JOURNAL OF FUNCTIONAL ANALYSIS 62, 73–86 (1985)

Sobolev Inequalities with Remainder Terms

HAÏM BREZIS

Département de Mathématiques, Université Paris VI,
4, Place Jussieu, 75230 Paris, Cedex 05, France

AND

ELLIOTT H. LIEB*

Departments of Mathematics and Physics, Princeton University,
Princeton, New Jersey 08544

Communicated by the Editors

Received September 14, 1984

The usual Sobolev inequality in $\mathbb{R}^n$, $n \geqslant 3$, asserts that $\|\nabla f\|_2^2 \geqslant S_n \|f\|_{2^*}^2$, with S_n being the sharp constant. This paper is concerned, instead, with functions restricted to bounded domains $\Omega \subset \mathbb{R}^n$. Two kinds of inequalities are established: (i) If $f = 0$ on $\partial\Omega$, then $\|\nabla f\|_2^2 \geqslant S_n \|f\|_{2^*}^2 + C(\Omega) \|f\|_{p,w}^2$ with $p = 2^*/2$ and $\|\nabla f\|_2^2 \geqslant S_n \|f\|_{2^*}^2 + D(\Omega) \|\nabla f\|_{q,w}^2$ with $q = n/(n-1)$. (ii) If $f \neq 0$ on $\partial\Omega$, then $\|\nabla f\|_2 + C(\Omega) \|f\|_{q,\partial\Omega} \geqslant S_n^{1/2} \|f\|_{2^*}$ with $q = 2(n-1)/(n-2)$. Some further results and open problems in this area are also presented.

I. INTRODUCTION

The usual Sobolev inequality in $\mathbb{R}^n$, $n \geqslant 3$, for the L^2 norm of the gradient is

$$\|\nabla f\|_2^2 \geqslant S_n \|f\|_{2^*}^2, \qquad 2^* = 2n/(n-2), \tag{1.1}$$

for all functions f with $\nabla f \in L^2$ and with f vanishing at infinity in the weak sense that $\text{meas}\{x \mid |f(x)| > a\} < \infty$ for all $a > 0$ (see [12]). The sharp constant S_n is known to be

$$S_n = \pi n(n-2)[\Gamma(n/2)/\Gamma(n)]^{2/n}. \tag{1.2}$$

* Work partially supported by U.S. National Science Foundation Grant PHY-8116101-A02.

0022-1236/85 $3.00

The constant S_n is achieved in (1.1) if and only if

$$f(x) = a[\varepsilon^2 + |x - y|^2]^{(2-n)/2} \tag{1.3}$$

for some $a \in \mathbb{C}$, $\varepsilon \neq 0$, and $y \in \mathbb{R}^n$ [1, 2, 6, 7, 9, 11].

In this paper we consider appropriate modifications of (1.1) when $\mathbb{R}^n$ is replaced by a *bounded* domain $\Omega \subset \mathbb{R}^n$. There are two main problems:

PROBLEM A. If $f = 0$ on $\partial\Omega$, then (1.1) still holds (with L^p norms in Ω, of course), since f can be extended to be zero outside of Ω. In this case (1.1) becomes a strict inequality when $f \neq 0$ (in view of (1.3)). However, S_n is still the sharp constant in (1.1) (since $\|\nabla f\|_2/\|f\|_{2^*}$ is scale invariant). Our goal, in this case, is to give a lower bound to the difference of the two sides in (1.1) for $f \in H_0^1(\Omega)$. In Section II we shall prove the following inequalities (1.4) and (1.6):

$$\|\nabla f\|_2^2 \geqslant S_n \|f\|_{2^*}^2 + C(\Omega) \|f\|_{p,w}^2, \tag{1.4}$$

where $C(\Omega)$ depends on Ω (and n), $p = n/(n-2) = 2^*/2$, and w denotes the weak L^p norm defined by

$$\|f\|_{p,w} = \sup_A |A|^{-1/p'} \int_A |f(x)| \, dx,$$

with A being a set of finite measure $|A|$.

