A Very Singular Solution of the Heat Equation with Absorption

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1. Introduction

Consider the Cauchy problem

$$u_t - \Delta u + u^p = 0 \quad \text{on } \mathbb{R}^N \times (0, \infty) \tag{1.1}$$

$$u > 0 \quad \text{on } \mathbb{R}^N \times (0, \infty) \tag{1.2}$$

$$u(x, 0) = c\,\delta(x) \quad \text{on } \mathbb{R}^N, \tag{1.3}$$

where $N \geqq 1$, $c > 0$ is a constant and $\delta(x)$ denotes the Dirac mass at the origin.

A result of BREZIS and FRIEDMAN [6] asserts that if $1 < p < (N+2)/N$, then for every $c > 0$ there exists a unique[1] solution u_c of (1.1)–(1.3). When $p \geqq (N+2)/N$ there is *no* solution of (1.1)–(1.3) and in fact any solution u of (1.1) such that $u \geqq 0$ on $\mathbb{R}^N \times (0, \infty)$ and

$$\lim_{t \downarrow 0} \int_{\mathbb{R}^N} u(x, t)\, \chi(x)\, dx = 0 \qquad \forall \chi \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$$

must vanish identically. Therefore we deal only with the case $p < (N+2)/N$.

The function $u_c(x, t)$ is smooth in $\mathbb{R}^N \times [0, \infty)$ *except at the point* (0, 0). Near (0, 0) the singular behaviour of u_c is essentially like that of cE, where $E(x, t)$ is the fundamental solution of the heat equation, that is

$$E(x, t) = \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x|^2}{4t}}.$$

In particular it can be shown that

$$u_c \leqq cE,$$

$$\lim_{t \downarrow 0} \int_{\mathbb{R}^N} |u_c(x, t) - cE(x, t)|\, dx = 0$$

[1] The solution is unique, even without prescribing any condition at infinity (see BREZIS [5]).

and, more precisely, as is shown in the Appendix,

$$|u_c(x,t) - cE(x,t)| \leqq Ct^{\nu}E(x,t) \quad \forall x \in \mathbb{R}^N, \quad t > 0,$$

for some constants $C > 0$ and $\nu = \frac{1}{2}N\{1 + (2/N) - p\} > 0$.

In this paper we establish that there exists a function $W(x, t)$, which satisfies (1.1), (1.2) and

$$W \text{ is smooth in } \mathbb{R}^N \times [0, \infty) \text{ except at } (0, 0) \tag{1.4}$$

$$W(x, 0) = 0 \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\}, \tag{1.5}$$

and W is more singular that E at (0, 0).

We look for a function W of the form

$$W(x,t) = \frac{1}{t^{1/(p-1)}} f\left(\frac{|x|}{t^{1/2}}\right). \tag{1.6}$$

It is readily verified that W satisfies (1.1), (1.2), (1.4) and (1.5) if and only if $f(\eta)$ satisfies

$$f'' + \left(\frac{N-1}{\eta} + \tfrac{1}{2}\eta\right) f' + \frac{1}{p-1} f - f^p = 0 \quad \text{on } (0, \infty) \tag{1.7}$$

$$f > 0 \quad \text{and } f \text{ is smooth on } [0, \infty) \tag{1.8}$$

$$f'(0) = 0 \quad \text{and} \quad \lim_{\eta \to \infty} \eta^{\frac{2}{p-1}} f(\eta) = 0. \tag{1.9}$$

Our main result is the following:

Theorem. *If* $1 < p < 1 + (2/N)$, *then there is a unique function* f *satisfying* (1.7), (1.8) *and* (1.9). *In addition the asymptotic behaviour of* f *as* $\eta \to \infty$ *is given by*

$$f(\eta) = Ae^{-\frac{1}{4}\eta^2} \eta^{\alpha - N} \left\{1 - (\alpha - N)(\alpha - 2)\frac{1}{\eta^2} + o\left(\frac{1}{\eta^2}\right)\right\},$$

where $\alpha = 2/(p-1)$ *and* A *is a certain positive constant.*

One sees easily that the singularity of W at (0, 0) is stronger than the singularity of E. Note for example that

$$E(0, t) = (4\pi t)^{-N/2},$$

while

$$W(0, t) = t^{-\frac{1}{p-1}} f(0)$$

and $1/(p-1) > N/2$ because $p < (N+2)/N$. Also observe that for any $\varepsilon > 0$ there exists a constant $K_\varepsilon > 0$ such that

$$\|W(\cdot, t)\|_{L^\infty(|x| > \varepsilon)} \leqq e^{-\frac{K_\varepsilon}{t}} \quad \text{as } t \downarrow 0,$$

and

$$\int_{\mathbb{R}^N} W(x, t)\, dx = c_p t^{-\gamma},$$

where

$$\gamma = \frac{1}{p-1} - \frac{N}{2} > 0$$

and $c_p > 0$ is a certain constant.

It follows in particular that

$$\lim_{t \downarrow 0} \int_{\mathbb{R}^N} W(x, t)\, dx = \infty$$

and $\lim_{t \downarrow 0} W(\cdot, t)$ does not exist, even in the weak sense of distributions (in $\mathscr{D}'(\mathbb{R}^N)$).

There is a certain analogy with a nonlinear elliptic problem. It is known that for each constant $c > 0$ the problem

$$\begin{cases} -\Delta u + u^p = c\delta & \text{on } \mathbb{R}^N \quad (1.10) \\ u > 0 & \text{on } \mathbb{R}^N \quad (1.11) \end{cases}$$

has a unique solution u_c if $p < N/(N-2)$ (no restriction on p if $N = 1$ or 2) and that *no* solution exists if $p \geqq N/(N-2)$. The function u_c is smooth in $\mathbb{R}^N \setminus \{0\}$, and near the origin the singular behaviour of u_c is essentially like that of cE, where E is the fundamental solution of $-\Delta$:

$$E(x) = \begin{cases} \dfrac{1}{(N-2)\,\omega_N} \cdot \dfrac{1}{|x|^{N-2}} & \text{if } N > 2 \\[2ex] \dfrac{1}{2\pi} \log \dfrac{1}{|x|} & \text{if } N = 2, \end{cases}$$

where ω_N is the area of the surface of the unit ball.

On the other hand if $p < N/(N-2)$ there exists a function $W(x)$ which satisfies

$$\begin{cases} -\Delta W + W^p = 0 \quad \text{on } \mathbb{R}^N \setminus \{0\} & (1.12) \\ W > 0 \quad \text{and } W \text{ is smooth on } \mathbb{R}^N \setminus \{0\} & (1.13) \\ W \text{ is more singular than } E \text{ at } 0. \end{cases}$$

In fact, W is given explicitly by

$$W(x) = c(p, N)\, |x|^{-2/(p-1)}$$

for some appropriate positive constant $c(p, N)$. In addition one knows that:

(i) W coincides with the (increasing) limit of u_c as $c \uparrow \infty$.

(ii) The functions (u_c) and W are the only functions satisfying (1.12) and (1.13).

(iii) The functions (u_c) and W provide a *complete classification* of the local behaviour near $x = 0$ of *any* function u satisfying

$$\begin{cases} -\Delta u + u^p = 0 & \text{in } \Omega \setminus \{0\} \qquad (1.14) \\ u > 0 \quad \text{and } u \text{ is smooth} & \text{in } \Omega \setminus \{0\}, \qquad (1.15) \end{cases}$$

where Ω is a neighbourhood of $x = 0$ in $\mathbb{R}^N$.

