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A PROPERTY OF SOBOLEV SPACES

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Introduction.

In the present paper, we study a property of the Sobolev space $W_o^{1,p}(\Omega)$ for an arbitrary domain Ω in $\mathbb{R}^N$ which plays a very useful role in the study of singular second order elliptic (and parabolic equations), singular either because of a strong nonlinearity or because of singularities in the coefficients.

In an earlier paper [1] the authors proved the following result. Let Ω be an open set in $\mathbb{R}^N$. Assume $T \in H^{-1}(\Omega) \cap L^1_{loc}(\Omega)$ and $u \in H^1_o(\Omega)$ are such that

$$T(x)u(x) \geq g(x) \quad \text{a.e. on } \Omega$$

with $g \in L^1(\Omega)$. Then $T.u \in L^1(\Omega)$ and

$$< T,u > = \int T(x).u(x)\, dx$$

where $< .,. >$ denotes the scalar product in the duality of H^{-1} with H^1_o.

We extend this result here and in particular, replace the assumption " $T \in L^1_{loc}(\Omega)$ " by " T is a measure ".

We indicate some open problems and describe various examples.

We thank Professor J. Dieudonné for providing us with the example quoted in § 3 .

We note also that the result of [1] has been applied to the study of the essential self-adjointness of Schrödinger operators with singular potentials in [2] .

§ 1 - PRELIMINARIES ON CAPACITIES

We briefly recall the definition and some properties of capacities.

Let $\Omega \subset \mathbb{R}^N$ be an (arbitrary) open set and let $1 < p < \infty$. The Sobolev space $W_0^{1,p}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ for the norm

$$\|u\|^p_{W_0^{1,p}(\Omega)} = \sum_{|\alpha| \leq 1} \int |D^\alpha u|^p \, dx .$$

Its dual space is $W^{-1,p'}(\Omega)$ and the scalar product in the duality $W^{-1,p'}$, $W_0^{1,p}$ is denoted by $\langle .,. \rangle$.

For a compact subset $K \subset \mathbb{R}^N$ we set

$$\text{cap } K = \text{Inf } \left\{ \|\alpha\|^p_{W^{1,p}(\mathbb{R}^N)} ; \alpha \in \mathcal{D}(\mathbb{R}^N) , \begin{array}{l} \alpha \geq 0 \text{ on } \mathbb{R}^N \\ \alpha \geq 1 \text{ on } K \end{array} \right\}$$

and for an arbitrary set $A \subset \mathbb{R}^N$ we set

$$\text{cap } A = \text{Sup}\{\text{cap } K, \ K \subset A , \ K \text{ compact}\} .$$

When $p = 2$ this coincides with the usual definition of capacities (see [4]).

We recall (see [4]) that if $u_n \in \mathcal{D}(\Omega)$ is a Cauchy sequence in $W_0^{1,p}(\Omega)$, then there is a subsequence u_{n_k} which converges for all $x \in \Omega$, except for a set of zero capacity. Hence every function $u \in W_0^{1,p}(\Omega)$ is defined pointwise except for a set of zero capacity.

Let $\mathcal{M}(\Omega)$ denote the space of all regular Borel measures on Ω (not necessarily bounded measures) ; $\mathcal{M}^+(\Omega)$ consists of nonnegative measures.

We shall use the following

<u>*LEMMA 1.*</u> *([3]). Assume* $\mu \in W^{-1,p'}(\Omega) \cap \mathcal{M}(\Omega)$. *Let* $A \subset \Omega$ *be such that* cap $A = 0$. *Then* A *is* μ*-measurable and* $|\mu|(A) = 0$ *(*$|\mu|(A)$ *denotes the measure of* A *with respect to* $|\mu|$*).*

§ 2 - THE MAIN RESULT

Let $\mu \in \mathcal{M}^+(\Omega)$ be such that :

(1) for every $A \subset \Omega$ with $\operatorname{cap} A = 0$, then $|\mu|(A) = 0$.