The inequality (1.4) was motivated by the weaker inequality in [3],

$$\|\nabla f\|_2^2 \geqslant S_n \|f\|_{2^*}^2 + C_p(\Omega) \|f\|_p^2, \tag{1.5}$$

which holds for all $p < n/(n-2)$ (with $C_p(\Omega) \to 0$ as $p \to n/(n-2)$). The proof of (1.5) in [3] was very indirect compared to the proof of (1.4) given here. Inequality (1.4) is best possible in the sense that (1.5) cannot hold with $p = n/(n-2)$; this can be shown by taking the f in (1.3), applying a cutoff function to make f vanish on the boundary, and then expanding the integrals (as in [3]) near $\varepsilon = 0$.

An inequality stronger than (1.4), and involving the gradient norm is

$$\|\nabla f\|_2^2 \geqslant S_n \|f\|_{2^*}^2 + D(\Omega) \|\nabla f\|_{q,w}^2, \tag{1.6}$$

with $q = n/(n-1)$. (The reason that (1.6) is stronger than (1.4) is that the Sobolev inequality has an extension to the weak norms, by Young's inequalities in weak L^p spaces.)

Among the open questions concerning (1.4)–(1.6) are the following:

(a) What are the sharp constants in (1.4)–(1.6)? Are they achieved? Except in one case, they are not known, even for a ball. If $n=3$, Ω is a ball of radius R and $p=2$ in (1.5), then $C_2(\Omega)=\pi^2/(4R^2)$; however, this constant is not achieved [3].

(b) What can replace the right side of (1.4)–(1.6) when Ω is unbounded, e.g., a half-space?

(c) Is there a natural way to bound $\|\nabla f\|_2^2 - S_n \|f\|_{2^*}^2$ from below in terms of the "distance" of f from the set of optimal functions (1.3)?

PROBLEM B. If $f\neq 0$ on $\partial\Omega$, then (1.1) does not hold in Ω (simply take $f=1$ in Ω). Let us assume now that Ω is not only bounded but that $\partial\Omega$ (the boundary of Ω) has enough smoothness. Then (1.1) might be expected to hold if suitable boundary integrals are added to the left side. In Section III we shall prove that for $f=\text{constant}\equiv f(\partial\Omega)$ on $\partial\Omega$

$$\|\nabla f\|_2^2 + E(\Omega)\,|f(\partial\Omega)|^2 \geqslant S_n\,\|f\|_{2^*}^2. \tag{1.7}$$

On the other hand, if f is not constant on $\partial\Omega$, then the following two inequalities hold.

$$\|\nabla f\|_2^2 + F(\Omega)\,\|f\|_{H^{1/2}(\partial\Omega)}^2 \geqslant S_n\,\|f\|_{2^*}^2, \tag{1.8}$$

$$\|\nabla f\|_2 + G(\Omega)\,\|f\|_{q,\partial\Omega} \geqslant S_n^{1/2}\,\|f\|_{2^*}, \tag{1.9}$$

with $q=2(n-1)/(n-2)$, which is sharp. (Note the absence of the exponent 2 in (1.9).)

In addition to the obvious analogues of questions (a)–(c) for Problem B, one can also ask whether (1.9) can be improved to

$$\|\nabla f\|_2^2 + H(\Omega)\,\|f\|_{q,\partial\Omega}^2 \geqslant S_n\,\|f\|_{2^*}^2. \tag{1.10}$$

We do not know.

If Ω is a ball of radius R, we shall establish that the sharp constant in (1.7) is $E(\Omega)=\sigma_n R^{n-2}/(n-2)$, where σ_n is the surface area of the ball of unit radius in $\mathbb{R}^n$. With this $E(\Omega)$, (1.7) is a strict inequality. Given this fact, one suspects (in view of the solution to Problem A) that some term could be added to the right side of (1.7). However, such a term cannot be any $L^p(\Omega)$ norm of f, as will be shown.

To conclude this Introduction, let us mention two related inequalities. First, if one is willing to replace S_n on the right side of (1.10) by the smaller constant $2^{-2/n}S_n$, then for a ball one can obtain the inequality

$$\int |\nabla f|^2 + I(\Omega)\,\|f\|_{2,\partial\Omega}^2 \geqslant 2^{-2/n} S_n\,\|f\|_{2^*}^2. \tag{1.11}$$

This is proved in Section III. Inequalities related to (1.11) were derived by Cherrier [4] for general manifolds.

