x - x_\lambda, x^* - x_\lambda^*) = \left(x - x_\lambda, x^* + \frac{F(x_\lambda - x)}{\lambda}\right)$$

$$\leqq -\frac{\|x - x_\lambda\|^2}{\lambda} + \|x - x\| \, \|x^*\|,$$

and thus

$$\|x_\lambda^*\| = \|B_\lambda x\| = \frac{\|x - x_\lambda\|}{\lambda} \leqq \|x^*\| .$$

Since $x^* \in Bx$ is arbitrary, $\|B_\lambda x\| \leqq |Bx|$. Let $[v, v^*] \in B$, then

$$\begin{aligned} \|x_\lambda - x\|^2 &= (F(x_\lambda - x), x_\lambda - x) \\ &= (F(x_\lambda - x), x_\lambda - v) + (F(x_\lambda - x), v - x) \qquad (1.8) \\ &\leqq \lambda(v^*, v - x_\lambda) + (F(x_\lambda - x), v - x) . \end{aligned}$$

It follows from (1.8) that $\|x_\lambda\|$ is bounded as $\lambda \to 0$, therefore $\|F(x_\lambda - x)\|$ is bounded. Let $\lambda_n \to 0$ be a sequence such that $F(x_{\lambda_n} - x) \rightharpoonup \eta$; then (1.8) implies

$$\overline{\lim_{n\to\infty}} \|x_{\lambda_n} - x\|^2 \leqq (\eta, v - x) \quad \text{for every} \quad v \in D(B)\,, \tag{1.9}$$

and therefore also for every $v \in \overline{\text{conv}\,(D(B))}$. If $x \in \overline{\text{conv}\,(D(B))}$, (1.9) yields $x_{\lambda_n} \to x$ which implies that $x_\lambda \to x$. Moreover, a simple argument shows that the convergence is uniform on compact subset of $\overline{\text{conv}\, D(B)}$.

(e) Since $[J_{\lambda_n}x_n\,, B_{\lambda_n}x_n] \in B$, part (e) follows directly from Lemma 1.2 applied to $u_n = J_{\lambda_n}x_n$ and $v_n^* = B_{\lambda_n}x_n$, noting that $\|B_{\lambda_n}x_n\| \leqq M$ implies that $\|u_n - x_n\| = \|J_{\lambda_n}x_n - x_n\| \to 0$ as $n \to \infty$ and that

$$\lim_{n,m\to\infty} (x_n - x_m\,, B_{\lambda_n}x_n - B_{\lambda_m}x_m) = \lim_{n,m\to\infty} (u_n - u_m\,, B_{\lambda_n}x_n - B_{\lambda_m}x_m) = 0\,.$$

The proof of Lemma 1.3 is complete.

Remarks. 1. If, in part (c) of Lemma 1.3, X has the property that $x_n \rightharpoonup x$ together with $\|x_n\| \to \|x\|$ imply $x_n \to x$, then J_λ is strongly continuous from X to X.

2. Lemma 1.3 part (d) clearly implies that if X is any reflexive Banach space and B is maximal monotone in $X \times X^*$, then $\overline{D(B)}$ is convex. Moreover, if X and X^* are strictly convex and X satisfies the condition of Remark 1, then $D(B)$ is virtually convex in the sense of Rockafellar [16]. Thus Lemma 1.3 part (d) provides us with a simple proof of the results of [16].

3. Rockafellar [16] considers for every $y^* \in X^*$ the equation

$$\lambda F(x_\lambda) + x_\lambda^* = y^*\,, \qquad\qquad [x_\lambda\,, x_\lambda^*] \in B\,,$$

and then denotes: $x_\lambda^* = P_\lambda(y^*)$. He raises the question whether P_λ is continuous in some natural topologies. It is easy to see that P_λ is the operator J_λ : $X^* \to X^*$ which corresponds to B^{-1} (maximal monotone in $X^* \times X$). Therefore by part (c), P_λ is continuous from X^* with the strong topology to X^* with the weak topology. In addition, if X^* has the property that $y_n^* \rightharpoonup y$, $\|y_n\| \to \|y\|$ imply $y_n \to y^*$, then P_λ is strongly continuous from X^* to X^*.

4. Let X and X^* be strictly convex and let A and B be maximal monotone sets in $X \times X^*$. According to Browder [4] (Theorem 2), $A + B_\lambda$ is maximal monotone for every $\lambda > 0$.

Our last lemma is a generalization of Lemma 2.4 of Crandall–Pazy [6].

LEMMA 1.4. *Let X and X^* be reflexive and strictly convex. Let $\{x_n\} \subset X$ and let $\{r_n\}$ be a monotonic sequence of positive real numbers. Further, let*

$$(x_n - x_m\,, r_nF(x_n) - r_mF(x_m)) \leqq 0\,.$$

Then:

(i) *If* $r_n \to \infty$, *then* $\|x_n\|$ *is non-increasing and* $x = \text{w-lim}\, x_n$ *exists. Moreover,* $\lim \|x_n\| = \|x\|$.

(ii) *If* $r_n \to 0$, *then* $\|x_n\|$ *is non-decreasing. If* $\{\|x_n\|\}$ *is bounded,* $x = \text{w-}\lim x_n$ *exists and* $\|x\| = \lim \|x_n\|$.

