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Periodic solutions of a nonlinear wave equation

Abbas Bahri and **Haïm Brezis**
Analyse Numerique, Université Paris VI, Tour 55, 4 pl. Jussieu,
75230 Paris Cedex 05

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SYNOPSIS

We provide a sufficient and "almost" necessary condition for the existence of a periodic solution of the equation

$$u_{tt} - u_{xx} + F(x, t, u) = 0$$

where F is nondecreasing in u and has a small linear growth as $|u| \to \infty$.

1. INTRODUCTION

Consider the nonlinear wave equation

$$u_{tt} - u_{xx} \pm F(x, t, u) = 0 \quad 0 < x < \pi, \quad t \in \mathbb{R} \tag{1}$$

under the boundary condition

$$u(0, t) = u(\pi, t) = 0 \quad t \in \mathbb{R}. \tag{2}$$

We seek solutions which are 2π-periodic in time. Set $\Omega = (0, \pi) \times (0, 2\pi)$. We make the following assumptions on F:

$F(x, t, u)$ is measurable in (x, t) for fixed u and continuous nondecreasing in u almost every (x, t); F is 2π-periodic in t. (3)

$|F(x, t, u)| \leqq \gamma |u| + C$ for almost every (x, t), for every $u \in \mathbb{R}$ where C and γ are constants such that $\gamma < 3$ or $\gamma < 1$ according to whether we have $+$ or $-$ in (1). (4)

The generalized nullspace of the operator $A = \dfrac{\partial^2}{\partial t^2} - \dfrac{\partial^2}{\partial x^2}$ acting on functions satisfying (2) and 2π-periodic in t consists of the functions ψ of the form

$$N = \{\psi(x, t) = p(t + x) - p(t - x);\ p \in L^2_{\mathrm{loc}}(\mathbb{R}),\ p \text{ is } 2\pi\text{-periodic}\}.$$

Finally assume there exists $\varphi \in L^\infty(\Omega)$ and constants M, $\delta > 0$ such that

$$\varphi \in N^\perp, \quad \text{i.e.} \quad \int_\Omega \varphi\psi \, dx \, dt = 0 \quad \text{for every} \quad \psi \in N \tag{5}$$

$$\varphi(x, t) + F(x, t, M) \geqq \delta, \quad \varphi(x, t) + F(x, t, -M) \leqq -\delta \quad \text{almost everywhere on } \Omega. \tag{6}$$

Our main result is the following:

THEOREM 1. *Assume* (3), (4), (5), (6). *Then there exists a* (*weak*) *solution* $u \in L^\infty$ *of* (1), (2).

Remarks. 1. Theorem 1 is a generalization of one of the results in [**2**]. In [**2**] the assumptions are (3), (4) and $|F(x, t, u)| \to \infty$ as $|u| \to \infty$ for every (x, t); thus (5), (6) are satisfied with $\varphi = 0$. The proof of Theorem 1 relies heavily on some of the techniques introduced in [**1, 2**].

2. Note that (5)–(6) is "almost" necessary for the solvability of (1), (2). Indeed if (1), (2) has a solution $u \in L^\infty$, set $\varphi(x, t) = -F(x, t, u)$, $M > \|u\|_{L^\infty}$.

Then we have (5) (since $\varphi = Au$) and almost everywhere on Ω.

$$\varphi(x, t) + F(x, t, M) \geqq 0, \quad \varphi(x, t) + F(x, t, -M) \leqq 0. \tag{6'}$$

In fact when $F(x, t, u)$ is continuous on $\bar{\Omega} \times \mathbb{R}$ and strictly increasing in u for each $(x, t) \in \bar{\Omega}$, then (5) (6) is a necessary and sufficient condition for solvability.

3. Assume now $F(x, t, u)$ has the special form $F(x, t, u) = g(u) - f(x, t)$ where f is continuous and nondecreasing in u, and $f \in L^\infty$. Then (1) (2) has a solution provided

$$|g(u)| \leqq \gamma |u| + C \quad \text{with} \quad \gamma < 3 \quad \text{or} \quad \gamma < 1, \tag{7}$$

f can be split as a sum

$$f(x, t) = f'(x, t) + f''(x, t)$$

with

$$f' \in L^\infty \quad \int_\Omega f' \psi \, dx \, dt = 0 \quad \text{for every} \quad \psi \in N, \tag{8}$$

$$g(-\infty) < \inf_\Omega f'' \leqq \sup_\Omega f'' < g(+\infty). \tag{9}$$

(Choose $\varphi = f'$ and M large enough so that $g(M) > \operatorname{Sup}_\Omega f''$, $g(-M) < \inf_\Omega f''$).