Second, one can consider the doubly weighted Hardy–Littlewood–Sobolev inequality [7, 10] which in some sense is the dual of (1.1), namely,

$$\left|\iint f(x)\, f(y)\, |x-y|^{-\lambda}\, |x|^{-\alpha}\, |y|^{-\alpha}\, dx\, dy\right| \leqslant P_{\alpha,\lambda,n}\, \|f\|_{p}^{2}, \tag{1.12}$$

with $p' = 2n/(\lambda + 2\alpha)$, $0 < \lambda < n$, $0 \leqslant \alpha < n/p'$. If f is restricted to have support in a bounded domain Ω and if P is (by definition) the sharp constant in $\mathbb{R}^n$, one should expect to be able to add some additional term to the left side of (1.12). When $p = 2$ this is indeed possible, and the additional term is

$$J_n\, |\Omega|^{-\lambda/n} \left\{\int f(x)\, |x|^{-\alpha}\, dx\right\}^2. \tag{1.13}$$

This was proved in [5] for $n = 3$, $\lambda = 2$, $\alpha = \frac{1}{2}$, and Ω being a ball, but the method easily extends (for a ball) to other n, λ. The result (1.13) further extends to general Ω (with the same constant J_n) by using the Riesz rearrangement inequality. On the other hand, when $p \neq 2$, it does not seem to be easy to find the additional term on the left side of (1.12): at least we have not succeeded in doing so. This is an open problem. In particular, in Section III we prove that when $p = \frac{6}{5}$, $n = 3$, $\lambda = 1$, $\alpha = 0$, one cannot even add $\|f\|_1^2$ to the left side of (1.12).

II. Proof of Inequalities (1.4) and (1.6)

Proof of Inequality (1.4). By the rearrangement inequality for the L^2 norm of the gradient we have

$$\|\nabla f^*\|_2 \leqslant \|\nabla f\|_2 \tag{2.1}$$

(see, e.g., [8]); in addition we have

$$\begin{aligned} \|f^*\|_{2^*} &= \|f\|_{2^*}, \\ \|f^*\|_{p,w} &= \|f\|_{p,w}. \end{aligned} \tag{2.2}$$

Here, f^* denotes the symmetric decreasing rearrangement of the function f extended to be zero outside Ω. Therefore, it suffices to consider the case in which Ω is a ball of radius R (chosen to have the same volume as the original domain) and f is symmetric decreasing.

Let $g \in L^\infty(\Omega)$ and define u to be the solution of

$$\begin{aligned} \Delta u &= g \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{2.3}$$

Let

$$\phi(x) = \begin{cases} f(x) + u(x) + \|u\|_\infty & \text{in} \quad \Omega, \\ \|u\|_\infty (R/|x|)^{n-2} & \text{in} \quad \Omega^c. \end{cases} \tag{2.4}$$

The Sobolev inequality in all of $\mathbb{R}^n$ applied to ϕ yields

$$\int_\Omega |\nabla(f+u)|^2 + \|u\|_\infty^2 R^{n-2}(n-2)\,\sigma_n \geqslant S_n \|f\|_{2^*}^2 \tag{2.5}$$

since $f \geqslant 0$ and $u + \|u\|_\infty \geqslant 0$. Here

$$\sigma_n = 2(\pi)^{n/2}/\Gamma(n/2)$$

is the surface area of the unit ball in $\mathbb{R}^n$. Therefore, we find

$$\int |\nabla f|^2 - 2\int fg + \int |\nabla u|^2 + k\,\|u\|_\infty^2 \geqslant S_n \|f\|_{2^*}^2, \tag{2.6}$$

where $k = R^{n-2}(n-2)\,\sigma_n$. Replacing g by λg and u by λu and optimizing with respect to λ we obtain

$$\int |\nabla f|^2 \geqslant S_n \|f\|_{2^*}^2 + \left(\int fg\right)^2 \Big/ \left[\int |\nabla u|^2 + k\,\|u\|_\infty^2\right]. \tag{2.7}$$