Proof: We have

$$
\begin{aligned}
2(x_n - x_m, r_nF(x_n) - r_mF(x_m)) &= (r_n - r_m)(\|x_n\|^2 - \|x_m\|^2) \\
&\quad + r_n(\|x_n\|^2 - 2(x_m, F(x_n)) + \|x_m\|^2) \\
&\quad + r_m(\|x_n\|^2 - 2(x_n, F(x_m)) + \|x_m\|^2)\,.
\end{aligned} \tag{1.10}
$$

The last two terms are non-negative; hence,

$$(r_n - r_m)(\|x_n\|^2 - \|x_m\|^2) \leqq 0\,,$$

and the monotonicity of $\{\|x_n\|\}$ follows. Let us prove (i). Divide (1.10) by $r_n + r_m$ to find

$$
\begin{aligned}
&\frac{r_n - r_m}{r_n + r_m}(\|x_n\|^2 - \|x_m\|^2) \\
&+ \frac{r_n}{r_n + r_m}(\|x_n\|^2 - 2(x_m, F(x_n)) + \|x_m\|^2) \\
&\qquad + \frac{r_m}{r_n + r_m}(\|x_n\|^2 - 2(x_n, F(x_m)) + \|x_m\|^2) \leqq 0\,.
\end{aligned} \tag{1.11}
$$

Let $\|x_n\| \to L$ and assume $F(x_{n_i}) \rightharpoonup \eta$. Fix m and let $n \to \infty$ through $\{n_i\}$ in (1.11). Since $r_n \to +\infty$, we obtain

$$(L^2 - \|x_m\|^2) + (L^2 - 2(x_m, \eta) + \|x_m\|^2) \leqq 0 \quad \text{for every} \quad m\,.$$

Letting $m \to \infty$, we see that

$$\lim_{m\to\infty} (x_m, \eta) = L^2\,.$$

Since $\|\eta\| \leqq L$, $\|x_m\| \leqq L$; this implies $x_m \rightharpoonup F^{-1}(\eta)$ and $\|\eta\| = L$. The proof of (ii) is similar.

We finish this section with some remarks on a special kind of maximal monotone set in $X \times X^*$. Let f be a convex lower semi-continuous function from X

into $R \cup \{+\infty\}$. We assume that f is proper (i.e., not identically $+\infty$). We recall that the subdifferential

$$\partial f(u) = \{w^*: w^* \in X^* \text{ and } f(v) \geqq f(u) + (w^*, v - u) \text{ for every } v \in X\}$$

is a maximal monotone set (see e.g. [14]).

Note that if K is a non-void closed convex set in X and ψ_k is the indicator function of K (i.e., $\psi_K(x) = 0$ for $x \in K$ and $\psi_K(x) = +\infty$ for $x \notin K$), then ψ_K is a proper, convex and lower semi-continuous function. It follows that $\partial\psi_K$ is a maximal monotone set in $X \times X^*$. It is easy to verify that the domain of $\partial\psi_K$ is K and that $w^* \in \partial\psi_K(u)$ if and only if $(w^*, u - v) \geqq 0$ for every $v \in K$.

2. Perturbation Theorems

We begin this section with a theorem which turns out to be very useful in proving perturbation results. Let X be strictly convex with a strictly convex dual X^*. Let A and B be maximal monotone sets in $X \times X^*$. According to the discussion in Section 1, $A + B_\lambda$ is maximal monotone in $X \times X^*$ and it follows that the conditions

$$F(x_\lambda) + x_\lambda^* + B_\lambda x_\lambda = f^*, \qquad [x_\lambda, x_\lambda^*] \in A, \tag{2.1}$$

determine a unique $x_\lambda \in X$ for every $f^* \in X^*$.

THEOREM 2.1. *Let X be strictly convex with a strictly convex dual X^*. Let A and B be maximal monotone sets in $X \times X^*$, and let x_λ be the solution of equation* (2.1). *Then $f^* \in R(F + A + B)$ if and only if $\|B_\lambda x_\lambda\|$ is bounded as λ tends to zero.*

Proof: Let $f^* \in R(F + A + B)$. Then there exists an $x \in X$ such that

$$F(x) + x_1^* + x_2^* = f^*, \qquad [x, x_1^*] \in A, [x, x_2^*] \in B. \tag{2.2}$$

Let x_λ be the solution of equation (2.1); then

$$\begin{aligned} 0 &\leqq (F(x_\lambda) - F(x), x_\lambda - x) = (x_1^* - x_\lambda^*, x_\lambda - x) + (x_2^* - B_\lambda x_\lambda, x_\lambda - x) \\ &\leqq (x_2^* - B_\lambda x_\lambda, x_\lambda - x), \end{aligned}$$

since A is monotone. Using

$$x_\lambda = J_\lambda x_\lambda + \lambda F^{-1}(B_\lambda x_\lambda),$$

we obtain

$$0 \leqq (x_2^* - B_\lambda x_\lambda, J_\lambda x_\lambda - x) + (x_2^* - B_\lambda x_\lambda, \lambda F^{-1}(B_\lambda x_\lambda))$$
$$\leqq (x_2^* - B_\lambda x_\lambda, \lambda F^{-1}(B_\lambda x_\lambda)),$$

since B is monotone and $B_\lambda x_\lambda \in BJ_\lambda x$. Thus, $\|B_\lambda x_\lambda\|^2 \leqq (x_2^*, F^{-1}(B_\lambda x_\lambda))$ or $\|B_\lambda x_\lambda\| \leqq \|x_2^*\|$, and the condition is necessary.

To prove that the condition is sufficient we show that the equation (2.2) has a solution if $\|B_\lambda x_\lambda\|$ is bounded. Let $[x_0, x_0^*] \in A$ and multiply equation (2.1) by $x_\lambda - x_0$. After rearrangement we obtain

$$\|x_\lambda\|^2 \leqq (f^*, x_\lambda - x_0) + (F(x_\lambda), x_0) - (x_\lambda^*, x_\lambda - x_0) - (B_\lambda x_\lambda, x_\lambda - x_0)$$
$$\leqq (f^*, x_\lambda - x_0) + (F(x_\lambda), x_0) - (x_0^*, x_\lambda - x_0) - (B_\lambda x_\lambda, x_\lambda - x_0)$$
$$\leqq C_1 \|x_\lambda\| + C_2,$$