Note that (8) (9) is again an "almost" necessary and sufficient condition for solvability.

4. Assumption (5)–(6) is equivalent to

$$F(x, t, M) - F(x, t, -M) \geqq 2\delta \quad \text{almost everywhere on } \Omega \tag{10}$$

$$\int_\Omega [F(x, t, M)\psi^+ - F(x, t, -M)\psi^-] \, dx \, dt \geqq \delta \, \|\psi\|_{L^1} \quad \text{for every} \quad \psi \in N. \tag{11}$$

It is clear that (5), (6) implies (10), (11). Conversely (11) can be written as

$$-\int_\Omega [F(x, t, M) + F(x, t, -M)]\psi \, dx \, dt \leqq \int_\Omega [F(x, t, M) - F(x, t, -M) - 2\delta] \, |\psi| \, dx \, dt$$

for every $\psi \in N$.

We deduce from the Hahn-Banach theorem that there exists $h \in L^\infty(\Omega)$ such that

$$\int_\Omega h\psi \, dx \, dt \leqq \int [F(x, t, M) - F(x, t, -M) - 2\delta] \, |\psi| \, dx \, dt \quad \text{for every} \quad \psi \in L^1(\Omega) \tag{12}$$

and

$$-\int_\Omega [F(x,t,M)+F(x,t,-M)]\psi\,dx\,dt=\int_\Omega h\psi\,dx\,dt \quad \text{for every} \quad \psi\in N. \tag{13}$$

By (12) we have

$$F(x,t,M)-F(x,t,-M)-2\delta\geqq\pm h$$

and we obtain (5), (6) by choosing

$$\varphi(x,t)=-\tfrac{1}{2}[h(x,t)+F(x,t,M)+F(x,t,-M)].$$

Assumption (11) is of the same nature as the assumption introduced by Landesman and Lazer [**4**] in order to solve some nonlinear elliptic at resonance. Our equation has similar features, except for the fact that the nullspace is infinite dimensional – this leads to some major new difficulties. Nonlinear wave equations at resonance have also been considered in [**6**] and [**7**].

2. Proof of Theorem 1

We shall suppose that we have the + sign in (1) (the argument is essentially the same for the − sign).

We first recall some well known facts concerning the operator A under (2) (see [**3, 5, 8**]).

For $u\in L^2(\Omega)$ we set

$$u=u_1+u_2 \quad \text{with} \quad u_1\in N^\perp, \quad u_2\in N.$$

We have $R(A)=N^\perp$ and

$$\|u_1\|_{H^1}\leqq C\,\|Au\|_{L^2} \quad \text{for every} \quad u\in D(A) \tag{14}$$

$$\|u_1\|_{L^\infty}=\leqq C\,\|Au\|_{L^2} \quad \text{for every} \quad u\in D(A). \tag{15}$$

Given a function $u(x,t)$ which is 2π-periodic in t we set

$$(Qu)(t)=\frac{1}{2\pi}\int_0^\pi [u(x,t-x)-u(x,t+x)]\,dx. \tag{16}$$

Clearly

$$\|Qu\|_{L^1(0,2\pi)}\leqq\frac{1}{\pi}\|u\|_{L^1(\Omega)}. \tag{17}$$

Also

$$Qu=0 \quad \text{for every} \quad u\in N^\perp; \tag{18}$$

indeed let $p\in L^2_{\text{loc}}(\mathrm{R})$ be 2π-periodic in t: we have

$$\int_0^{2\pi}(Qu)(t)p(t)\,dt=\frac{1}{2\pi}\int_\Omega u(x,t)[p(t+x)-p(t-x)]\,dx\,dt=0.$$

Any $\psi \in N$ can be written in a unique way as

$$\psi(x, t) = p(t+x) - p(t-x) \quad \text{with} \quad \int_0^{2\pi} p(t)\, dt = 0$$

and in fact $p = Q\psi$.

Finally, recall that

$$(Au, u) \geqq -\tfrac{1}{3} \|Au\|_{L^2}^2 \quad \text{for every} \quad u \in D(A). \tag{19}$$

We shall prove Theorem 1 by considering for $\varepsilon > 0$ small, the equation

$$\varepsilon u_\varepsilon + Au_\varepsilon + F(x, t, u_\varepsilon) = 0 \tag{20}$$

and then letting $\varepsilon \to 0$.

According to [1 Th. 1.8], there exists a solution $u_\varepsilon \in L^2$ of (20) provided $\varepsilon + \gamma < 3$.