More precisely, if u satisfies (1.14), (1.15) then *either* u is smooth at $x = 0$,
or $u \sim cE$ as $x \to 0$ for some constant $c > 0$,
or $u \sim W$ as $x \to 0$.
For all these properties we refer the reader to [4, 7, 8, 9, 15]. We believe that similar results hold for (1.1), (1.2).[1]

The plan of the paper is the following. In section 2 we formulate Problem (1.7)–(1.9) in the phase plane and in the subsequent three sections we establish the existence of a solution of (1.7)–(1.9) by means of a shooting argument. We refer to a solution of (1.7)–(1.9) as a *fast orbit.*

The proof also demonstrates the existence of a one-parameter family of functions which satisfy (1.7), (1.8) and the boundary conditions

$$f'(0) = 0, \quad f(\infty) = 0,$$

but which do not satisfy (1.9). In fact they satisfy the boundary conditions

$$f'(0) = 0 \quad \text{and} \quad \lim_{\eta \to \infty} \eta^{\frac{2}{p-1}} f(\eta) = A > 0.$$

We refer to these solutions as the *slow orbits*. Their existence (for any given $A > 0$) has been established earlier by KAMIN and PELETIER [11] in connection with an investigation of the large time behaviour of solutions of (1.1).

In section 6 we analyze the asymptotic behaviour as $\eta \to \infty$ of both the fast and slow orbits. Using the results of section 6, we prove in section 7 that there exists at most one fast orbit, and thus, at most one solution of (1.7)–(1.9).

Finally we mention a study of HARAUX and WEISSLER [10] of solutions W of the form (1.6) of the equation

$$u_t - \Delta u - u^p = 0 \quad \text{on } \mathbb{R}^N \times (0, \infty)$$

$$u > 0 \quad \text{on } \mathbb{R}^N \times (0, \infty).$$

They are led to an equation similar to (1.7) in which however the nonlinear term has a different sign ($-f^p$ is replaced $+f^p$). Their methods and conclusion are quite different from ours*.

* After the manuscript was completed, we received two preprints from WEISSLER [16, 17]. In [16] he extend his analysis with HARAUX [10]. and in [17] he proves the existence of a solution f of Problem (1.7)–(1.8)–(1.9) when $p < 1 + (2/N)$. The asymptotic behaviour of $f(\eta)$ as $\eta \to \infty$ is shown to be $O\left(e^{-\frac{1}{8}\eta^2} \eta^{(1-N)/2}\right)$. See also [20] and [22].

[1] Thus has been established in KAMIN–PELETIER [18] and L. OSWALD [19].

2. Preliminaries

We consider the problem

$$\text{(A)}\qquad \begin{cases} u'' + \left(\dfrac{N-1}{x} + \dfrac{x}{2}\right) u' + \dfrac{\alpha}{2} u - u^p = 0, \quad \forall x > 0, & (2.1) \\ u > 0 \quad \forall x \geqq 0, & \\ u'(0) = 0, \quad u(\infty) = 0 & (2.2) \end{cases}$$

in which $\alpha > 0$ and $p > 1$. Problem (1.7)–(1.8)–(1.9) is a special case of Problem A, in which $\alpha = 2/(p-1)$ and the behaviour of $u(x)$ as $x \to \infty$ is further prescribed to be $o(x^{-\alpha})$. Note that

$$p < \frac{N+2}{N} \Rightarrow \alpha > N.$$

In this section, and the next three sections we shall prove that there exists a solution u of Problem A such that $u(x) = o(x^{-\alpha})$ as $x \to \infty$.

Theorem 1. *Suppose $p > 1$ and $\alpha > N$. Then there exists a solution of Problem A with the property*

$$\lim_{x \to \infty} x^\alpha u(x) = 0. \tag{2.3}$$

The proof of this theorem is based on a shooting argument in the phase plane. Thus, we write equation (2.1) as a first order system

$$\text{(B)}\qquad \begin{cases} u' = v \\ v' = -\left(\dfrac{N-1}{x} + \dfrac{x}{2}\right) v - f(u), \end{cases}$$

where

$$f(u) = \frac{\alpha}{2} u - u^p. \tag{2.4}$$

This system has two critical points: (0, 0) and $(A, 0)$, where $A = (\alpha/2)^{1/(p-1)}$. For each $\gamma \in [0, A]$, let $(u(x, \gamma), v(x, \gamma))$ be the solution of (B) which satisfies

$$(u(0, \gamma), v(0, \gamma)) = (\gamma, 0).$$

It was shown in [13] that this solution is well defined and twice continuously differentiable in some interval $[0, x_0]$, $x_0 > 0$, and that

$$\lim_{x \to 0} \frac{u'(x, \gamma)}{x} = u''(0, \gamma) = -\frac{1}{N} f(\gamma). \tag{2.5}$$

We wish to prove that there exists a number $\gamma_0 \in (0, A)$ such that $u(x, \gamma_0)$ exists and is positive for all $x \geqq 0$, and satisfies (2.3). The proof is broken up in a number of steps. We first prove in section 3, that if γ is sufficiently close to

A, then $u(x, \gamma) > 0$ for $x \geqq 0$. We shall see that this implies that $u(x, \gamma) \to 0$ as $x \to \infty$. Then, in section 4, we discuss the rates at which $u(x, \gamma)$ may approach zero and finally, in section 5, we complete the proof of Theorem 1, showing that

$$\gamma_0 = \inf\{\gamma \in (0, A) : u(x, \gamma) > 0 \quad \text{for all } x \geqq 0\} > 0$$

and that $u(x, \gamma_0)$ has the asymptotic behaviour prescribed in (2.3).

3. Global Behaviour in the Phase Plane

We begin with some notation. For $\lambda > 0$, let

$$\mathcal{S} = \{(u, v) : 0 \leqq u \leqq A, v \leqq 0\}$$

$$\mathcal{L}_\lambda = \{(u, v) \in \mathcal{S} : v \geqq -\lambda u\}$$

$$\ell_\lambda = \{(u, v) \in \mathcal{S} : v = -\lambda u\},$$

and for $\varepsilon > 0$ we define

$$\mathcal{A}_\varepsilon = \{(u, v) \in \mathcal{S} : \|(u, v) - (A, 0)\| < \varepsilon\}.$$

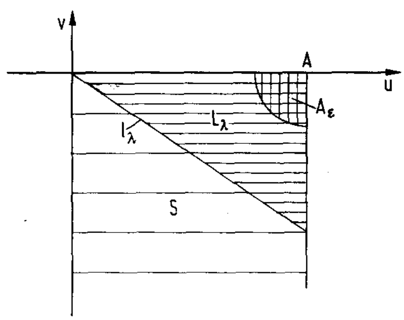

Fig. 1. The sets $\mathcal{S}$, $\mathcal{L}_\lambda$, ℓ_λ and $\mathcal{A}_\varepsilon$.

Lemma 1. *For every* $\gamma \in (0, A)$ *there exists a number* $x_0 > 0$ *such that*

$$(u(x, \gamma), v(x, \gamma)) \in \mathcal{S} \quad \text{for } 0 \leqq x \leqq x_0.$$

If $(u(x, \gamma), v(x, \gamma))$ *ever leaves* $\mathcal{S}$ *it must do so through the half line* $\{(u, v) : u = 0, v < 0\}$.

Proof. The existence of a number $x_0 > 0$ follows from (2.5). If $(u(x, \gamma), v(x, \gamma))$ leaves $\mathcal{S}$ it cannot do so through the top or the right side of $\mathcal{S}$ because on those sides the vector field determined by (B) points into $\mathcal{S}$ for all $x > 0$, and not through the corners since they are equilibrium points.

Thus, for any $\gamma \in (0, A)$, (u, v) enters $\mathcal{S}$, and we now need to find conditions on γ which ensure that it does not leave it.