In inequality (2.7) we can obviously maximize the right side with respect to g. In view of the definition of the weak norm we shall in fact restrict our attention to $g = 1_A$, namely, the characteristic function of some set A in Ω. We shall now establish some simple estimates for all the quantities in (2.7) in which C_n generically denotes constants depending only on n,

$$\int fg = \int_A f, \tag{2.8}$$

$$\int |\nabla u|^2 \leqslant C_n |A|^{1+2/n}, \tag{2.9}$$

$$\|u\|_\infty \leqslant C_n |A|^{2/n}. \tag{2.10}$$

Indeed we have, by multiplying (2.3) by u and using Hölder's inequality,

$$\begin{aligned}\int |\nabla u|^2 = -\int_A u &\leqslant \|u\|_{2^*} |A|^{(1/2)+(1/n)} \\ &\leqslant S_n^{-1/2} \|\nabla u\|_2 |A|^{(1/2)+(1/n)}\end{aligned} \tag{2.11}$$

which implies (2.9). Next we have, by comparison with the solution in $\mathbb{R}^n$,

$$\begin{aligned}|u| &\leqslant C_n |x|^{-n+2} * (1_A) \\ &\leqslant C_n' |A|^{2/n}\end{aligned} \tag{2.12}$$

since the function $|x|^{-n+2}$ belongs to $L_w^{n/(n-2)}$. Since $|A| \leqslant |\Omega| = \sigma_n R^n/n$ we obtain

$$\int |\nabla u|^2 + k \|u\|_\infty^2 \leqslant C_n |A|^{4/n} R^{n-2}. \tag{2.13}$$

Hence (1.4) has been proved (for all Ω) with a constant

$$C(\Omega) = C_n |\Omega|^{(2-n)/n}. \tag{2.14}$$

Proof of Inequality (1.6). To a certain extent the previous proof can be imitated except for one important ingredient, namely, the rearrangement technique cannot be used since it is not true that $\|\nabla f\|_{q,w} \leqslant \|\nabla f^*\|_{q,w}$. (However, it is still true that we can replace f by $|f|$ without changing any of the norms in (1.6), and thus we may and still assume that $f \geqslant 0$.) Consequently we have to use a direct approach and the constant $D(\Omega)$ in (1.6) will not depend only on $|\Omega|$; it will in fact depend on the capacity of Ω. It is an open question whether (1.6) holds with $D(\Omega)$ depending only on $|\Omega|$. Our result is that

$$D(\Omega) = C_n/\mathrm{cap}(\Omega). \tag{2.15}$$

We begin as before with (2.3), but (2.4) is replaced by

$$\phi = \begin{cases} f + u + \|u\|_\infty & \text{in} \quad \Omega, \\ \|u\|_\infty v & \text{in} \quad \Omega^c, \end{cases} \tag{2.16}$$

where v is the solution of

$$\begin{aligned} \Delta v &= 0 \quad \text{in} \quad \Omega^c, \\ v &= 1 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.17}$$

with $v \to 0$ at infinity. By definition,

$$\operatorname{cap}(\Omega) = \int |\nabla v|^2. \tag{2.18}$$

Inequality (2.7) still holds but with the constant k replaced by $k = \operatorname{cap}(\Omega)$. Also we note that (2.7) can be written as

$$\int |\nabla f|^2 \geqslant S_n \|f\|_{2^*}^2 + \left(\int \nabla f \cdot \nabla u\right)^2 \Big/ \left[\int |\nabla u|^2 + k \|u\|_\infty^2\right], \tag{2.19}$$

which holds for any $u \in C_0^\infty(\Omega)$. By density, (2.19) still holds for every u in $H_0^1 \cap L^\infty$ (the reason is that for every such u there is a sequence $u_j \in C_0^\infty(\Omega)$ with $u_j \to u$ in H_0^1 and $\|u_j\|_\infty \to \|u\|_\infty$).

We now choose u to be the solution of (2.3) with

$$g = \frac{\partial}{\partial x_i}\left[\left(\operatorname{sgn} \frac{\partial f}{\partial x_i}\right) 1_A\right]. \tag{2.20}$$