which implies that $\|x_\lambda\| \leqq C_1^2 + 2C_2$, i.e., $\|x_\lambda\|$ is bounded. By our assumption, $\|B_\lambda x_\lambda\| \leqq C$ and therefore in equation (2.1) we have $\|x_\lambda^*\| \leqq C$. We choose a sequence $\lambda_n \to 0$ such that $x_{\lambda_n} \rightharpoonup x_0$, $x_{\lambda_n}^* \rightharpoonup x_1^*$, $B_{\lambda_n} x_{\lambda_n} \rightharpoonup x_2^*$ and $F(x_{\lambda_n}) \rightharpoonup z^*$. Using equation (2.1) for λ_n and λ_m, we obtain

$$0 = (F(x_{\lambda_n}) + x_{\lambda_n}^* - (F(x_{\lambda_m}) + x_{\lambda_m}^*), x_{\lambda_n} - x_{\lambda m})$$
$$+ (B_{\lambda_n} x_{\lambda_n} - B_{\lambda_m} x_{\lambda_m}, x_{\lambda_n} - x_{\lambda m});$$

since $F + A$ is monotone, the last equation implies

$$\overline{\lim_{n,m\to\infty}} (B_{\lambda_n} x_{\lambda_n} - B_{\lambda_m} x_{\lambda_m}, x_{\lambda_n} - x_{\lambda_m}) \leqq 0$$

and hence, by Lemma 1.3(e), $[x_0, x_2^*] \in B$ and

$$\lim_{n,m\to\infty} (B_{\lambda_n} x_{\lambda_n} - B_{\lambda_m} x_{\lambda_m}, x_{\lambda_n} - x_{\lambda_m}) = 0 .$$

Consequently,

$$\lim_{n,m\to\infty} (F(x_{\lambda_n}) + x_{\lambda_n}^* - (F(x_{\lambda_m}) + x_{\lambda_m}^*), x_{\lambda_n} - x_{\lambda_m}) = 0$$

and, since A is monotone,

$$\overline{\lim_{n,m\to\infty}} (F(x_{\lambda_n}) - F(x_{\lambda_m}), x_{\lambda_n} - x_{\lambda_m}) \leqq 0 .$$

Therefore, by Lemma 1.2, $z^* = F(x_0)$ and

$$\lim_{n,m\to\infty} (x^*_{\lambda_n} - x^*_{\lambda_m}, x_{\lambda_n} - x_{\lambda_m}) = 0 ,$$

which again by Lemma 1.2 implies that $[x_0, x_1^*] \in A$. Passing to the limit through the sequence $\{\lambda_n\}$ in equation (2.1), we obtain $F(x_0) + x_1^* + x_2^* = f^*$, $[x_0, z_1^*] \in A$, $[x_0, x_2^*] \in B$, i.e., $f^* \in R(F + A + B)$.

Remarks. 1. If $f^* \in R(F + A + B)$, then there exist a unique $x \in X$ and an $x^* \in (A + B)x$ such that $f^* = F(x) + x^*$. Let $x^* = x_1^* + x_2^*$, where $x_1^* \in Ax$ and $x_2^* \in Bx$. This decomposition of x^* is in general not unique. We can choose any x_2^* in the non-void convex set $Hx = Bx \cap (f - F(x) - Ax)$ and then take the corresponding x_1^*. If, however, the solution of (2.2) is obtained as in the proof of Theorem 2.1, i.e., by a sequence of solutions of (2.1) for which x_λ, $F(x_\lambda)$, x_λ^* and $B_\lambda x_\lambda$ converge weakly, then x_2^* and x_1^* are uniquely determined, x_2^* is the element of minimum norm in Hx and $x_1^* = f^* - F(x) - x_2^*$. This is a direct consequence of the first part of the proof of Theorem 2.1 in which we took for x_2^* any element in Hx and obtained $\|B_\lambda x_\lambda\| \leqq \|x_2^*\|$.

2. In the sufficient part of Theorem 2.1, we obtained a solution of (2.2) by considering a sequence of solutions of (2.1) with some special properties. Using Remark 1, we shall show that, if $f^* \in R(F + A + B)$ and

$$F(x_\lambda) + x_\lambda^* + B_\lambda x_\lambda = f^* , \qquad [x_\lambda, x_\lambda^*] \in A ,$$

then $x_\lambda \rightharpoonup x_0$, $F(x_\lambda) \rightharpoonup F(x_0)$, $B_\lambda x_\lambda \rightharpoonup \bar{x}_2^*$ and $x_\lambda^* \rightharpoonup x_1^*$, where x_0 is the unique solution of equation (2.2), $\bar{x}_2^*$ is the element of minimum norm in Hx_0 and $x_1^* = f^* - F(x_0) - \bar{x}_2^*$. Moreover, if X and X^* are uniformly convex, then the above weak limits are strong limits.

Proof: Any sequence $\lambda_n \to 0$ has a subsequence λ'_n such that $x_{\lambda'_n}$, $F(x^*_{\lambda'_n})$, $x^*_{\lambda'_n}$ and $B_{\lambda'_n} x_{\lambda'_n}$ converge weakly. As in the proof of Theorem 2.1, we then have $x_{\lambda'_n} \rightharpoonup x_0$, $F(x_{\lambda'_n}) \rightharpoonup F(x_0)$, $B_{\lambda'_n} x_{\lambda'_n} \rightharpoonup \bar{x}_2^*$ (by Remark 1) and $x^*_{\lambda'_n} \rightharpoonup x_1^* = f^* - F(x_0) - \bar{x}_2^*$. Since the limits are uniquely determined and λ_n was an arbitrary sequence converging to zero, this implies $x_\lambda \rightharpoonup x_0$, $F(x_\lambda) \rightharpoonup F(x_0)$, $B_\lambda x_\lambda \rightharpoonup \bar{x}_2^*$ and $x_\lambda^* \rightharpoonup x_1^* = f^* - F(x_0) - \bar{x}_2^*$.