Lemma 2. *For any* $\lambda > 0$ *there exists a* $\xi_\lambda > 0$ *such that* $\mathscr{L}_\lambda$ *is positively invariant for* $x \geqq \xi_\lambda$. *That is, if* $(u_0, v_0) \in \mathscr{L}_\lambda$, $x_0 \geqq \xi_\lambda$ *and* $(u(x), v(x))$ *is the solution of* (B) *which satisfies* $(u(x_0), v(x_0)) = (u_0, v_0)$, *then* $(u(x), v(x)) \in \mathscr{L}_\lambda$ *for all* $x \geqq x_0$.

Proof. We shall show that, given $\lambda > 0$, there exists a $\xi_\lambda > 0$ such that if $x > \xi_\lambda$, then the vector field determined by (B) points into $\mathscr{L}_\lambda$, except at the critical points $(0, 0)$ and $(A, 0)$.

On the top $(v = 0)$:

$$v' = -\left(\frac{N-1}{x} + \frac{x}{2}\right) v - f(u) < 0 \qquad \text{for all } x > 0,$$

while on the right side $(u = A)$,

$$u' = v < 0 \qquad \text{for all } x > 0.$$

Thus, on the top and on the right side, the vector field points into $\mathscr{L}_\lambda$ for all $x > 0$. On l_λ we must prove that $v'/u' < -\lambda$ for x sufficiently large. This is true because on l_λ:

$$\frac{v'}{u'} = -\left(\frac{N-1}{x} + \frac{x}{2}\right) - \frac{f(u)}{v} = -\left(\frac{N-1}{x} + \frac{x}{2}\right) + \frac{\alpha}{2\lambda} - \frac{u^{p-1}}{\lambda}$$

$$< -\left(\frac{N-1}{x} + \frac{x}{2}\right) + \frac{\alpha}{2\lambda} < -\lambda$$

if $x > \xi_\lambda$, where $\xi_\lambda = 2\lambda + (\alpha/\lambda)$.

Lemma 3. *Given* $\xi, \varepsilon > 0$, *there exists a* $\delta > 0$ *such that if* $\gamma \in (A - \delta, A)$ *then* $(u(x, \gamma), v(x, \gamma)) \in \mathscr{A}_\varepsilon$ *for all* $x \in [0, \xi]$.

Proof. Note that $(u(x, A), v(x, A)) = (A, 0)$ for all $x \geqq 0$. Thus, the result follows from Lemma 1 and the continuous dependence on initial data.

Lemma 4. *Given* $\lambda > 0$, *there exists a* $\delta_\lambda > 0$ *such that if* $\gamma \in (A - \delta_\lambda, A)$, *then* $(u(x, \gamma), v(x, \gamma)) \in \mathscr{L}_\lambda$ *for all* $x \geqq 0$.

Proof. Choose ε so small that $\mathscr{A}_\varepsilon \subset \mathscr{L}_\lambda$. By Lemma 2, $\mathscr{L}_\lambda$ is positively invariant for $x \geqq \xi_\lambda$ and by Lemma 3 it is possible to choose a $\delta_\lambda > 0$ so that if $\gamma \in (A - \delta_\lambda, A)$, then $(u(x, \gamma), v(x, \gamma)) \in \mathscr{A}_\varepsilon \subset \mathscr{L}_\lambda$ for $0 \leqq x \leqq \xi_\lambda$. Thus for $\gamma \in (A - \delta_\lambda, A)$ the solution does not leave $\mathscr{L}_\lambda$ for any $x \geqq 0$.

Lemma 4 implies that if γ is sufficiently close to A, then $u(x, \gamma) > 0$ for all $x \geqq 0$. We shall see in the next lemma that this implies that $u(x, \gamma) \to 0$ as $x \to \infty$.

Lemma 5. *Suppose that for some* $\gamma \in (0, A)$, $u(x, \gamma) > 0$ *for all* $x \geqq 0$. *Then* $\lim\limits_{x \to \infty} u(x, \gamma) = 0$.

Proof. By Lemma 1, $u'(x,\gamma) = v(x,\gamma) < 0$, whence $u(x,\gamma)$ is decreasing and bounded below. Therefore

$$\bar{u} \overset{\text{def}}{=} \lim_{x\to\infty} u(x,\gamma) \quad \text{exists.}$$

Suppose $\bar{u} > 0$. Then

$$u'' + \left(\frac{N-1}{x} + \frac{x}{2}\right) u' = -f(u) \to -f(\bar{u}) < 0 \quad \text{as } x \to \infty.$$

Hence there exist numbers $\varepsilon > 0$ and $x_0 > 0$ such that

$$u'' + \left(\frac{N-1}{x} + \frac{x}{2}\right) u' < -\varepsilon \quad \text{for } x > x_0. \tag{3.1}$$

Write

$$a(x) = \frac{N-1}{x} + \frac{x}{2}$$

and

$$w(x) = \exp\left\{\int_{x_0}^{x} a(s)\, ds\right\}.$$

Then, setting $u(x) = u(x,\gamma)$, (3.1) can be written as

$$(w(x)\, u'(x))' < -\varepsilon w(x),$$

which yields, upon integration from x_0 to x

$$w(x)\, u'(x) < u'(x_0) - \varepsilon \int_{x_0}^{x} w(s)\, ds < -\varepsilon \int_{x_0}^{x} w(s)\, ds.$$

Hence

$$u'(x) < -\varepsilon \frac{\int_{x_0}^{x} w(s)\, ds}{w(s)}. \tag{3.2}$$

Note that

$$\begin{aligned}
\lim_{x\to\infty} \frac{\int_{x_0}^{x} w(s)\, ds}{\frac{1}{a(x)} w(x)} &= \lim_{x\to\infty} \frac{w(x)}{w(x) - \frac{a'(x)}{a^2(x)} w(x)} \\
&= \lim_{x\to\infty} \left\{1 - \frac{a'(x)}{a^2(x)}\right\}^{-1} \\
&= 1.
\end{aligned}$$

Using this in (3.2) we conclude that

$$u'(x) < -\frac{\varepsilon}{2a(x)} < -\frac{\varepsilon}{2x} \quad \text{for } x \text{ large},$$

which implies that $u(x) \to -\infty$ as $x \to \infty$. This contradicts the assumption that $u(x) > 0$ for all $x \geqq 0$.

It remains to prove that there exists a $\gamma \in (0, A)$ for which $u(x, \gamma) > 0$ for all $x \geqq 0$ and, *in addition*, $x^\alpha u(x, \gamma) \to 0$ as $x \to \infty$. Before doing this we investigate the behaviour of solutions near the origin in the phase plane.

4. Local Behaviour near (0, 0)

For convenience we write $(u(x, \gamma), v(x, \gamma)) = (u(x), v(x))$.

Lemma 6. *Assume that* $u(x) > 0$ *for all* $x \geqq 0$. *Then*

$$\lim_{x\to\infty} \frac{v(x)}{u(x)} \quad \textit{exists in } [-\infty, 0].$$

Proof. Lemma 1, together with the assumption that $u(x) > 0$ for all $x > 0$, implies that $v(x) < 0$ for all $x > 0$. Let

$$\begin{aligned} \limsup_{x\to\infty} \frac{v(x)}{u(x)} &= -\lambda_1, \\ \liminf_{x\to\infty} \frac{v(x)}{u(x)} &= -\lambda_2. \end{aligned} \tag{4.1}$$

Suppose that $\lambda_1 \neq \lambda_2$ and fix λ_3 so that $\lambda_1 < \lambda_3 < \lambda_2$. Then there exists a sequence (x_k) such that $x_k \to \infty$ as $k \to \infty$ and

$$\frac{v(x_k)}{u(x_k)} > -\lambda_3 \quad \text{for all } k \geqq 1.$$

Choose k^* so that $x_{k^*} \geqq \xi_{\lambda_3}$. Then $(u(x_{k^*}), v(x_{k^*})) \in \mathscr{L}_{\lambda_3}$, and hence, by Lemma 2, $(u(x), v(x)) \in \mathscr{L}_{\lambda_3}$ for all $x \geqq x_{k^*}$. This is impossible by (4.1) and the fact that $-\lambda_2 < -\lambda_3$.

