MATHEMATICAL ANALYSIS AND APPLICATIONS, PART A.
ADVANCES IN MATHEMATICS SUPPLEMENTARY STUDIES, VOL. 7A

A Note on Isolated Singularities for Linear Elliptic Equations

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Set $B_R = \{x \in \mathbb{R}^N; |x| < R\}$ with $N \geq 2$. Our main result is the following:

THEOREM 1. *Let $u \in L^1_{\text{loc}}(B_R\backslash\{0\})$ be such that*

$$\Delta u \in L^1_{\text{loc}}(B_R\backslash\{0\}) \quad \text{in the sense of distributions on } B_R\backslash\{0\}, \tag{1}$$

$$u \geq 0 \qquad \text{a.e. in } B_R, \tag{2}$$

$$\Delta u(x) \leq a u(x) + f(x) \quad \text{a.e. in } B_R, \tag{3}$$

where a is a positive constant and $f \in L^1_{\text{loc}}(B_R)$. Then $u \in L^1_{\text{loc}}(B_R)$ and there exists $\varphi \in L^1_{\text{loc}}(B_R)$ and $\alpha \geq 0$ such that

$$-\Delta u = \varphi + \alpha\delta_0 \quad \text{in } \mathscr{D}'(B_R), \tag{4}$$

where δ_0 is the Dirac mass at the origin. In particular, it follows that $u \in M^p_{\text{loc}}(B_R)$[†] where $p = N/(N-2)$ when $N > 2$ and $p < \infty$ is arbitrary when $N = 2$.

Remarks. (1) Theorem 1 turns out to be useful in the study of isolated singularities in semilinear elliptic equations. For example, it implies the following fact—which extends a result of [1]. Assume f is a continuous function from $\mathbb{R}_+$ into $\mathbb{R}$ such that $\liminf_{t\to+\infty} f(t)/t > -\infty$; assume $u \in L^1_{\text{loc}}(B_R\backslash\{0\})$, $f(u) \in L^1_{\text{loc}}(B_R\backslash\{0\})$, $u \geq 0$ a.e. and $-\Delta u = f(u)$ in $\mathscr{D}'(B_R\backslash\{0\})$. Then $f(u) \in L^1_{\text{loc}}(B_R)$, and $-\Delta u = f(u) + \alpha\delta_0$ in $\mathscr{D}'(B_R)$ for some $\alpha \geq 0$.

(2) Suppose $N \geq 3$; if instead of (2), we assume that

$$u \geq g \quad \text{a.e. in } B_R \qquad \text{with} \quad g \in M^{N/N-2}_{\text{loc}}(B_R). \tag{2'}$$

[†] M^p denotes the weak L^p (or Marcinkiewicz) space for $1 < p < \infty$.

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ISBN 0-12-512801-0

Then (4) holds with some $\alpha \in \mathbb{R}$. If in addition $g \in L^{N/(N-2)}_{\text{loc}}(B_R)$, then $\alpha \geq 0$. The proof is an easy modification of the proof of Theorem 1, and we shall omit it.

The proof of Theorem 1 is divided into three steps.

Step 1. Let $\bar{u}(r)$ be the average of u over the sphere $|x| = r$, i.e.,

$$u(r) = \frac{1}{s_N r^{N-1}} \int_{|x|=r} u(x)\, ds,$$

where s_N denotes the measure of the unit sphere in $\mathbb{R}^N$. It follows from (1), (2), and (3) that

$$\frac{1}{r^{N-1}} (r^{N-1}\bar{u}_r)_r \in L^1_{\text{loc}}(0, R), \qquad \bar{u} \geq 0,$$

and

$$\frac{1}{r^{N-1}} (r^{N-1}\bar{u}_r)_r \leq a\bar{u} + \bar{f} \quad \text{on } (0, R). \tag{5}$$

In particular $\bar{u} \in C^1(0, R)$; let $R' < R$ be fixed. Integrating (5) over (r, R') we find

$$-r^{N-1}\bar{u}_r(r) \leq a\psi(r) + C \qquad \text{for} \quad r \in (0, R'), \tag{6}$$

where $\psi(r) = \int_r^{R'} s^{N-1}\bar{u}(s)\, ds$ is a nonincreasing function and C is a constant.

Dividing (6) through by r^{N-1} and integrating we obtain

$$\bar{u}(r) \leq a \int_r^{R'} \frac{\psi(s)}{s^{N-1}}\, ds + \frac{C}{r^{N-2}} + C \tag{7}$$

(when $N = 2$, C/r^{N-2} must be replaced by $C|\log r|$). In particular we have

$$r^{N-1}\bar{u}(r) \leq a \int_r^{R'} \psi(s)\, ds \leq aR'\psi(r),$$

and integrating again we see that

$$\psi(r) \leq a(R')^2\psi(r) + C \qquad \text{for} \quad 0 < r < R'.$$

Choosing R' small enough we obtain that $\psi(r)$ remains bounded as $r \to 0$ and consequently $u \in L^1_{\text{loc}}(B_R)$. Going back to (7) we also find that

$$\begin{aligned} &\bar{u}(r) \leq \frac{C}{r^{N-2}} + C \qquad \text{if} \quad N \geq 3, \\ &\bar{u}(r) \leq C|\log r| + C \qquad \text{if} \quad N = 2. \end{aligned} \tag{8}$$