Assume now that X and X^* are uniformly convex. Since $B_\lambda x_\lambda \rightharpoonup \bar{x}_2^*$ and $[x_0, \bar{x}_2^*] \in B$, we have

$$\|\bar{x}_2^*\| \leqq \underline{\lim}\, \|B_\lambda x_\lambda\| \leqq \overline{\lim}\, \|B_\lambda x_\lambda\| \leqq \|\bar{x}_2^*\| ;$$

therefore, $\lim \|B_\lambda x_\lambda\| = \|\bar{x}_2^*\|$ and the uniform convexity of X^* implies $B_\lambda \bar{x}_\lambda \to \bar{x}_2^*$. Subtracting (2.1) from

$$F(x_0) + x_1^* + \bar{x}_2^* = f^*$$

and multiplying by $x_\lambda - x_0$ yields, after passing to the limit as $\lambda \to 0$,

$$\overline{\lim}\, (F(x_\lambda) - F(x_0), x_\lambda - x_0) \leqq 0 ,$$

which implies $\|x_\lambda\| \to \|x_0\|$ and we conclude that $x_\lambda \to x_0$ and $F(x_\lambda) \to F(x_0)$. The proof is complete.

3. A direct proof of Remark 2 can be obtained using Lemma 1.4.

Using Theorem 2.1 we obtain an alternative proof of the following theorem of Rockafellar [15].

THEOREM 2.2. *Let X be a reflexive Banach space. Let A and B be maximal monotone in $X \times X^*$. If* $\operatorname{int}(D(A)) \cap D(B) \neq \varnothing$, *then $A + B$ is maximal monotone in $X \times X^*$.*

Proof: We choose in X and X^* any strictly convex equivalent dual norms (see Theorem 1.1). Clearly, we may assume without loss of generality that $0 \in \operatorname{int}(D(A)) \cap D(B)$ and $0 \in A0$, $0 \in B0$. This can be achieved by shifting $D(A)$, $D(B)$ and $R(A)$, $R(B)$. Let f^* be any element of X^* and consider the equation

$$F(x_\lambda) + x_\lambda^* + B_\lambda x_\lambda = f^* , \qquad x_\lambda^* \in Ax_\lambda . \tag{2.3}$$

Since A and B are monotone, $0 \in A0$ and $0 \in B0$, we see by multiplying (2.3) by x_λ that

$$\|x_\lambda\| \leqq \|f^*\| , \tag{2.4}$$

and

$$(x_\lambda^* , x_\lambda) \leqq \|f^*\|^2 . \tag{2.5}$$

Moreover, since $0 \in \operatorname{int}(D(A))$, A is locally bounded at 0 (see e.g. Rockafellar [15]). Hence there exist constants $\alpha > 0$ and $K > 0$ such that if $\|x\| < \alpha$, then $x \in D(A)$ and if $x^* \in \bigcup_{\|x\|<\alpha} Ax$, then $\|x^*\| \leqq K$.

For $\lambda > 0$, define $z_\lambda = \frac{1}{2}\alpha F^{-1}(x_\lambda^*)/\|x_\lambda^*\|$. Since $\|z_\lambda\| = \frac{1}{2}\alpha < \alpha$, $z_\lambda \in D(A)$. Let $[z_\lambda , z_\lambda^*] \in A$. Then $\|z_\lambda^*\| \leqq K$ and we have

$$0 \leqq (x_\lambda^* - z_\lambda^* , x_\lambda - z_\lambda) = (x_\lambda^* , x_\lambda) - (x_\lambda^* , z_\lambda) - (z_\lambda^* , x_\lambda) + (z_\lambda^* , z_\lambda) ;$$

therefore,

$$\begin{aligned} \tfrac{1}{2}\alpha \|x_\lambda^*\| &\leqq (x_\lambda^* , z_\lambda) \leqq (x_\lambda^* , x_\lambda) + (z_\lambda^* , z_\lambda) - (z_\lambda^* , x_\lambda) \\ &\leqq \|f^*\|^2 + K(\tfrac{1}{2}\alpha + \|f^*\|) . \end{aligned}$$

This implies that $\|x_\lambda^*\| \leqq C$. Using this together with (2.3) and (2.4) we see that $\|B_\lambda x_\lambda\| \leqq C$ and therefore, by Theorem 2.1, $f^* \in R(F + A + B)$. Since f^* was arbitrary, $R(F + A + B) = X^*$ and $A + B$ is maximal monotone.

We now turn to our main result.

THEOREM 2.3. *Let X be a reflexive Banach space. Let A and B be maximal monotone sets in $X \times X^*$ such that*

(i) $D(A) \subset D(B)$,

(ii) $|Bx| \leqq k(\|x\|)\,|Ax| + C(\|x\|)$, *where $k(r)$ and $C(r)$ are non-decreasing functions of r and $k(r) < 1$ for every r.*

Then $A + B$ is maximal monotone in $X \times X^*$.

Proof: Without loss of generality we may assume that $0 \in D(A)$, $0 \in A0$ and $0 \in B0$. This can be achieved by shifting the domains and ranges of A and B.

Let $\{\|\ \|_a\}$ be the family of equivalent norms on X introduced in Theorem 1.1. In view of Lemma 1.1, $A + B$ is maximal monotone if for every $f^* \in X^*$ and $u \in X$ there exists an a such that

$$f^* + F_a(u) \in R(F_a + A + B)\ .$$

To show that this is indeed the case, consider the equation

$$F_a(x_\lambda) + x_\lambda^* + B_\lambda^a x_\lambda = f^* + F_a(u)\ , \qquad [x_\lambda\, , x_\lambda^*] \in A\ . \tag{2.6}$$