Lemma 7. *Suppose* $u(x) > 0$ *for all* $x \geqq 0$. *Then*

$$\textit{either} \quad \lim_{x\to\infty} \frac{v(x)}{u(x)} = 0 \quad \textit{or} \quad \lim_{x\to\infty} \frac{v(x)}{u(x)} = -\infty.$$

Proof. Let us assume that

$$\lim_{x\to\infty} \frac{v(x)}{u(x)} \neq -\infty.$$

Then, by Lemma 6,

$$L \overset{\text{def}}{=} \lim_{x\to\infty} \frac{v(x)}{u(x)} \quad \text{exists in } (-\infty, 0].$$

By l'Hôpital's rule

$$\begin{aligned} L &= \lim_{x\to\infty} \frac{v'(x)}{u'(x)} \\ &= \lim_{x\to\infty} \frac{1}{v}\left\{-\left(\frac{N-1}{x}+\frac{x}{2}\right)v - f(u)\right\} \\ &= -\tfrac{1}{2}\lim_{x\to\infty} x - \frac{\alpha}{2}\lim_{x\to\infty}\frac{u(x)}{v(x)} + \lim_{x\to\infty}\frac{u(x)}{v(x)}\cdot u^{p-1}(x) \\ &= -\tfrac{1}{2}\lim_{x\to\infty} x - \frac{\alpha}{2L}. \end{aligned}$$

This is impossible unless $L = 0$.

5. Proof of Theorem 1

Define the set

$$S = \left\{\gamma \in (0, A) : u(x,\gamma) > 0 \qquad \text{for all } x > 0 \text{ and } \lim_{x\to\infty} \frac{v(x,\gamma)}{u(x,\gamma)} = 0\right\} \tag{5.1}$$

and let

$$\gamma_0 = \inf S. \tag{5.2}$$

We assert that the solution $u(x, \gamma_0)$ has the desired asymptotic behaviour as $x \to \infty$.

Lemma 8. *S is nonempty.*

Proof. By Lemma 4, if $\gamma \in (0, A)$ is sufficiently close to A, then $u(x,\gamma) > 0$ and $\lim_{x\to\infty} v(x,\gamma)/u(x,\gamma) \neq -\infty$. This means, by Lemma 7, that $\lim_{x\to\infty} v(x,\gamma)/u(x,\gamma) = 0$ and thus that $\gamma \in S$.

Lemma 9. *S is open.*

Proof. Let $\gamma^* \in S$. Then, given any $\lambda > 0$ there exists a $x_\lambda > 0$ such that

$$\frac{v(x,\gamma^*)}{u(x,\gamma^*)} > -\lambda \quad \text{for } x > x_\lambda.$$

Define $x^* = 2\max\{\xi_{2\lambda}, x_\lambda\}$ and

$$B_\varrho = \{(u,v) : \|(u,v) - u((x^*,\gamma^*), v(x^*,\gamma^*))\| < \varrho\}.$$

Choose ϱ so small that $B_\varrho \subset \mathscr{L}_{2\lambda}$. This is possible because $(u(x^*, \gamma^*), v(x^*, \gamma^*)) \subset \mathscr{L}_\lambda \subset \mathscr{L}_{2\lambda}$. By continuous dependence of the solution on initial data, there exists an $\varepsilon > 0$ such that

$$|\gamma - \gamma^*| < \varepsilon \Rightarrow (u(x^*, \gamma), v(x^*, \gamma)) \in B_\varrho .$$

Clearly, $u(x, \gamma) > 0$ when $|\gamma - \gamma^*| < \varepsilon$, because for $x \in [0, x^*]$, $u(x, \gamma) \geqq u(x^*, \gamma) > 0$ and for $x > x^*$, $u(x, \gamma) \in \mathscr{L}_{2\lambda}$.

Hence

$$\lim_{x\to\infty} \frac{v(x, \gamma)}{u(x, \gamma)} \geqq -2\lambda > -\infty$$

which implies, by Lemma 7, that

$$\lim_{x\to\infty} \frac{v(x, \gamma)}{u(x, \gamma)} = 0 .$$

Thus $(\gamma^* - \varepsilon, \gamma^* + \varepsilon) \subset S$.

Lemma 10. *If* $\alpha > N$, *then* $\gamma_0 > 0$.

Proof. Choose $\gamma \in (0, \gamma_1)$, where $\gamma_1 = \{(\alpha - N)/2\}^{1/(p-1)}$. We shall show that $u(x) = u(x, \gamma)$ vanishes for some $x > 0$.

Suppose to the contrary that $u(x) > 0$ for all $x \geqq 0$. Then

$$u'' + \left(\frac{N-1}{x} + \frac{x}{2}\right) u' + \frac{\alpha}{2} u - u^p = 0 \qquad \text{for } 0 \leqq x < \infty,$$

or, when we multiply by x^{N-1},

$$(x^{N-1}u')' + \tfrac{1}{2} x^N u' + \frac{\alpha}{2} x^{N-1} u - x^{N-1} u^p = 0 \qquad \text{for } 0 \leqq x < \infty .$$

If we integrate over $(0, x)$, and integrate by parts, we find that

$$x^{N-1}u'(x) + \tfrac{1}{2} x^N u(x) = \frac{N-\alpha}{2} \int_0^x s^{N-1} u(s)\, ds + \int_0^x s^{N-1} u^p(s)\, ds . \tag{5.3}$$

Since $u(x) > 0$ it follows from Lemma 1 that $u'(x) < 0$ for all $x > 0$. Therefore

$$\int_0^x s^{N-1} u^p(s)\, ds < \gamma^{p-1} \int_0^x s^{N-1} u(s)\, ds ,$$

and hence, by (5.3),

$$x^{N-1}u'(x) + \tfrac{1}{2} x^N u(x) < -\nu \int_0^x s^{N-1} u(s)\, ds , \tag{5.4}$$

where $\nu = \gamma_1^{p-1} - \gamma^{p-1} > 0$. Therefore

$$u'(x) + \tfrac{1}{2} x u(x) < 0 \qquad \text{for } x > 0$$

which implies that

$$u(x) < \gamma e^{-\frac{1}{4}x^2} \quad \text{for } x > 0. \tag{5.5}$$

Now let $x \to \infty$ in (5.4). Then, in view of (5.5),

$$\limsup_{x\to\infty} x^{N-1}u'(x) \leqq -\nu \int_0^\infty s^{N-1}u(s)\,ds \overset{\text{def}}{=} -2\delta.$$

Hence, there exists a number $R_0 > 0$ such that

$$x^{N-1}u'(x) < -\delta \quad \text{for } x > R_0,$$

or

$$u'(x) < -\delta x^{1-N} \quad \text{for } x > R_0. \tag{5.6}$$

Choose $R > R_0$, and integrate (5.6) over (x, R), $x > R_0$. If $N \neq 2$ this yields

$$u(x) > u(R) + \frac{\delta}{2-N}(R^{2-N} - x^{2-N}). \tag{5.7}$$

If $N < 2$, the right hand side of (5.7) becomes unbounded if $R \to \infty$, which is of course impossible, and if $N > 2$, (5.7) yields, in the limit as $R \to \infty$:

$$u(x) > \frac{\delta}{N-2} x^{2-N} \quad \text{for } x > R_0.$$

This is incompatible with the earlier estimate (5.5). Finally, if $N = 2$ we obtain, instead of (5.7),

$$u(x) > u(R) + \delta \log \frac{R}{x}$$

which yields a contradiction again when we let $R \to \infty$.