Step 2. We define a measurable function φ by

$$\varphi(x) = -\Delta u(x) \quad \text{a.e. in } B_R.$$

By (1) we know that $\varphi \in L^1_{\mathrm{loc}}(B_R\backslash\{0\})$; we shall now prove that in fact $\varphi \in L^1_{\mathrm{loc}}(B_R)$. Consider functions $\zeta_\varepsilon(x)$ satisfying the following properties:

$$\zeta_\varepsilon \in C^\infty(B_R), \qquad 0 \le \zeta_\varepsilon \le 1 \quad \text{on } B_R, \tag{9}$$

$$\zeta_\varepsilon(x) \to 1 \quad \text{as } \varepsilon \to 0 \quad \text{for every} \quad x \neq 0$$

and

$$\nabla\zeta_\varepsilon \to 0 \quad \text{as } \varepsilon \to 0, \quad \text{uniformly on compact subsets of } B_R\backslash\{0\}, \tag{10}$$

$$\zeta_\varepsilon(x) = 0 \qquad \text{for} \quad |x| < \varepsilon, \tag{11}$$

$$\Delta\zeta_\varepsilon \ge 0 \quad \text{on } B_R. \tag{12}$$

To see that such functions exist it suffices to choose ζ_ε of the form

$$\zeta_\varepsilon(x) = \begin{cases} \Phi((\varepsilon/|x|)^{N-2}) & \text{when} \quad N \ge 3, \\ \Phi(\log|x|/\log\varepsilon) & \text{when} \quad N = 2, \end{cases}$$

where Φ is a C^∞ convex function on $[0, \infty)$ such that $\Phi(0) = 1$, $\Phi(t) = 0$ for $t \ge 1$. Let $\eta \in \mathcal{D}(B_R)$, $0 \le \eta \le 1$, $\eta(x) = 1$ near $x = 0$. We have

$$\begin{aligned} \int_{B_R} \varphi\zeta_\varepsilon\eta &= -\int_{B_R} u\Delta(\zeta_\varepsilon\eta) \\ &= -\int_{B_R} u(\Delta\zeta_\varepsilon)\eta - 2\int_{B_R} u\nabla\zeta_\varepsilon\nabla\eta - \int_{B_R} u\zeta_\varepsilon\Delta\eta \\ &\le -2\int_{B_R} u\nabla\zeta_\varepsilon\nabla\eta - \int_{B_R} u\zeta_\varepsilon\Delta\eta \quad \text{by (2) and (12).} \end{aligned}$$

From Fatou's lemma—which is valid since $\varphi \ge -au - f \in L^1_{\mathrm{loc}}(B_R)$—we deduce as $\varepsilon \to 0$ that $\varphi \in L^1_{\mathrm{loc}}(B_R)$ and in addition

$$\int_{B_R} \varphi\eta \le -\int_{B_R} u\Delta\eta. \tag{13}$$

Step 3. Consider now the distribution

$$T = -\Delta u - \varphi \in \mathcal{D}'(B_R).$$

The support of T is contained in $\{0\}$; thus by a classical result about distributions (see [2, Theorem XXXV]) we know that

$$T = \sum_{|p| \le m} c_p D^p\delta_0. \tag{14}$$

We claim that $c_p = 0$ when $|p| \ge 1$. Indeed let $\zeta \in \mathcal{D}(B_R)$ be any fixed function such that

$$(-1)^{|p|}(D^p\zeta)(0) = c_p \qquad \text{for every} \quad |p| \le m.$$

Multiplying (14) by $\zeta_\varepsilon(x) = \zeta(x/\varepsilon)$ we obtain

$$-\int_{B_R} u\,\Delta\zeta_\varepsilon = \int_{B_R} \varphi\zeta_\varepsilon + \sum_{|p|\le m} \frac{c_p^2}{\varepsilon^{|p|}}. \tag{15}$$

On the other hand we have

$$\int_{B_R} u\,\Delta\zeta_\varepsilon = \frac{1}{\varepsilon^2}\int_{|x|<R\varepsilon} u(x)(\Delta\zeta)\left(\frac{x}{\varepsilon}\right)$$

and therefore

$$\left|\int_{B_R} u\,\Delta\zeta_\varepsilon\right| \le \frac{C}{\varepsilon^2}\int_{|x|<R\varepsilon} u = \frac{C}{\varepsilon^2}\int_0^{R\varepsilon} \bar{u}(r)r^{N-1}\,dr.$$

We deduce from (8) that

$$\begin{aligned} &\left|\int_{B_R} u\,\Delta\zeta_\varepsilon\right| \le C && \text{when} \quad N \ge 3, \\ &\left|\int_{B_R} u\,\Delta\zeta_\varepsilon\right| \le C|\log\varepsilon| + C && \text{when} \quad N = 2. \end{aligned} \tag{16}$$

Comparing (15) and (16) we conclude that $c_p = 0$ when $|p| \ge 1$. In conclusion we have proved that

$$-\Delta u = \varphi + c_0\delta_0 \quad \text{in } \mathscr{D}'(B_R). \tag{17}$$

Finally we choose $\eta \in \mathscr{D}(B_R)$, $0 \le \eta \le 1$, and $\eta(x) = 1$ near $x = 0$. By (13) we have

$$\int_{B_R} \varphi\eta \le -\int_{B_R} u\,\Delta\eta$$

while by (17) we have

$$\int_{B_R} \varphi\eta = -\int_{B_R} u\,\Delta\eta - c_0.$$

It follows that $c_0 \ge 0$.

REFERENCES

1. P. L. LIONS, Isolated singularities in semilinear problems, *J. Differential Equations* (1981), to appear.
2. L. SCHWARTZ, "Theorie des distributions," Hermann, Paris, 1966.