For every $f^* \in X^*$, $u \in X$ and any fixed a, this equation has a unique solution x_λ. If $\|B_\lambda^a x_\lambda\|_a$ is bounded as λ tends to zero, then $f^* + F_a(u) \in R(F_a + A + B)$ by Theorem 2.1. To prove the theorem it is therefore sufficient to show that, for every $f^* \in X^*$ and $u \in X$, there exists an a such that $\|B_\lambda^a x_\lambda\|_a$ is bounded as λ tends to zero. Multiplying (2.6) by x_λ yields

$$\|x_\lambda\|_a \leqq \|f^*\|_a + \|u\|_a \leqq a(\|f^*\| + \|u\|)\ ,$$

since $B_\lambda^a 0 = 0$. Let $R = 2(\|f^*\| + \|u\|)$ and choose a such that $1 < a < 2$ and $k(R)a^2 < 1$. Using equation (2.6) again, we obtain

$$\begin{aligned} a^{-1}\,|Ax_\lambda| \leqq |Ax_\lambda|_a \leqq \|x_\lambda^*\|_a &\leqq \|f^*\|_a + \|B_\lambda^a x_\lambda\|_a + \|u\|_a + \|x_\lambda\|_a \\ &\leqq 2a(\|f^*\| + \|u\|) + |Bx_\lambda|_a \\ &\leqq 2a(\|f^*\| + \|u\|) + a\,|Bx_\lambda| \\ &\leqq 2a(\|f^*\| + \|u\|) + ak(R)\,|Ax_\lambda| + aC(R)\ . \end{aligned} \tag{2.7}$$

Thus $|Ax_\lambda| \leqq a^2k(R)\,|Ax_\lambda| + C$, which implies that $|Ax_\lambda|$ is bounded and therefore, by (2.7), $|Bx_\lambda|_a$ is bounded. Hence, $\|B_\lambda^a x_\lambda\|_a$ is bounded and the proof is complete.

Remarks. 1. If X and X^* are uniformly convex, condition (ii) of Theorem 2.3 can be replaced by the following local condition:

(ii)′ For every $x \in \overline{D(A)}$ there exist a neighborhood V_x of x, a $k_x < 1$ and a constant C_x such that

$$|By| \leqq k_x\,|Ay| + C_x \quad \text{for} \quad \text{every} \quad y \in D(A) \cap V_x .$$

We do not know whether or not this is true in a general reflexive Banach space.

2. In the case that X is a Hilbert space (and the case of accretive operators in Banach space), Theorem 2.3 was first proved by Crandall and Pazy [6]. For these cases, Kato [8] observed that condition (ii) of Theorem 2.3 can be replaced by the local condition (ii)′.

Corollary 2.1. *Let X be a reflexive Banach space. Let A be a maximal set in $X \times X^*$, and let B be a single-valued monotone hemicontinuous operator with convex domain $D(B)$ in X. If $D(A) \subset D(B)$ and*

$$\|Bu\| \leqq k(\|u\|)\,|Au| + C(\|u\|) \quad \textit{for} \quad \textit{every} \quad u \in D(A) ,$$

where $k(r)$ and $C(r)$ are non-decreasing functions and $k(r) < 1$ for every r, then $A + B$ is maximal monotone in $X \times X^*$.

Proof: Let $\bar{B}$ be a maximal monotone extension of B. Let $K = \overline{D(B)}$. Clearly, $|\bar{B}u| \leqq \|Bu\|$ for every $u \in D(A)$ and $D(\bar{B}) \supset D(B) \supset D(A)$. Therefore, $A + \bar{B}$ is maximal monotone by Theorem 2.3. We shall prove that, for every $u \in D(B)$,

$$\bar{B}u \subset Bu + \partial\psi_K(u) . \tag{2.8}$$

Let $u \in D(B)$ and $f \in \bar{B}u$; then

$$(Bv - f, v - u) \geqq 0 \quad \text{for} \quad \text{every} \quad v \in D(B) . \tag{2.9}$$

Let $w \in D(B)$ and define $v_t = (1 - t)u + tw$, $0 < t \leqq 1$. Substituting v_t in place of v in (2.9) yields

$$(f - Bv_t\,, u - w) \geqq 0 .$$

Letting t tend to zero and using the hemicontinuity of B, we obtain

$$(f - Bu, u - w) \geqq 0$$

for every $w \in D(B)$ and therefore also for every $w \in \overline{D(B)} = K$. Thus,

$$f - Bu \in \partial\psi_K(u) \Leftrightarrow f \in Bu + \partial\psi_K(u)$$

and (2.8) is proved. From (2.8) it follows that $A + \tilde{B} \subset A + B + \partial\psi_K$. But $A = A + \partial\psi_K$, since $D(A) \subset K$ and A is maximal monotone. Therefore, $A + \tilde{B} \subset A + B$ which implies $A + \tilde{B} = A + B$ and hence $A + B$ is maximal monotone.

3. Applications

In this section we give three simple examples in which the previous theory is applied to partial differential equations. Our main interest is in the technique used to solve the problems rather than in the specific results. We denote by Ω a bounded domain in R^n with smooth boundary $\partial\Omega$, and by $H^m(\Omega)$, $H_0^m(\Omega)$ the usual Sobolev spaces.

EXAMPLE 1. Let $\beta \subset R \times R$ be a maximal monotone set in $R \times R$ such that $0 \in D(\beta)$. Let $V(x) \in L^p(\Omega)$, $p \geqq 2$, and $V(x) \geqq 0$ a.e. in Ω.

THEOREM 3.1. *Let $p > \frac{1}{2}n$; then for every $f \in L^2(\Omega)$ there exists a unique solution $u \in H^2(\Omega)$ of the equation*

$$\begin{aligned} f \in (-\Delta u + \beta(u) + Vu) \quad &\text{in} \quad \Omega\,, \\ u = 0 \quad &\text{on} \quad \partial\Omega\,. \end{aligned} \tag{3.1}$$

More precisely, there exists a $g \in L^2(\Omega)$ such that $g(x) \in \beta(u(x))$ a.e. in Ω and the equation

$$\begin{aligned} -\Delta u + g + Vu = f \quad &\text{in} \quad \Omega\,, \\ u = 0 \quad &\text{on} \quad \partial\Omega\,. \end{aligned} \tag{3.1$'$}$$

is satisfied.