Remark. It follows from the proof of Lemma 10 that if $\alpha > N$, then

$$\gamma_0 \geqq \left(\frac{\alpha - N}{2}\right)^{\frac{1}{p-1}}.$$

In the last lemma of this section we shall show that the function $u(x, \gamma_0)$ is a positive solution of Problem A, which has the asymptotic behaviour, as $x \to \infty$, prescribed in (2.3). Thus, Lemma 11 completes the proof of Theorem 1.

Lemma 11. *The function $u(x, \gamma_0)$ has the properties*

(i) $u(x, \gamma_0) > 0$ *for all* $x \geqq 0$,

(ii) *for each* $\lambda > 0$ *there exists a constant* $C_\lambda > 0$ *such that*

$$u(x, \gamma_0) \leqq C_\lambda e^{-\lambda x} \quad \textit{for } x \geqq 0.$$

Proof. (i) Choose $\{\gamma_k\} \subset S$ such that $\gamma_0 = \lim_{k\to\infty} \gamma_k$. Then, for all $x \geqq 0$,

$$u(x, \gamma_0) = \lim_{k\to\infty} u(x, \gamma_k) \geqq 0. \tag{5.8}$$

Suppose that $u(x_0, \gamma_0) = 0$ for some $x_0 = 0$. Then by uniqueness $u'(x_0, \gamma_0) < 0$ whence there must exist a $\delta > 0$ such that

$$u(x, \gamma_0) < 0 \quad \text{for } x_0 < x < x_0 + \delta.$$

This is impossible because of (5.8).

(ii) Since $\gamma_0 \notin S$, it follows from Lemma 7 that

$$\lim_{x\to\infty} \frac{v(x, \gamma_0)}{u(x, \gamma_0)} = -\infty.$$

Hence, for any $\lambda > 0$ there exists a number $R_\lambda > 0$ such that

$$\frac{u'(x, \gamma_0)}{u(x, \gamma_0)} \leqq -\lambda \quad \text{for } x \geqq R_\lambda$$

and hence

$$u(x, \gamma_0) \leqq C_\lambda\, e^{-\lambda x} \quad \text{for } x \geqq R_\lambda$$

where $C_\lambda = u(R_\lambda, \gamma_0)\, e^{\lambda R_\lambda}$.

Corollary 12. $\lim\limits_{x\to\infty} x^\alpha u(x, \gamma_0) = 0.$

6. Behaviour as $x \to \infty$

Recall from Lemma 7 that if $u(x, \gamma) > 0$ for x large, then

$$\textit{either} \quad \lim_{x\to\infty} \frac{v(x, \gamma)}{u(x, \gamma)} = -\infty \quad \textit{or} \quad \lim_{x\to\infty} \frac{v(x, \gamma)}{u(x, \gamma)} = 0.$$

We shall say that $u(x, \gamma)$ is a *fast orbit* if the limit is $-\infty$. If the limit is zero we shall say that $u(x, \gamma)$ is a *slow orbit*. In this section we determine the asymptotic behaviour of both the fast and the slow orbits. This analysis will be used in the next section when we prove that the fast orbit is unique.

The estimates are all obtained by a very elementary method, the main tool being l'Hôpital's rule. The power of this method was first observed by Serrin [14].

1. *Fast orbits.* Let $(u(x), v(x)) = (u(x, \gamma), v(x, \gamma))$ where $u(x, \gamma)$ is a fast orbit.

Lemma 13. $\lim\limits_{x\to\infty} \dfrac{v(x)}{xu(x)} = -\tfrac{1}{2}.$

Proof. By Lemma 11, $xu(x) \to 0$ as $x \to \infty$, and

$$v' + \left(\frac{N-1}{x} + \frac{x}{2}\right) v = O(e^{-\lambda x})$$

for any $\lambda > 0$. Hence $v(x) \to 0$ as $x \to \infty$. Therefore, using l'Hôpital's rule and the fact that $u(x)/v(x) \to 0$ as $x \to \infty$ we obtain

$$\lim_{x\to\infty} \frac{v(x)}{xu(x)} = \lim_{x\to\infty} \frac{v'(x)}{xu'(x) + u(x)}$$

$$= \lim_{x\to\infty} \frac{-\left(\frac{N-1}{x} + \frac{x}{2}\right) v - \frac{\alpha}{2} u + u^p}{xv + u}$$

$$= \lim_{x\to\infty} \frac{-\frac{N-1}{x^2} - \frac{1}{2} - \frac{\alpha}{2x} \cdot \frac{u}{v} + u^{p-1} \frac{u}{v}}{1 + \frac{1}{x} \frac{u}{v}} = -\tfrac{1}{2}.$$

Define the function

$$E(x) = xv(x) + \tfrac{1}{2} x^2 u(x). \tag{6.1}$$

Lemma 14. $\lim_{x\to\infty} \frac{E(x)}{u(x)} = \alpha - N.$

Proof. Note that, by l'Hôpital's rule,

$$\lim_{x\to\infty} \frac{E(x)}{u(x)} = \lim_{x\to\infty} \frac{E'(x)}{v(x)},$$

if the second limit exists. However, an elementary computation shows that

$$\frac{E'}{v} = (2-N) + \frac{xu}{v}\left(-\frac{\alpha}{2} + 1 + u^{p-1}\right) \tag{6.2}$$

whence, by Lemma 13

$$\lim_{x\to\infty} \frac{E'(x)}{v(x)} = 2 - N - 2\left(-\frac{\alpha}{2} + 1\right) = \alpha - N.$$

Define the function

$$G(x) = x^2 E(x) - (\alpha - N)\, x^2 u. \tag{6.3}$$

Lemma 15. $\lim_{x\to\infty} \frac{G(x)}{u(x)} = 2(\alpha - N)(\alpha - 2).$

Proof. By Lemmas 11 and 13, $G(x) \to 0$ as $x \to \infty$. Thus by l'Hôpital's rule

$$\lim_{x\to\infty} \frac{G(x)}{u(x)} = \lim_{x\to\infty} \frac{G'(x)}{v(x)} \tag{6.4}$$

if the second limit exists. But

$$\frac{G'}{v} = \frac{2xu}{v}\left\{\frac{E}{u} - (\alpha - N)\right\} + x^2\left\{\frac{E'}{v} - (\alpha - N)\right\}, \tag{6.5}$$

whence, by Lemma 14,

$$\lim_{x\to\infty} \frac{G'(x)}{v(x)} = \lim_{x\to\infty} x^2\left\{\frac{E'}{v} - (\alpha - N)\right\} \tag{6.6}$$

if the limit exists. Remembering (6.2) we find

$$\begin{aligned}\frac{E'}{v} - (\alpha - N) &= 2 - \alpha + \left(1 - \frac{\alpha}{2}\right)\frac{xu}{v} + \frac{xu}{v}\cdot u^{p-1}\\ &= \left(1 - \frac{\alpha}{2}\right)\left(2 + \frac{xu}{v}\right) + \frac{xu}{v}u^{p-1}\\ &= (2 - \alpha)\left(\frac{v}{xu} + \tfrac{1}{2}\right)\cdot\frac{xu}{v} + \frac{xu}{v}u^{p-1}\\ &= \frac{2-\alpha}{x^2}\frac{E}{v}\cdot\frac{xu}{v} + \frac{xu}{v}u^{p-1}.\end{aligned} \tag{6.7}$$

Thus, by (6.4), (6.6) and (6.7), and Lemmas 13 and 14

$$\lim_{x\to\infty} \frac{G(x)}{u(x)} = (2 - \alpha)\lim_{x\to\infty} \frac{E}{v}\cdot\frac{xu}{v} = 2(\alpha - N)(\alpha - 2).$$

The following theorem describes the asymptotic behaviour of the fast orbit as $x \to \infty$.