This function u is in L^∞ as we now verify. We can write

$$u = w + h,$$

where w satisfies $\Delta w = g$ in all of $\mathbb{R}^n$, namely,

$$w = C_n |x|^{2-n} * g. \tag{2.21}$$

Clearly h is harmonic and $h = -w$ on $\partial\Omega$. Therefore $\|h\|_\infty \leqslant \|w\|_{\infty,\partial\Omega} \leqslant \|w\|_\infty$ and hence $\|u\|_\infty \leqslant 2 \|w\|_\infty$. On the other hand,

$$w = C_n \left(\frac{\partial}{\partial x_i} |x|^{2-n}\right) * \left[\left(\operatorname{sgn} \frac{\partial f}{\partial x_i}\right) 1_A\right],$$

and thus

$$|w| \leqslant C_n(n-2) |x|^{1-n} * 1_A. \tag{2.22}$$

Since $|x|^{1-n} \in L_w^{n/(n-1)}$ we obtain

$$\|u\|_\infty \leqslant 2 \|w\|_\infty \leqslant C_n' |A|^{1/n}. \tag{2.23}$$

Next, let us estimate $\int |\nabla u|^2$. Multiplying (2.3) by u we have

$$\int |\nabla u|^2 = \int (\operatorname{sgn} \partial f/\partial x_i)\, 1_A (\partial u/\partial x_i) \leqslant \left[\int |\nabla u|^2\right]^{1/2} |A|^{1/2}$$

and thus

$$\int |\nabla u|^2 \leqslant |A|. \tag{2.24}$$

Finally, since $f=0$ on $\partial\Omega$,

$$\int \nabla f \cdot \nabla u = -\int f\, \Delta u = \int |\partial f/\partial x_i|\, 1_A. \tag{2.25}$$

Using these estimates in (2.19) we find

$$\int |\nabla f|^2 \geqslant S_n \, \|f\|_{2^*}^2 + C_n \left(\int_A |\partial f/\partial x_i|\right)^2 \Big/ (\mathrm{cap}(\Omega)\, |A|^{2/n}),$$

since $|A|^{1-(2/n)} \leqslant |\Omega|^{1-(2/n)} \leqslant S_n^{-1} \,\mathrm{cap}(\Omega)$ by Sobolev's inequality applied to the function $\tilde{v} = v$ in Ω^c and $\tilde{v} = 1$ in Ω. This completes the proof of (1.6) with the constant given in (2.15).

III. Proofs of (1.7)–(1.9) and Related Matters

Proof of (1.8). Let us define

$$\phi = \begin{cases} f & \text{in } \Omega, \\ w & \text{in } \Omega^c, \end{cases} \tag{3.1}$$

where w is the harmonic function that vanishes at infinity and agrees with f on $\partial\Omega$. Using ϕ in (1.1) we find

$$\int_\Omega |\nabla f|^2 + \int_{\Omega^c} |\nabla w|^2 \geqslant S_n \, \|f\|_{2^*}^2. \tag{3.2}$$

On the other hand, we have

$$\int_{\Omega^c} |\nabla w|^2 \sim \|f\|^2_{H^{1/2}(\partial\Omega)}. \tag{3.3}$$

This concludes the proof of (1.8).

Proof of (1.7). Now suppose that f is a constant on $\partial\Omega$. We shall first investigate the case that Ω is a ball of radius R centered at zero. In this case $w(x) = f(\partial\Omega)\, R^{n-2}\, |x|^{2-n}$. Inequality (3.2) then yields (1.7) with

$$\begin{aligned} E(\Omega) = \mathrm{cap}(\Omega) &= \sigma_n R^{n-2}/(n-2) \\ &= \frac{n\,|\Omega|}{n-2} \left\{\frac{\sigma_n}{n\,|\Omega|}\right\}^{2/n}. \end{aligned} \tag{3.4}$$

Furthermore, (1.7) is a strict inequality with this $E(\Omega)$ because the function ϕ in (3.1) is not of the form (1.3). Also, $E(\Omega)$ given by (3.4) is the sharp constant in (3.4). To see this we apply (1.7) with $f=f_\varepsilon$ given by (1.3) with $a=1$ and $y=0=$ center of the ball. We have

$$\int_{\mathbb{R}^n} |\nabla f_\varepsilon|^2 = S_n \, \|f_\varepsilon\|_{2^*,\mathbb{R}^n}^2. \tag{3.5}$$

On the other hand, as $\varepsilon \to 0$

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla f_\varepsilon|^2 &= \int_\Omega |\nabla f_\varepsilon|^2 + \int_{\Omega^c} |\nabla f_\varepsilon|^2 \\ &= \int_\Omega |\nabla f_\varepsilon|^2 + \mathrm{cap}(\Omega)\, |f_\varepsilon(\partial\Omega)|^2 + o(1).\end{aligned} \tag{3.6}$$