To prove Theorem 3.1, let $X = X^* = L^2(\Omega)$ and let $\|\ \|$ be the $L^2(\Omega)$ norm. We introduce the following operators:

$$\tilde{\beta} = \{[u, v] : u, v \in L^2(\Omega) \text{ and } v(x) \in \beta(u(x)) \text{ a.e. in } \Omega\}\,.$$

Clearly $\tilde{\beta}$ is maximal monotone in $X \times X^*$. There is no loss of generality in assuming that $0 \in \tilde{\beta}(0)$ and we shall henceforth assume this.

Let $D(A) = H^2(\Omega) \cap H_0^1(\Omega) \cap D(\tilde{\beta})$ and let $Au = -\Delta u + \tilde{\beta}(u)$ for $u \in D(A)$. Using Theorem 2.1, we shall show that A is maximal monotone. It is well known (see e.g. Nirenberg [13]) that $-\Delta$ with domain $H^2(\Omega) \cap H_0^1(\Omega)$ is maximal monotone in $L^2(\Omega) \times L^2(\Omega)$. Hence the equation

$$(3.2)\qquad \begin{aligned} -\Delta u_\lambda + \bar\beta_\lambda(u_\lambda) + u_\lambda &= f \quad \text{in} \quad \Omega\,, \\ u_\lambda &= 0 \quad \text{on} \quad \partial\Omega\,, \end{aligned}$$

has a solution $u_\lambda \in H^2(\Omega) \cap H_0^1(\Omega)$. Multiplying (3.2) by $\bar\beta_\lambda(u_\lambda)$, integrating over Ω and noting that $\bar\beta_\lambda(u_\lambda) \cdot u_\lambda \geqq 0$ a.e. in Ω and that

$$-\int_\Omega \Delta u_\lambda \cdot \bar\beta_\lambda(u_\lambda)\, dx \geqq 0$$

(since $\bar\beta_\lambda = (\bar\beta)_\lambda$ and β_λ is a monotone Lipschitz function), we obtain

$$(3.3)\qquad \|\bar\beta_\lambda(u_\lambda)\| \leqq \|f\|\,.$$

Therefore (by Theorem 2.1), the equation

$$\begin{aligned} f \in (-\Delta u + \bar\beta(u) + u) &\quad \text{in} \quad \Omega\,, \\ u = 0 &\quad \text{on} \quad \partial\Omega\,, \end{aligned}$$

has a solution. Since $f \in L^2(\Omega)$ is arbitrary, A is maximal monotone. Moreover, for any $f \in Au$ we have (see [13])

$$\|u\|_{H^2(\Omega)} \leqq C\,\|\Delta u\| \leqq C\,\|f\| \quad \text{for every} \quad u \in D(A)\,,$$

and therefore,

$$(3.4)\qquad \|u\|_{H^2(\Omega)} \leqq C\,|Au| \quad \text{for every} \quad u \in D(A)\,.$$

Let $D(B) = \{u : u \in L^2(\Omega) \text{ such that } Vu \in L^2(\Omega)\}$. Since $V \geqq 0$, B is monotone. It is maximal monotone, since the equation

$$u + \lambda V u = f\,, \qquad \lambda > 0\,,$$

has the solution $u = f/(1 + \lambda V)$ which is in $L^2(\Omega)$ for every $f \in L^2(\Omega)$.

We now use Theorem 2.3 to show that $A + B$ is maximal monotone. We start by showing that $D(A) \subset D(B)$. For this it is sufficient to show that $H^2(\Omega) \subset D(B)$. Consider

$$\|Bu\|^2 = \int_\Omega V^2 u^2\, dx \leqq \left(\int_\Omega V^{2\cdot p/2}\right)^{2/p} \left(\int_\Omega u^{2q}\right)^{1/q},$$

where $1/q + 2/p = 1$, i.e., $q = p/(p-2)$. Thus,

$$\|Bu\| \leqq \|V\|_{L^p(\Omega)} \|u\|_{L^{2q}(\Omega)} . \tag{3.5}$$

But $p > \frac{1}{2}n$ implies $1/2q > \frac{1}{2} - 2/n$ and, therefore, by Sobolev's theorem, $H^2(\Omega) \subset L^{2q}(\Omega)$. Moreover, the embedding is compact. This implies that $H^2(\Omega) \subset D(B)$ and that for every $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that

$$\|u\|_{L^{2q}(\Omega)} \leqq \varepsilon \|u\|_{H^2(\Omega)} + C(\varepsilon)\|u\| .$$

Using this estimate together with (3.4) and (3.5) we see that

$$\|Bu\| \leqq \varepsilon C \|V\|_{L^p(\Omega)} |Au| + C(\varepsilon) \|V\|_{L^p(\Omega)} \|u\| .$$

Choosing ε so small that $\varepsilon C \|V\|_{L^p(\Omega)} < 1$, we obtain the estimate which is needed in Theorem 2.3, and hence $A + B$ is maximal monotone.

To complete the proof note that $A + B$ is also coercive and, therefore, $R(A + B) = X^* = L^2(\Omega)$.

Theorem 3.1, together with the results of Crandall and Pazy [6], yields the following corollary.

COROLLARY 3.1. *Let β and V be the same as in Theorem 3.1 and let*

$$u_0 \in H^2(\Omega) \cap H_0^1(\Omega) \cap D(\bar{\beta}).$$

Then the equation

$$\begin{aligned} 0 \in \frac{\partial u}{\partial t} - \Delta u + \bar{\beta}(u) + Vu \quad & \textit{in} \quad \Omega \times (0, +\infty) , \\ u(x, t) = 0 \quad & \textit{on} \quad \partial\Omega \times (0, +\infty) , \\ u(x, 0) = u_0(x) \quad & \textit{in} \quad \Omega , \end{aligned} \tag{3.6}$$

has a unique solution $u(x, t) \in C(0, +\infty; L^2(\Omega))$ such that

$$u(x, t) \in H^2(\Omega) \cap H_0^1(\Omega) \cap D(\bar{\beta})$$

for every fixed $t \geqq 0$ and $\partial u/\partial t \in L^\infty(0, +\infty; L^2(\Omega))$.