Theorem 2. *Let u be a positive solution of Problem* A, *which corresponds to a fast orbit. Then there exists a constant $A > 0$ such that*

$$u(x) = Ae^{-\frac{1}{4}x^2}x^{\alpha - N}\left\{1 - (\alpha - N)(\alpha - 2)\frac{1}{x^2} + o\left(\frac{1}{x^2}\right)\right\} \quad \textit{as } x \to \infty.$$

Proof. By Lemma 15,

$$\frac{G(x)}{u(x)} = 2(\alpha - N)(\alpha - 2) + \varepsilon(x),$$

where $\varepsilon(x) \to 0$ as $x \to \infty$. Hence, recalling the definition (6.3) of G,

$$\frac{E(x)}{u(x)} = (\alpha - N) + 2(\alpha - N)(\alpha - 2)\frac{1}{x^2} + \frac{\varepsilon(x)}{x^2},$$

or, in view of the definition (6.1) of E:

$$\frac{u'(x)}{u(x)} = -\tfrac{1}{2}x + \frac{\alpha - N}{x} + 2(\alpha - N)(\alpha - 2)\frac{1}{x^3} + \frac{\varepsilon(x)}{x^3}.$$

Integration from $x = 1$ to $x > 1$ yields

$$\log \frac{u(x)}{u(1)} = -\tfrac{1}{4}(x^2 - 1) + \log x^{\alpha - N} - (\alpha - N)(\alpha - 2)\left(\frac{1}{x^2} - 1\right) + \int_1^x s^{-3}\varepsilon(s)\,ds,$$

whence

$$u(x) = Ae^{-\frac{1}{4}x^2} x^{\alpha - N}\left\{1 - (\alpha - N)(\alpha - 2)\frac{1}{x^2} + o\left(\frac{1}{x^2}\right)\right\} \quad \text{as } x \to \infty,$$

where

$$A = u(1)\exp\left\{\tfrac{1}{4} + (\alpha - N)(\alpha - 2) + \int_1^\infty s^{-3}\varepsilon(s)\,ds\right\}.$$

2. *Slow orbits.* Let $(u(x), v(x)) = (u(x, \gamma), v(x, \gamma))$, where $u(x, \gamma)$ is a slow orbit.

Lemma 16. $\lim\limits_{x \to \infty} \dfrac{xv(x)}{u(x)} = -\alpha.$

Proof. Set $z = v/u$. Then, using the differential equation (2.1) for u, we find that

$$z' + \tfrac{1}{2}xz = -\frac{\alpha}{2} + \varrho(x), \tag{6.8}$$

where

$$\varrho(x) = u^{p-1}(x) - \frac{N - 1}{x}z(x) - z^2(x).$$

Because u is a slow orbit, $z(x) \to 0$ as $x \to \infty$, and hence $\varrho(x) \to 0$ as $x \to \infty$.

Multiply (6.8) by $e^{\frac{1}{4}x^2}$ and integrate over $(0, x)$. This yields, after dividing by $e^{\frac{1}{4}x^2}$ again and remembering that $u(0) = \gamma > 0$ and $v(0) = u'(0) = 0$,

$$z(x) = -e^{-\frac{1}{4}x^2}\int_0^x \left\{\frac{\alpha}{2} - \varrho(s)\right\} e^{\frac{1}{4}s^2}\,ds. \tag{6.9}$$

By l'Hôpital's rule,

$$\lim_{x \to \infty} \frac{\int_0^x \left\{\frac{\alpha}{2} - \varrho(s)\right\} e^{\frac{1}{4}s^2}\,ds}{\frac{1}{x}e^{\frac{1}{4}x^2}} = \lim_{x \to \infty} \frac{\left\{\frac{\alpha}{2} - \varrho(x)\right\} e^{\frac{1}{4}x^2}}{\left(\frac{1}{2} - \frac{1}{x^2}\right) e^{\frac{1}{4}x^2}} = \alpha.$$

Using this in (6.9) we finally obtain

$$\lim_{x \to \infty} xz(x) = -\alpha.$$

Corollary 17. *For every* $\varepsilon > 0$ *there exists a constant* $K_\varepsilon > 0$ *such that*

$$u(x) \leqq K_\varepsilon x^{-(\alpha-\varepsilon)} \quad \textit{for } x \geqq 0.$$

Proof. By Lemma 16,

$$\frac{u'(x)}{u(x)} = -\frac{\alpha}{x}[1 + \varepsilon(x)],$$

where $\varepsilon(x) \to 0$ as $x \to \infty$. The result now follows upon integration.

Lemma 18. *Suppose that*

$$\lim_{x\to\infty} x^{\frac{2}{p-1}} u(x) = L \quad \textit{exists}. \tag{6.10}$$

Then

$$\lim_{x\to\infty} x^2\{xz(x) + \alpha\} = -2\alpha(\alpha - N + 2) + 2L^{p-1}.$$

Proof. Write, using (6.9),

$$x^2\{xz(x) + \alpha\} = \frac{-\int\limits_0^x \left\{\frac{\alpha}{2} - \varrho(s)\right\} e^{\frac{1}{4}s^2}\, ds + \frac{\alpha}{x} e^{\frac{1}{4}x^2}}{x^{-3}e^{\frac{1}{4}x^2}}.$$

Thus, by l'Hôpital's rule,

$$\lim_{x\to\infty} x^2\{xz(x) + \alpha\} = \lim_{x\to\infty} \frac{\varrho(x) - \dfrac{\alpha}{x^2}}{\dfrac{1}{2x^2} - \dfrac{3}{x^4}} = -2\alpha + 2\lim_{x\to\infty} x^2\varrho(x).$$

Recall that

$$\varrho(x) = u^{p-1}(x) - \frac{N-1}{x} z(x) - z^2(x).$$

Hence, by Lemma 16 and (6.10),

$$\lim_{x\to\infty} x^2\{xz(x) + \alpha\} = -2\alpha(\alpha - N + 2) + 2L^{p-1}.$$

We now consider two cases:

$$\text{I. } p > 1 + \frac{2}{\alpha} \quad \text{and} \quad \text{II. } 1 < p \leqq 1 + \frac{2}{\alpha}.$$

Case I. $p > 1 + \dfrac{2}{\alpha}$.

Lemma 19. *If* $p > 1 + \dfrac{2}{\alpha}$, *then*

$$\lim_{x\to\infty} x^2\{xz(x) + \alpha\} = -2\alpha(\alpha - N + 2). \tag{6.11}$$

Proof. By Corollary 17, given $\varepsilon > 0$ there exists a constant $K_\varepsilon > 0$ such that

$$x^2 u^{p-1}(x) \leqq K_\varepsilon^{p-1} x^{2-(p-1)(\alpha-\varepsilon)}.$$

Because $p > 1 + (2/\alpha)$ and thus $(p-1)\,\alpha > 2$, it is possible to choose $\varepsilon > 0$ so that $(p-1)(\alpha-\varepsilon) > 2$. Therefore

$$L^{p-1} = \lim_{x\to\infty} x^2 u^{p-1}(x) = 0.$$

The assertion now follows from Lemma 18.