Here we have to note that as $\varepsilon \to 0$ for $|x| > R$

$$f_\varepsilon(x) \to |x|^{2-n}$$

in the appropriate topologies. On the other hand,

$$\int_{\mathbb{R}^n} |f_\varepsilon|^{2^*} - \int_\Omega |f_\varepsilon|^{2^*} = \int_{\Omega^c} |f_\varepsilon|^{2^*} \to C.$$

Thus

$$\|f_\varepsilon\|_{2^*,\mathbb{R}^n}^2 = \|f_\varepsilon\|_{2^*,\Omega}^2 + o(1). \tag{3.7}$$

This proves that $E(\Omega)$ in (1.7) is greater than or equal to $\mathrm{cap}(\Omega)$ when Ω is a ball, and thus that (3.4) is sharp.

The same calculation with f_ε as above shows that if Ω is a ball there is no inequality of the type

$$\int_\Omega |\nabla f|^2 + \mathrm{cap}(\Omega)\, |f(\partial\Omega)|^2 \geqslant S_n \, \|f\|_{2^*}^2 + d\, \|f\|_1^2 \tag{3.8}$$

with $d > 0$, because the additional term $\|f_\varepsilon\|_1 = O(1)$ as $\varepsilon \to 0$.

Now we consider a general domain with $f(\partial\Omega) = \text{constant} = C$. We can assume $C \geqslant 0$ and note that we can also assume $f \geqslant C$ in Ω. (This is so because replacing f by $|f - C| + C \geqslant f$ does not decrease the L^{2^*} norm and leaves $\|\nabla f\|_2$ invariant.) Consider the function $g = f - C \geqslant 0$ which vanishes on $\partial\Omega$ and hence can be extended to be zero on Ω^c. Apply to g the rearrangement inequality for the L^2 norm of the gradient, as was done in

Section II. Finally consider $\tilde{f} \equiv g^* + C$ in the ball Ω^* whose volume is $|\Omega|$. Since $\tilde{f}(\partial\Omega^*) = C = f(\partial\Omega)$ we have

$$\int_{\Omega^*} |\nabla \tilde{f}|^2 + E(\Omega^*)\,|f(\partial\Omega)|^2 \geqslant S_n\,\|\tilde{f}\|_{2^*,\Omega^*}^2.$$

As we remarked, $\|\nabla f\|_2 \geqslant \|\nabla \tilde{f}\|_2$. Also since $f \geqslant C$, it is easy to check that $\|f\|_{2^*} = \|\tilde{f}\|_{2^*}$.

The conclusion to be drawn from this exercise is that (1.7) holds for general Ω with $E(\Omega)$ given by (3.4), namely, $\operatorname{cap}(\Omega^*)$. We also note that (1.7), with this $E(\Omega)$, is strict, since it is strict for a ball.

QUESTION. Is $E(\Omega)$ given by (3.4) the sharp constant in general?

Proof of (1.9). Given f in Ω we consider the harmonic function h in Ω which equals f on $\partial\Omega$. We write

$$f = h + u \tag{3.9}$$

with $u = 0$ on $\partial\Omega$ and thus

$$\int |\nabla u|^2 \geqslant S_n\,\|u\|_{2^*}^2. \tag{3.10}$$

On the one hand

$$\int |\nabla u|^2 = \int |\nabla(f-h)|^2 = \int |\nabla f|^2 - \int |\nabla h|^2 \tag{3.11}$$

(note that $\int_\Omega |\nabla h|^2 = \int_{\partial\Omega} h(\partial h/\partial n) = \int_{\partial\Omega} f(\partial h/\partial n) = \int_\Omega \nabla f\,\nabla h$). On the other hand, by the triangle inequality,

$$\|u\|_{2^*} \geqslant \|f\|_{2^*} - \|h\|_{2^*}. \tag{3.12}$$

Inserting (3.11) and (3.12) in (3.10) we obtain

$$\|\nabla f\|_2 + \|h\|_{2^*} \geqslant S_n^{1/2}\,\|f\|_{2^*}. \tag{3.13}$$

Next we claim that

$$\|h\|_{2^*} \leqslant G(\Omega)\,\|f\|_{q,\partial\Omega} \tag{3.14}$$

with $q = 2(n-1)/(n-2)$, which will complete the proof of (1.9). The proof of (3.14) is a standard duality argument. Indeed, let ψ be the solution of

$$\begin{aligned} \Delta\psi &= Y \quad \text{in} \quad \Omega, \\ \psi &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{3.15}$$

where Y is some arbitrary function in L^t. We have, by multiplying by h and integrating by parts,

$$\int_\Omega hY = \int_{\partial\Omega} f \frac{\partial\psi}{\partial n}. \tag{3.16}$$