Remark. It can be shown that $\partial u/\partial t \in L^2(0, +\infty; H_0^1(\Omega))$.

EXAMPLE 2. Let $\psi_1, \psi_2 \in H^2(\Omega)$ satisfy $\psi_1 \leqq \psi_2$ in Ω and $\psi_1 \leqq 0 \leqq \psi_2$ on $\partial\Omega$. The set

$$K = \{v : v \in L^2(\Omega), \psi_1 \leqq v \leqq \psi_2 \quad \text{a.e. on} \quad \Omega\}$$

is clearly a closed convex subset of $L^2(\Omega)$. Let P_K be the projection on K in $L^2(\Omega)$. For every u in $L^2(\Omega)$ we have

$$P_K u = u + (\psi_1 - u)^+ - (u - \psi_2)^+ , \tag{3.7}$$

where $r^+ = \max(r, 0)$.

Let $V(x) \in L^p(\Omega)$, $p \geqq 2$, and $V(x) \geqq 0$ a.e. in Ω and consider the following problem: Given any $f \in L^2(\Omega)$, find a function $u \in H^2(\Omega) \cap H_0^1(\Omega) \cap K$ such that

$$\int_\Omega (f + \Delta u - Vu)(u - v)\, dx \geqq 0 \quad \text{for} \quad \text{every} \quad v \in K . \tag{3.8}$$

This elliptic inequality is equivalent to the problem

$$\begin{aligned} &f \in -\Delta u + \partial\psi_K(u) + Vu , \\ &u \in H^2(\Omega) \cap H_0^1(\Omega) , \end{aligned} \tag{3.9}$$

where ψ_K is the indicator function of the convex set K.

THEOREM 3.2. *The elliptic inequality* (3.8) *has a unique solution* $u \in H^2(\Omega) \cap H_0^1(\Omega) \cap K$ *for every* $f \in L^2(\Omega)$ *provided that* $p > \frac{1}{2}n$.

To prove Theorem 3.2, let $X = X^* = L^2(\Omega)$ and let $\|\ \|$ be the $L^2(\Omega)$ norm. We introduce the operators: A, with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega) \cap K$, defined by $A = -\Delta u + \partial\psi_K(u)$ and B, multiplication by $V(x)$, as in Example 1. From the results of [2] it follows that A is maximal monotone. Nevertheless, we give here a direct proof of this result which seems to be simpler and is based on Theorem 2.1. Using Theorem 2.3 we conclude that $A + B$ is maximal monotone. Finally we note that $A + B$ is also coercive and, therefore, $R(A + B) = L^2(\Omega)$. We start with a lemma.

LEMMA 3.1. *Let* L *be a linear operator,* $L : H^2(\Omega) \to L^2(\Omega)$, *such that*

$$\int_\Omega Lw \cdot w^+ dx \geqq 0$$

for every $w \in H^2(\Omega)$ *which satisfies* $w \leqq 0$ *on* $\partial\Omega$. *Then*

$$\int_\Omega Lv \cdot (v - P_K v)\, dx \geqq -C_K \|v - P_K v\|$$

for every $v \in H^2(\Omega)$ *and* $C_K = \|L\psi_1\| + \|L\psi_2\|$.

Proof: We have

$$\begin{aligned}\int_\Omega Lv(v - P_K v)\, dx &= \int_\Omega Lv[(v-\psi_2)^+ - (\psi_1 - v)^+]dx \\ &\geqq \int_\Omega L\psi_2 \cdot (v-\psi_2)^+\, dx - \int_\Omega L\psi_1(\psi_1 - v)^+\, dx \\ &\geqq -\|L\psi_2\|\, \|(v-\psi_2)^+\| - \|L\psi_1\|\, \|(\psi_1 - v)^+\| \\ &\geqq -(\|L\psi_1\| + \|L\psi_2\|)\, \|v - P_K v\|\,,\end{aligned}$$

since $\|(v-\psi_2)^+\|^2 + \|(\psi_1 - v)^+\|^2 = \|v - P_K v\|^2$.

We now prove that A is maximal monotone. For f given in $L^2(\Omega)$, consider the equation

$$\text{(3.10)}\qquad \begin{aligned} -\Delta u_\lambda + (\partial\,\psi_K)_\lambda u_\lambda + u_\lambda &= f \quad \text{in} \quad \Omega\,, \\ u_\lambda &= 0 \quad \text{on} \quad \partial\Omega\,. \end{aligned}$$

It is easy to verify that

$$(\partial\,\psi_K)_\lambda v = \frac{1}{\lambda}(v - P_K v) \quad \text{for every} \quad v \in L^2(\Omega)\,.$$

Multiplying equation (3.10) by $(\partial\,\psi_K)_\lambda u_\lambda$ and integrating over Ω yields

$$\frac{1}{\lambda}\int_\Omega(-\Delta u_\lambda + u_\lambda)(u_\lambda - P_K u_\lambda)\, dx + \|(\partial\,\psi_K)_\lambda u_\lambda\|^2 = \frac{1}{\lambda}\int_\Omega f\cdot(\partial\,\psi_K)_\lambda u_\lambda\, dx\,.$$

Using Lemma 3.1 with $L = -\Delta + I$, we obtain

$$\|(\partial\,\psi_K)_\lambda u_\lambda\|^2 - C_K\,\|(\partial\,\psi_K)_\lambda u_\lambda\| \leqq \|f\|\,\|(\partial\,\psi_K)_\lambda u_\lambda\|\,,$$

and hence

$$\text{(3.11)}\qquad \|(\partial\,\psi_K)_\lambda u_\lambda\| \leqq \|f\| + C_K\,.$$

By Theorem 2.1 we conclude that $f \in R(I + A)$ and, since f was arbitrary, A is maximal monotone.