In the following theorem we translate (6.11) in terms of the behaviour of $u(x)$ as $x \to \infty$.

Theorem 3. *Suppose that* $p > 1 + (2/\alpha)$. *Let* u *be a solution of Problem* A, *which corresponds to a slow orbit. Then there exists a constant* $A > 0$ *such that*

$$u(x) = Ax^{-\alpha}\left\{1 + \alpha(\alpha - N + 2)\cdot\frac{1}{x^2} + o\left(\frac{1}{x^2}\right)\right\} \quad \textit{as } x \to \infty.$$

Proof. By Lemma 19,

$$\frac{u'(x)}{u(x)} = -\frac{\alpha}{x} - \frac{2\alpha(\alpha - N + 2)}{x^3}[1 + \varepsilon(x)],$$

where $\varepsilon(x) \to 0$ as $x \to \infty$. The proof is completed as in the proof of Theorem 2.

Case II. $1 < p \leqq 1 + \dfrac{2}{\alpha}$. We begin with a preliminary estimate.

Lemma 20. *Suppose* $1 < p \leqq 1 + (2/\alpha)$. *Then for any* $k < \alpha(p-1)$

$$\lim_{x\to\infty} x^k\{xz(x) + \alpha\} = 0.$$

Proof. As in the proof of Lemma 18, we write

$$x^k\{xz(x) + \alpha\} = \frac{-\int\limits_0^x \left\{\frac{\alpha}{2} - \varrho(s)\right\} e^{\frac{1}{4}s^2}\, ds + \frac{\alpha}{x} e^{\frac{1}{4}x^2}}{x^{-1-k}\, e^{\frac{1}{4}x^2}}$$

and deduce by means of l'Hôpital's rule

$$\lim_{x\to\infty} x^k\{xz(x) + \alpha\} = \lim_{x\to\infty} \frac{\varrho(x) - \frac{\alpha}{x^2}}{\frac{1}{2}\, x^{-k} - (1+k)\, x^{-2-k}} = 2 \lim_{x\to\infty} x^k \varrho(x)$$

if the last limit exists.

By Lemma 16, $z(x) \sim -\alpha/x$ as $x \to \infty$, whence, since $k < 2$,

$$\lim_{x\to\infty} x^k z^2(x) = 0$$

and

$$\lim_{x\to\infty} x^k \left\{\frac{N-1}{x} z(x)\right\} = 0.$$

By assumption, $k < \alpha(p-1)$, so that it is possible to choose $\varepsilon > 0$ so small that $k < (\alpha - \varepsilon)(p-1)$, and hence, by Corollary 17,

$$x^k u^{p-1}(x) \leqq K_\varepsilon x^{k-(\alpha-\varepsilon)(p-1)} \quad \text{for all } x \geqq 0$$

for some constant $K_\varepsilon > 0$. Therefore,

$$\lim_{x\to\infty} x^k u^{p-1}(x) = 0,$$

as well, and we have shown that

$$\lim_{x\to\infty} x^k \varrho(x) = 0.$$

This completes the proof.

Lemma 21. *Suppose* $1 < p \leqq 1 + (2/\alpha)$ *and* $k < \alpha(p-1)$. *Then there exists a constant* $A > 0$ *such that*

$$u(x) = Ax^{-\alpha}[1 + o(x^{-k})] \quad \textit{as } x \to \infty.$$

Proof. By Lemma 20,

$$\frac{u'(x)}{u(x)} = -\frac{\alpha}{x} + \frac{\varepsilon(x)}{x^{k+1}},$$

where $\varepsilon(x) \to 0$ as $x \to \infty$. The result now follows upon integration.

Corollary 22. *If* $1 < p \leqq 1 + (2/\alpha)$, *then*

$$\lim_{x\to\infty} x^\alpha u(x) = A \quad \textit{exists}.$$

We are now ready to prove the desired asymptotic estimate when $1 < p \leqq 1 + (2/\alpha)$.

Theorem 4. *Suppose* $1 < p \leqq 1 + (2/\alpha)$. *Let u be a solution of Problem* A, *which corresponds to a slow orbit. Then there exists a constant* $A > 0$ *such that*

$$u(x) = Ax^{-\alpha}\left\{1 + \frac{B}{\alpha(p-1)} x^{-\alpha(p-1)} + o(x^{-\alpha(p-1)})\right\} \quad \textit{as } x \to \infty,$$

where

$$B = \begin{cases} 2\alpha(\alpha - N + 2) - 2A^{p-1} & \textit{if } p = 1 + \dfrac{2}{\alpha} \\[2ex] -2A^{p-1} & \textit{if } p < 1 + \dfrac{2}{\alpha}. \end{cases}$$

Proof. We proceed as in the proof of Lemma 20. Setting $k = \alpha(p-1)$ we find that

$$\begin{aligned}\lim_{x\to\infty} x^k\{xz(x)+\alpha\} &= 2\lim_{x\to\infty} x^k\varrho(x) - 2\alpha \lim_{x\to\infty} x^{k-2} \\ &= 2\lim_{x\to\infty} x^k u^{p-1}(x) - 2(N-1)\lim_{x\to\infty} x^{k-1}z(x) \\ &\quad - 2\lim_{x\to\infty} x^k z^2(x) - 2\alpha\lim_{x\to\infty} x^{k-2}\end{aligned}$$

Note that $p \le 1 + (2/\alpha)$ implies $k \le 2$. Thus, by Lemma 20, we have

$$\lim_{x\to\infty} xz(z) = -\alpha,$$

it follows that

$$\lim_{x\to\infty} x^k\{xz(x)+\alpha\} = 2A^{p-1} - 2\alpha(\alpha - N + 2) \quad \text{if } k = 2$$

and

$$\lim_{x\to\infty} x^k\{xz(x)+\alpha\} = 2A^{p-1} \quad \text{if } k < 2.$$

Thus

$$\frac{u'(x)}{u(x)} = -\frac{\alpha}{x} - \frac{B}{x^{k+1}}[1+\varepsilon(x)],$$

where $\varepsilon(x) \to 0$ as $x \to \infty$. Proceeding as in the proof of Theorem 2, we obtain the desired asymptotic estimate.

Remark 1. We see from Theorems 2, 3 and 4 that the first two terms in the asymptotic expansion of $u(x)$ for the fast orbit, and also for the slow orbit if $p > 1 + (2/\alpha)$, are not affected by the nonlinear term, but that for the slow orbit the second one is if $1 < p \leqq 1 + (2/\alpha)$.

Remark 2. It is interesting to observe that the asymptotic behaviour of solutions of the linear equation corresponding to (2.1),

$$u'' + \left(\frac{N-1}{x} + \frac{x}{2}\right)u' + \frac{\alpha}{2}u = 0 \tag{6.12}$$

can be obtained by transforming this equation to a standard form. We set

$$u(x) = x^{-\frac{N}{2}} e^{-\frac{x^2}{8}} y(t), \quad \text{and} \quad x^2 = 4t.$$

Then, in terms of the new variables we obtain

$$y'' + \left(-\tfrac{1}{4} + \frac{\varkappa}{t} + \frac{\frac{1}{4}-\mu^2}{t^2}\right)y = 0 \tag{6.13}$$

in which

$$\varkappa = \tfrac{1}{4}(2\alpha - N), \quad \mu = \tfrac{1}{4}(N-2).$$

Equation (6.13) is Whittaker's equation [1, p. 505]. One solution is given by the function

$$y(t) = e^{-\frac{1}{2}t}\, t^{\frac{1}{2}+\mu} U(a, b, t), \tag{6.14}$$

where

$$a = \tfrac{1}{2} + \mu - \varkappa = \frac{N - \alpha}{2}, \quad b = 1 + 2\mu = \frac{N}{2}$$

and

$$U(a, b, t) = \mathscr{C}_1 t^{-a} \left\{ 1 - a(1 + a - b) \frac{1}{t} + O(t^{-2}) \right\} \quad \text{as } t \to \infty.$$

The other solution is given by

$$y(t) = e^{-\frac{1}{2}t}\, t^{\frac{1}{2}+\mu}\, M(a, b, t) \tag{6.15}$$

in which

$$M(a, b, t) = \mathscr{C}_2 e^t t^{a-b} \left\{ 1 + (b - a)(1 - a) \frac{1}{t} + O(t^{-2}) \right\} \quad \text{as } t \to \infty.$$

Here $\mathscr{C}_1$ and $\mathscr{C}_2$ are appropriate constants.