However, the L^p regularity theory shows that $\psi \in W^{2,t}$ with $\|\psi\|_{W^{2,t}(\Omega)} \leqslant C\,\|Y\|_t$. In particular, $\|\nabla\psi\|_{W^{1,t}(\Omega)} \leqslant C\,\|Y\|_t$ and, by trace inequalities,

$$\left\|\frac{\partial\psi}{\partial n}\right\|_{r,\partial\Omega} \leqslant C\,\|Y\|_t, \tag{3.17}$$

where

$$\frac{1}{r} = \frac{n-t}{t(n-1)}. \tag{3.18}$$

Therefore, by (3.16) and Hölder's inequality,

$$\left|\int hY\right| \leqslant C\,\|f\|_{q,\partial\Omega}\,\|Y\|_t, \tag{3.19}$$

where $1/r + 1/q = 1$. Since (3.19) holds for all Y we conclude that

$$\|h\|_{t'} \leqslant C\,\|f\|_{q,\partial\Omega},$$

which coincides with (3.14) since $t' = 2^*$ when $q = 2(n-1)/(n-2)$.

Finally, we claim that there is no inequality of the type (1.9) with $q < 2(n-1)/(n-2)$. Indeed, suppose (1.9) holds with some such q. We choose $f = f_\varepsilon$ as in (1.3) with $a = 1$ and $y \in \partial\Omega$. It is obvious that as $\varepsilon \to 0$

$$\int_\Omega |\nabla f_\varepsilon|^2 \Big/ \int_{\mathbb{R}^n} |\nabla f_\varepsilon|^2 = 1/2 + o(1),$$

$$\int_\Omega |f_\varepsilon|^{2^*} \Big/ \int_{\mathbb{R}^n} |f_\varepsilon|^{2^*} = 1/2 + o(1),$$

while

$$\int_{\mathbb{R}^n} |\nabla f_\varepsilon|^2 = S_n\,\|f_\varepsilon\|^2_{2^*,\mathbb{R}^n} \qquad \text{and} \qquad \|f_\varepsilon\|_{q,\partial\Omega}/\|f_\varepsilon\|_{2^*} = o(1).$$

This contradicts (1.9).

Remark. The last exercise with f_ε given above shows that it is not possible to apply rearrangement techniques when f is not constant on $\partial\Omega$, even if Ω is a ball. It also shows that there is *no* inequality for all $f \in H^1$ of the type

$$\|\nabla f\|_2^2 + C\|f\|_{q,\Omega}^2 \geqslant S_n \|f\|_{2^*}^2$$

with $q < 2^*$.

Proof of (1.11). Let Ω be a ball of radius R centered at zero. For simplicity, assume $R = 1$. Define

$$g(x) = \begin{cases} f(x), & |x| \leqslant 1, \\ |x|^{2-n} f(x\,|x|^{-2}), & |x| \geqslant 1, \end{cases} \tag{3.20}$$

and apply the usual Sobolev inequality (1.1) to g. We note (by a change of variables) that

$$\begin{aligned} \int_\Omega g^{2^*} &= \int_{\Omega^c} g^{2^*}, \\ \int_\Omega |\nabla g|^2 &= \int_{\Omega^c} |\nabla g|^2 - (n-2)\,\|f\|_{2,\partial\Omega}^2. \end{aligned} \tag{3.21}$$

Inserting (3.21) into (1.1) yields (1.11) with $I(\Omega) = (n-2)/2$.