We shall divide the proof into four steps:

Step 1. $\|Au_\varepsilon\|_{L^2} \leqq C$,

Step 2. $\|u_\varepsilon\|_{L^1} \leqq C$,

Step 3. $\|u_\varepsilon\|_{L^\infty} \leqq C$,

Step 4. Passage to the limit as $\varepsilon \to 0$,

where C denotes various constants independent of ε.

Step 1. $\|Au_\varepsilon\|_{L^2} \leqq C$.

For simplicity we drop ε. It follows from (6) that for some function

$$w \in L^\infty(\Omega)(\|w\|_{L^\infty} \leqq M)$$

we have

$$\varphi(x, t) + F(x, t, w(x, t)) = 0.$$

Since $\varphi \in N^\perp$, we have $\varphi = Av$ for some $v \in L^\infty$. Therefore we obtain

$$\varepsilon u + A(u - v) + F(x, t, u) - F(x, t, w) = 0.$$

Taking the L^2 scalar product with $u - v$ and using (19) we find

$$\varepsilon \int_\Omega u(u-v)\, dx\, dt + \int_\Omega (F(x, t, u) - F(x, t, w))(u - v)\, dx\, dt \leqq \tfrac{1}{3} \|Au - Av\|_{L^2}^2.$$

Hence

$$\varepsilon \int_\Omega u^2\, dxdt + \int_\Omega |F(x, t, u) - F(x, t, w)|\, |u - w|\, dx\, dt$$
$$\leqq \varepsilon \int uv\, dx\, dt + \tfrac{1}{3} \|Au - Av\|_{L^2}^2 + \|F(x, t, u) - F(x, t, w)\|_{L^2} \|v - w\|_{L^2}$$

and consequently

$$\frac{\varepsilon}{2}\int_\Omega u^2\,dx\,dt+\int_\Omega |F(x,t,u)-F(x,t,w)|\,|u|\,dx\,dt$$
$$\leqq \tfrac{1}{3}\|Au-Av\|_{L^2}^2+C\,\|F(x,t,u)\|_{L^2}+C.$$

Using now (4) and (20) we see that

$$\frac{\varepsilon}{2}\int_\Omega u^2\,dx\,dt+\frac{1}{\gamma}\int_\Omega |F(x,t,u)|^2\,dx\,dt$$
$$\leqq \tfrac{1}{3}\|F(x,t,u)+\varepsilon u+Av\|_{L^2}^2+C\,\|F(x,t,u)\|_{L^2}+C.$$

It follows easily that

$$\|F(x,t,u)\|_{L^2}\leqq C$$
$$\varepsilon\|u\|_{L^2}^2\leqq C$$
$$\|Au\|\leqq C$$

Step 2. $\|u_\varepsilon\|_{L^1}\leqq C$

From (6) we deduce that for any $h\in L^\infty(\Omega)$ with $\|h\|_{L^\infty}\leqq\delta$, there exists some function $w_h(x,t)\in L^\infty$ with $\|w_h\|_{L^\infty}\leqq M$ such that

$$\varphi(x,t)+F(x,t,w_h(x,t))=h(x,t).$$

hence

$$\varepsilon u+A(u-v)+F(x,t,u)-F(x,t,w_h)=-h.$$

Taking the L^2 scalar product with $u-w_h$ we obtain

$$\varepsilon(u,u-w_h)+(A(u-v),u-w_h)\leqq-(h,u-w_h).$$

By Step 1,

$$(A(u-v),u-w_h)=(A(u-v),u-v+v-w_h)$$
$$\geqq-\tfrac{1}{3}\|A(u-v)\|_{L^2}^2-\|A(u-v)\|_{L^2}\|v-w_h\|\geqq-C.$$

Therefore

$$(h,u)\leqq C\quad\text{for every}\quad h\in L^\infty\quad\text{with}\quad\|h\|_{L^\infty}\leqq\delta.$$

Hence

$$\delta\|u\|_{L^1}\leqq C.$$

Step 3. $\|u_\varepsilon|_{L^\infty}\leqq C.$

Write

$$u(x,t)=u_1(x,t)+u_2(x,t)\quad\text{with}\quad u_2(x,t)=p(t+x)-p(t-x).$$

By Step 1 and (15) we know that $\|u_1\|_{L^\infty} \leqq C$. By Step 2 and (17) we know that $\|p\|_{L^1} \leqq C$. Set $G(x, t, u) = F(x, t, u) + \varphi(x, t)$ for $u \in \mathbb{R}$. Since $Q\varphi = 0$, we deduce from (20) that

$$\varepsilon p + QG(x, t, u) = 0,$$

i.e.

$$\varepsilon p(t) + \frac{1}{2\pi} \int_0^\pi [G(x, t-x, u_1(x, t-x) + p(t) - p(t-2x)) \\ - G(x, t+x, u_1(x, t+x) + p(t+2x) - p(t))]\, dx = 0.$$