Returning to the original variables u and x, we find that the first solution (6.14) yields the asymptotic expansion for the *fast* orbit corresponding to the one given in Theorem 2, and the second solution (6.15), the asymptotic expansion for the *slow* orbit, corresponding to the one derived in Theorem 3 $(p > 1 + (2/\alpha))$.

7. Uniqueness

In this section we shall prove that the solution u of Problem A whose existence was asserted in Theorem 1 is also unique.

Theorem 5. *Suppose* $p > 1$ *and* $\alpha > N$. *Then there exists at most one solution of Problem* A *which has the property*

$$\lim_{x \to \infty} x^\alpha u(x) = 0.$$

In other words we shall prove that *the fast orbit is unique.*

Proof. We exploit the concavity of the function

$$f(u) = \frac{\alpha}{2} u - u^p.$$

It is well known that in many instances the concavity of the nonlinearity implies the uniqueness of positive solutions. (See for instance Krasnoselski [12] chapters 6 and 7 and Berestycki [2]).

Here we use a device inspired by an argument of BENGURIA, BREZIS and LIEB ([3] Lemmas 4 and 11). See also [21].

Suppose that u_1 and u_2 are positive solutions of Problem A and

$$\lim_{x \to \infty} x^\alpha u_i(x) = 0 \qquad i = 1, 2.$$

We know by Corollary 17, Theorem 2 and Lemma 13 that

$$u_i(x) = e^{-x^2/4} x^{\alpha - N}[1 + O(x^{-2})] \quad \text{as } x \to \infty$$

and

$$\frac{u_i'(x)}{u_i(x)} = x[-\tfrac{1}{2} + o(1)] \quad \text{as } x \to \infty.$$

Set $K(x) = x^{N-1} e^{x^2/4}$. If we multiply the equation for u_i by K/u_i and subtract the resulting expressions, we obtain

$$\frac{(Ku_1')'}{u_1} - \frac{(Ku_2')'}{u_2} + K\left\{\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2}\right\} = 0.$$

Now multiply this equation by $u_1^2 - u_2^2$ and integrate over $(0, R)$. This yields

$$K(R)\left\{\frac{u_1'(R)}{u_1(R)} - \frac{u_2'(R)}{u_2(R)}\right\}\{u_1^2(R) - u_2^2(R)\} + \int_0^R \left\{\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2}\right\}(u_1^2 - u_2^2)\, K\, dx$$
$$= \int_0^R \left\{u_1'\left(u_1' - \frac{2u_2u_2'}{u_1} + \frac{u_2^2 u_1'}{u_1^2}\right) + u_2'\left(u_2' - \frac{2u_1u_1'}{u_2} + \frac{u_1^2u_2'}{u_2^2}\right)\right\} K\, dx.$$

The integral on the right hand side may be written as

$$\int_0^R \left\{\left(u_1' - \frac{u_1u_2'}{u_2}\right)^2 + \left(u_2' - \frac{u_2u_1'}{u_1}\right)^2\right\} K\, dx \geqq 0.$$

Therefore we obtain

$$\int_0^R \left\{\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2}\right\}(u_1^2 - u_2^2)\, K\, dx \geqq o(1) \qquad \text{as } R \to \infty.$$

On the other hand the function $u \to f(u)/u$ is decreasing on $(0, \infty)$ and thus

$$\left\{\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2}\right\}(u_1^2 - u_2^2) < 0$$

almost everywhere on the set $\{x > 0 \colon u_1(x) \neq u_2(x)\}$. As we let R tend to infinity we see that $u_1 = u_2$.

Appendix

Here we give an estimate for the difference between the solution u of the problem

$$u_t - \Delta u + u^p = 0 \quad \text{on } \mathbb{R}^N \times (0, \infty)$$

$$u > 0 \quad \text{on } \mathbb{R}^N \times (0, \infty)$$

$$u(x, 0) = a\delta(x) \quad \text{on } \mathbb{R}^N$$

in which $a \in \mathbb{R}^+$, and the function aE, where E is the fundamental solution of the heat operator $(\partial/\partial t) - \Delta$:

$$E(x, t) = \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x|^2}{4t}}.$$

Proposition. *Suppose* $1 < p < 1 + (2/N)$ *and* $a > 0$. *Then there exists a constant* C, *which only depends on* p *and* N, *such that*

$$|u(x, t) - aE(x, t)| \leqq Ca^p t^\nu E(x, t) \quad \textit{for } x \in \mathbb{R}^N,\ t > 0$$

in which

$$\nu = \frac{N}{2}\left(\frac{N+2}{N} - p\right) > 0.$$

Proof. By the maximum principle we have

$$u(x, t) \leqq aE(x, t) \quad \text{for } x \in \mathbb{R}^N,\ t > 0. \tag{A1}$$

On the other hand,

$$u(\cdot, t) = S(t)(a\delta) - \int_0^t S(t - s)\, u^p(\cdot, s)\, ds,$$

where $S(t)$ denotes the semigroup generated by Δ, that is $S(t)\phi = E(\cdot, t) * \phi$, where $*$ denotes the convolution product and ϕ is some initial function.

We deduce from (A1) that

$$u^p(x, t) \leqq a^p E^p(x, t) = C_1 \frac{a^p}{t^{N(p-1)/2}} E\left(x, \frac{t}{p}\right),$$

where $C_1 = (4\pi)^{-N(p-1)/2} p^{-N/2}$. Hence, using the semigroup property, we obtain

$$\begin{aligned} |u(x, t) - aE(x, t)| &\leqq C_1 a^p \int_0^t S(t - s)\, S\left(\frac{s}{p}\right) \delta(x) \frac{ds}{s^{N(p-1)/2}} \\ &= C_1 a^p \int_0^t S\left(t - \frac{s}{p'}\right) \delta(x) \frac{ds}{s^{N(p-1)/2}}, \end{aligned} \tag{A2}$$

where p' is defined by $(1/p') + (1/p) = 1$. But

$$S\left(t - \frac{s}{p'}\right)\delta(x) = \left\{4\pi\left(t - \frac{s}{p'}\right)\right\}^{-\frac{N}{2}} e^{-|x|^2/4\left(t-\frac{s}{p'}\right)} \leqq \left(\frac{4\pi}{p}t\right)^{-N/2} e^{-\frac{|x|^2}{4t}}$$

when $0 \leqq s < t$. Using this in (A2) we find that

$$|u(x, t) - aE(x, t)| \leqq C_1 a^p \left(\frac{4\pi}{p}t\right)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}} \int\limits_0^t \frac{ds}{s^{N(p-1)/2}}$$

in which the integral exists because $p < 1 + (2/N)$. Thus

$$|u(x, t) - aE(x, t)| \leqq C a^p t^\nu E(x, t),$$

where ν is given in the Proposition and $C = (4\pi)^{-N(p-1)/2}\nu^{-1}$.

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