From equations (3.10) and (3.11) we have

$$\|-\Delta u_\lambda + u_\lambda\| \leqq 2\,\|f\| + C_K\,;$$

passing to the limit as $\lambda \to 0$ we obtain

$$\| - \Delta u + u \| \leqq 2 \, \|f\| + C_K \, . \tag{3.12}$$

Let $v \in D(A)$ and $g \in Av$; then $g + v \in -\Delta u + v + \partial \psi_K(v)$ and therefore, by (3.12),

$$\| - \Delta v + v \| \leqq 2 \, \|g\| + 2 \, \|v\| + C_K$$

from which we conclude that

$$\|v\|_{H^2(\Omega)} \leqq C' \, \| - \Delta v + v \| \leqq C(|Av| + \|v\| + 1) \quad \text{for every} \quad v \in D(A) \, . \tag{3.13}$$

From this point the proof proceeds exactly as the proof of Example 1. Equation (3.13) replaces equation (3.4) and after a simple computation we obtain $D(A) \subset D(B)$ and

$$\|Bu\| \leqq k \, |Au| + C\,(\|u\|) \quad \text{for every} \quad u \in D(A) \, ,$$

where $k < 1$.

Using the results of [6] in conjunction with Theorem 3.2 we obtain

COROLLARY 3.2. *Assume that the conditions of Theorem 3.2 are satisfied. Let $u_0 \in H^2(\Omega) \cap H_0^1(\Omega) \cap K$. Then the parabolic inequality*

$$\begin{aligned} &\int_\Omega \left(\Delta u - Vu - \frac{\partial u}{\partial t} \right) (u - v) \, dx \geqq 0 \quad \textit{for every} \quad v \in K \, , \\ &u(0) = u_0 \, , \end{aligned} \tag{3.14}$$

has a unique solution $u \in C(0, +\infty; L^2(\Omega))$ such that $u(x, t) \in H^2(\Omega) \cap H_0^1(\Omega) \cap K$ for every $t \geqq 0$ and $\partial u/\partial t \in L^\infty(0, +\infty; L^2(\Omega))$.

Remarks. 1. It can be shown that $\partial u/\partial t \in L^2(0, T; H_0^1(\Omega))$.

2. Weak solutions of (3.8) and (3.14) could be obtained using the results of Browder [5], Hartman and Stampacchia [7] and Lions-Stampacchia [12].

EXAMPLE 3. Let $\mathscr{V}$ be a reflexive Banach space, $\mathscr{H}$ be a Hilbert space and let

$$\mathscr{V} \subset \mathscr{H} \subset \mathscr{V}^* \, ,$$

where the embedding is dense and continuous. Let $\mathscr{K}$ be a closed convex set in $\mathscr{V}$ and $0 \in \mathscr{K}$. We denote by (,) the scalar product in $\mathscr{H}$ and in the duality $\mathscr{V}$, $\mathscr{V}^*$. Let $X = L^p(0, T; \mathscr{V})$, $p \geqq 2$, and $X^* = L^{p'}(0, T; \mathscr{V}^*)$ with

$1/p + 1/p' = 1$ and let

$$K = \{u \in X : u(t) \in \mathscr{K} \text{ a.e. on } (0, T)\} .$$

THEOREM 3.3. *Let* $B : K \cap C(0, T; \mathscr{H}) \to X^*$ *be a single-valued monotone hemicontinuous and coercive operator such that*

$$\|Bu\|_{X^*} \leqq \phi_1(\|u\|_X) \|u\|^\alpha_{C(0,T;\mathscr{H})} + \phi_2(\|u\|) \quad \text{for every} \quad u \in K \cap C(0, T; \mathscr{H}), \tag{3.15}$$

with $\alpha < 2$, ϕ_1 *and* ϕ_2 *non-decreasing functions. Then for every* $f \in X^*$ *there exists a* $u \in K \cap C(0, T; \mathscr{H})$ *such that* $u(0) = 0$ *(respectively* $u(0) = u(T)$*) which is a solution of*

$$\int_0^T \left(f - Bu - \frac{dv}{dt}, v - u\right) dt \leqq 0 \tag{3.16}$$

for every $v \in K$ *with* $dv/dt \in X^*$, $v(0) = 0$ *(respectively* $v(0) = v(T)$*).*

To prove Theorem 3.3 we show that the operators A and B defined below satisfy the conditions of Corollary 2.1. Let A be defined as follows: $g \in Au$ if and only if $u \in K$, $g \in X^*$ and

$$\int_0^T \left(g - \frac{dv}{dt}, v - u\right) dt \leqq 0$$

for every $v \in K$ with $dv/dt \in X^*$, $v(0) = 0$ (respectively $v(0) = v(T)$). It follows from a result of Brezis [3] that A is maximal monotone. Moreover, if $u \in D(A)$, then $u \in C(0, T; \mathscr{H})$, $u(0) = 0$ (respectively $u(0) = u(T)$) and

$$\|u\|^2_{C(0,T;\mathscr{H})} \leqq C_1 |Au|_{X^*} \|u\|_X + C_2 \|u\|^2_X \quad \text{for every} \quad u \in D(A) .$$

Using (3.15) we then have, for every $u \in D(A)$,

$$\begin{aligned} \|Bu\|^2_{X^*} &\leqq (C_1 |Au|_{X^*} \|u\|_X + C_2 \|u\|^2_X)^{\alpha/2} \phi_1(\|u\|_X) + \phi_2(\|u\|_X) \\ &\leqq C_1^{\alpha/2} |Au|^{\alpha/2}_{X^*} \|u\|^{\alpha/2}_X + \psi(\|u\|_X) \leqq \varepsilon |Au|_{X^*} + C_\varepsilon(\|u\|_X) \end{aligned}$$

(ε can be chosen arbitrarily small since $\alpha < 2$). Thus the conditions of Corollary 2.1 are satisfied and hence $A + B$ is maximal monotone. Finally since $0 \in A0$ and B is coercive, $A + B$ is coercive and $R(A + B) = X^*$.

Remark. Theorem 3.3 includes as a particular case the result of Lions [11].

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