Let $f_1, f_2 \dots f_k \in L^1_{loc}(\Omega ; \mu)$ and consider the measures

$$T_i = f_i \mu \qquad 1 \leq i \leq k .$$

Assume

$$T_i \in W^{-1,p'}(\Omega) \qquad 1 \leq i \leq k .$$

Let $u_1, u_2, \dots, u_k \in W_0^{1,p}(\Omega)$.

THEOREM 1. *Suppose that for some* $g \in L^1(\Omega ; \mu)$ *we have*

$$f.u = \sum_{i=1}^{k} f_i u_i \geq g \qquad \mu - \text{a.e.}$$

(note that each u_i *is defined* μ - a.e.*)*

Then $\quad f.u \in L^1(\Omega ; \mu)$ *and*

$$< T,u > = \sum_{i=1}^{k} < T_i, u_i > = \int f.u \, d\mu$$

Remarks.

1) Choosing μ to be the Lebesgue n-measure, we find exactly the result of [1].

2) Assume $T_1, T_2 \dots T_k$ are given in $W^{-1,p'}(\Omega) \cap \mathcal{M}(\Omega)$ and set $\mu = \sum_{i=1}^{k} |T_i|$.

It follows from Lemma 1, that μ satisfies (1).

On the other hand, since T_i is absolutely continuous with respect to μ we can write

$$T_i = f_i \mu \quad \text{with} \quad f_i \in L^1_{loc}(\Omega ; \mu)$$

and Theorem 1 may be applied.

Some open problems.

1) Let $W_o^{2,p}(\Omega)$ denote the closure of $\mathcal{D}(\Omega)$ for the norm

$$\| u \|^p = \sum_{|\alpha| \leq 2} \int |D^\alpha u|^p .$$

Let $W^{-2,p'}$ denote its dual space. Assume $T \in W^{-2,p'}(\Omega) \cap L^1_{loc}(\Omega)$ and let $u \in W_o^{2,p}(\Omega)$ be such that

$$T.u \geq g \quad \text{a.e. on } \Omega \text{ with } g \in L^1(\Omega)$$

Does it follow that $T.u \in L^1$ and $< T,u > = \int Tu \, dx$?

2) Assume $T \in W^{-1,p'}(\Omega) \cap L^1_{loc}(\Omega)$, $u \in W_o^{1,p}(\Omega)$ are such that

$$< T, \zeta u > \geq 0 \qquad \forall \zeta \in \mathcal{D}_+(\Omega)$$

Does it follow that $T(x)u(x) \geq 0$ a.e. ?

Proof of Theorem 1.

We use an extension of the technique developed in [1].Assume first, in addition to the assumptions of Theorem 1 that for each i , Supp u_i is a compact subset of Ω and that $|u_i(x)| \leq M$ a.e. (for Lebesgue measure). Then the conclusion of Theorem 1 holds.

Indeed let ζ_ε denote a sequence of mollifiers and let $u_\varepsilon = \zeta_\varepsilon \star u$. As $\varepsilon \to 0$, $u_\varepsilon \to u$ in $[W_o^{1,p}]^k$ and $u_\varepsilon(x) \to u(x)$ for all x except for a set of zero capacity ; in particular $u_\varepsilon(x) \to u(x)$ μ - a.e. On the other hand we have

$$< T, u_\varepsilon > = \int (f.u_\varepsilon) \, d\mu .$$

It follows from the dominated convergence theorem that

$$< T,u > = \int f.u \, d\mu \, .$$

In the general case let $v_n \in [\mathcal{D}(\Omega)]^k$ be a sequence such that $v_n \to u$ in $[W_o^{1,p}(\Omega)]^k$, $v_n(x) \to u(x)$ for all $x \in \Omega$, except for a set of zero capacity and so $v_n(x) \to u(x)$ μ - a.e.