Thus for $t \in (0, 2\pi)$,

$$\varepsilon p(t) + \frac{1}{2\pi} \int_0^\pi [G(x, t-x, -C + p(t) - p(t-2x)) - G(x, t+x, C + p(t+2x) - p(t))]\, dx \\ \leqq 0.$$

Finally we set for $u \in \mathbb{R}$

$$\underline{G}(u) = \operatorname*{Inf}_{x,t} G(x, t, u)$$

$$\bar{G}(u) = \operatorname*{Sup}_{x,t} G(x, t, u).$$

We find

$$\varepsilon p(t) + \frac{1}{2\pi} \int_0^\pi [\underline{G}(-C + p(t) - p(t-2x)) - \bar{G}(C + p(t+2x) - p(t))]\, dx \leqq 0,$$

i.e.

$$\varepsilon p(t) + \frac{1}{2\pi} \int_0^{2\pi} [\underline{G}(-C + p(t) - p(s)) - \bar{G}(C + p(s) - p(t))] \frac{ds}{2} \leqq 0.$$

Finally set for $u \in \mathbb{R}$

$$H(u) = \tfrac{1}{2}[\underline{G}(u) - \bar{G}(-u)].$$

So that $H(u) \geqq \delta$ for $u \geqq M$ (by (6)). We have

$$\varepsilon p(t) + \frac{1}{2\pi} \int_0^{2\pi} H(-C + p(t) - p(s))\, ds \leqq 0.$$

Let

$$\mu = \operatorname*{Sup}_{t \in (0, 2\pi)} p(t) \quad \left(\mu \geqq 0 \text{ since } \int_0^{2\pi} p(t)\, dt = 0\right).$$

We obtain

$$\int_0^{2\pi} H(-C + \mu - p(s))\, ds \leqq 0. \tag{21}$$

Let

$$\Sigma = \{s \in (0, 2\pi);\ p(s) \geqq \tfrac{1}{2}\mu\}$$
$$^c\Sigma = \{s \in (0, 2\pi);\ p(s) < \tfrac{1}{2}\mu\}.$$

Since $\int_0^{2\pi} |p(s)|\, ds \leqq C$ we have meas $\Sigma \leqq \dfrac{2C}{\mu}$ and meas $^c\Sigma \geqq 2\pi - \dfrac{2C}{\mu}$. On the other hand we have

$$\int_\Sigma H(-C+\mu-p(s))\, ds \geqq H(-C) \text{ meas } \Sigma \geqq -|H(-C)| \frac{2C}{\mu}, \tag{22}$$

$$\int_{\Sigma^c} H(-C+\mu-p(s))\, ds \geqq H\left(-C+\frac{\mu}{2}\right) \text{ meas } {}^c\Sigma \geqq H\left(-C+\frac{\mu}{2}\right)\left(2\pi - \frac{2C}{\mu}\right). \tag{23}$$

Combining (21), (22) (23) we find

$$H\left(-C+\frac{\mu}{2}\right)\left(2\pi - \frac{2C}{\mu}\right) \leqq |H(-C)| \frac{2C}{\mu}.$$

Clearly this leads to a bound for μ.

Step 4. Passage to the limit as $\varepsilon \to 0$.

We repeat an argument from [**1**, section 1.3] to show that u is a solution of (1) (2).

For any $\xi \in L^2(\Omega)$, we have by monotonicity of F,

$$(F(x, t, u_\varepsilon) - F(x, t, \xi), u_\varepsilon - \xi) \geqq 0.$$

Thus

$$(-\varepsilon u_\varepsilon - Au_\varepsilon - F(x, t, \xi), u_\varepsilon - \xi) \geqq 0.$$

We have

$$u_{\varepsilon_n} \to u \quad \text{in} \quad w^* - L^\infty$$
$$Au_{\varepsilon_n} \to Au \text{ weakly in } L^2$$
$$(Au_{\varepsilon_n}, u_{\varepsilon_n}) \to (Au, u);$$

indeed $(Au_\varepsilon, u_\varepsilon) = (Au_\varepsilon, u_{1\varepsilon})$ and $u_{1\varepsilon_n} \to u_1$ in L^2 by (14) imply $(Au_{\varepsilon_n}, u_{\varepsilon_n}) \to (Au, u_1) = (Au, u)$).

Hence we deduce that

$$(-Au - F(x, t, \xi), u - \xi) \geqq 0 \quad \text{for every} \quad \xi \in L^2.$$

We conclude, using Minty's trick, that $Au + F(x, t, u) = 0$.

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