Remark on the Hardy–Littlewood–Sobolev Inequality

Consider the inequality (in $\mathbb{R}^3$)

$$I(f) \leqslant P\,\|f\|_{6/5}^2, \tag{3.22}$$

with

$$I(f) = \iint f(x)\,f(y)\,|x-y|^{-1}\,dx\,dy \geqslant 0. \tag{3.23}$$

The sharp constant P is known to be [7]

$$P = 4^{5/3}/[3\pi^{1/3}]. \tag{3.24}$$

Let Ω be a ball of radius one centered at zero and assume that $f = 0$ outside Ω. In this case, (3.22) is strict because the only functions that give equality in (3.22) are of the form [7]

$$f_\varepsilon(x) = a[\varepsilon^2 + |x-y|^2]^{-5/2}. \tag{3.25}$$

For $f=0$ outside Ω, we ask whether (3.22) can be improved to

$$C \|f\|_1^2 + I(f) \leqslant P \|f\|_{6/5}^2. \tag{3.26}$$

Our conclusion is that (3.26) fails for any $C>0$.

Take $f=\bar{f}_\varepsilon \equiv f_\varepsilon 1_\Omega$ with f_ε given by (3.25) and with $y=0$ and with $a=a_\varepsilon$ chosen so that $\|f_\varepsilon\|_{6/5, \mathbb{R}^3}=1$. The function f_ε satisfies the following (Euler) equation on $\mathbb{R}^3$,

$$\frac{1}{|x|} * f_\varepsilon = P f_\varepsilon^{1/5}. \tag{3.27}$$

However, for $|x|<1$

$$\left(\frac{1}{|x|} * \bar{f}_\varepsilon\right)(x) + K_\varepsilon = \left(\frac{1}{|x|} * f_\varepsilon\right)(x), \tag{3.28}$$

where K_ε is a constant bounded above by $D_\varepsilon = \int_{|x|>1} f_\varepsilon$. Multiply (3.27) by $\bar{f}_\varepsilon$ and integrate over Ω. Then

$$I(\bar{f}_\varepsilon) + T_\varepsilon \|\bar{f}_\varepsilon\|_1^2 \geqslant I(\bar{f}_\varepsilon) + K_\varepsilon \int \bar{f}_\varepsilon = P \|\bar{f}_\varepsilon\|_{6/5}^{6/5} \geqslant P \|\bar{f}_\varepsilon\|_{6/5}^2, \tag{3.29}$$

where $T_\varepsilon = D_\varepsilon / \int \bar{f}_\varepsilon$. From (3.29), we see that (3.26) fails if $C > T_\varepsilon$ for any $\varepsilon > 0$. However, it is obvious that $T_\varepsilon \to 0$ as $\varepsilon \to 0$.

References

1. T. Aubin, Problèmes isopérimétriques et espaces de Sobolev, *C. R. Acad. Sci. Paris* **280A** (1975), 279–281; *J. Diff. Geom.* **11** (1976), 573–598.
2. G. A. Bliss, An integral inequality, *J. London Math. Soc.* **5** (1930), 40–46.
3. H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.* **36** (1983), 437–477.
4. P. Cherrier, Problèmes de Neumann nonlinéaires sur les variétés Riemanniennes, *J. Funct. Anal.* **57** (1984), 154–206.
5. I. Daubechies and E. Lieb, One-electron relativistic molecules with Coulomb interaction, *Comm. Math. Phys.* **90** (1983), 497–510.
6. B. Gidas, W. M. Ni, and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$, *in* "Mathematical Analysis and Applications" (L. Nachbin, Ed.), pp. 370–401, Academic Press, New York, 1981.
7. E. Lieb, Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities, *Ann. of Math.* **118** (1983), 349–374.
8. E. Lieb, Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation, *Stud. Appl. Math.* **57** (1977), 93–105.
9. G. Rosen, Minimum value for c in the Sobolev inequality $\|\phi\|_6 \leqslant c \|\nabla\phi\|_2$, *SIAM J. Appl. Math.* **21** (1971), 30–32.

10. E. Stein and G. Weiss, Fractional integrals in n-dimensional Euclidean space, *J. Math. Mech.* **7** (1958), 503–514. See also, G. Hardy and J. Littlewood, Some properties of fractional integrals (1), *Math. Z.* **27** (1928), 565–606.
11. G. Talenti, Best constant in Sobolev inequality, *Ann. Mat. Pura Appl.* **110** (1976), 353–372.
12. H. Brezis and E. Lieb, Minimum action solution of some vector field equations, *Comm. Math. Phys.* **96** (1984), 97–113. See Remark 3 on p. 100.