Set

$$\lambda_n = (|u|^2 + \frac{1}{n^2})^{-1/2} \text{ Min } \{(|u|^2 + \frac{1}{n^2})^{1/2} - \frac{1}{n} , (|v_n|^2 + \frac{1}{n^2})^{1/2} - \frac{1}{n}\} ,$$

so that $0 \leqslant \lambda_n \leqslant 1$ and set

$$w_n = \lambda_n u$$

(here $|\cdot|$ denotes the euclidean norm on $\mathbb{R}^k$).

Clearly $|w_n(x)| \leqslant |v_n(x)|$ and in particular Supp $w^n \subset$ Supp v_n . We deduce from the first step that

(2) $$< T,w_n > = \int (f.w_n) d\mu$$

Next, by the Lemma in [1], we have

$$|\frac{\partial w_n}{\partial x_i}| \leqslant 3 \text{ Max}\{|\frac{\partial u}{\partial x_i}| , |\frac{\partial v_n}{\partial x_i}|\} .$$

It follows that $w_n \to u$ weakly in $[W_o^{1,p}(\Omega)]^k$ and in particular $< T,w_n > \to < T,u >$.

On the other hand $w_n \to u$ pointwise, except on a set of zero capacity ; thus $w_n \to u$ μ - a.e.

Also

(3) $$f.w_n = \lambda_n(f.u) \geqslant \lambda_n g \geqslant -|g| \quad \mu\text{- a.e.}$$

We deduce from Fatou's Lemma (2) and (3) that $f.u \in L^1(\Omega ; \mu)$ and

$$\int (f.u) d\mu \leqslant < T,u >$$

Finally, since $|f.w_n| \leqslant |f.u|$ we conclude using the dominated convergence Theorem that

$$< T,u > = \int f.u \, d\mu$$

§ 3 - EXAMPLES.

Example 1. (Dieudonné) Let $\Omega = \mathbb{R}$; there exists some $T \in H^{-1}(\mathbb{R}) \cap C^{\infty}(\mathbb{R})$ and some $u \in H^{1}(\mathbb{R}) \cap C^{\infty}(\mathbb{R})$ such that $T.u \notin L^{1}(\mathbb{R})$.

Choose $T(x) = \frac{d}{dx}\left(\frac{\sin(e^x)}{1+x^2}\right)$, $u(x) = \frac{1}{1+x^2}$.

It is easy to check that $T.u \notin L^1$ using the fact that

$$\int_{-\infty}^{+\infty} \frac{|\cos e^x|}{(1+x^2)^2} e^x \, dx = \int_0^{\infty} \frac{|\cos t|}{(1+|\log t|^2)^2} \, dt = \infty$$

Example 2. $\Omega = \mathbb{R}^3$; there exists some $T \in H^{-1}(\mathbb{R}^3) \cap L^1_{loc}(\mathbb{R}^3)$ and some $u \in H^1(\mathbb{R}^3)$ such that $T.u \notin L^1_{loc}(\mathbb{R}^3)$.

Choose $T(x) = \frac{d}{dr}\left[\cos\left(\frac{1}{r^\alpha}\right)\zeta(r)\right]$, $u(x) = \frac{1}{r^\beta}\zeta(r)$ $(r = |x|)$.

where $\alpha < 2$, $\beta < \frac{1}{2}$ and $\alpha + \beta \geqslant 2$,

$\zeta \in \mathcal{D}(\mathbb{R})$ with $\zeta(r) = 1$ for $|r| < 1$.

It is clear that $T \in H^{-1}(\mathbb{R}^3)$, and that $T \in L^1_{loc}(\mathbb{R}^3)$ since $\alpha < 2$.

Also $u \in H^1(\mathbb{R}^3)$ provided $\beta < \frac{1}{2}$ and finally $T.u \notin L^1(|x| < 1)$ since $\int_0^1 \left|\sin\left(\frac{1}{r^\alpha}\right)\right| \frac{1}{r^{\alpha+1}} \frac{1}{r^\beta} r^2 \, dr = \frac{1}{\alpha}\int_1^{\infty} |\sin t| \, t^{\frac{\beta-2}{\alpha}} \, dt = \infty$

provided $\alpha + \beta \geqslant 2$.

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