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Asymptotic Behavior of Some Evolution Systems

Haim Brezis

Introduction

Let C be a closed convex subset of a Hilbert space H. We denote by $S(t)$ a semi-group of nonlinear contractions on C i.e. $\{S(t)\}_{t>0}$ is a family of mappings from C into itself satisfying: $S(0) = I$, $S(t_1)\circ S(t_2) = S(t_1 + t_2)$, $|S(t)x - S(t)y| \leq |x - y|$ $\forall t > 0$, $\forall x,y \in C$ and $\lim_{t\downarrow 0}|x - S(t)x| = 0$ $\forall x \in C$.

We say that $p \in C$ is an equilibrium point of $S(t)$ provided $S(t)p = p$ $\forall t > 0$ and we set

$$F = \{p \in C;\ S(t)p = p\ \forall t > 0\}\ .$$

It is well known that F is convex; we assume that $F \neq \phi$.

Since, in general, $S(t)x$ does not converge to a limit as $t \to \infty$, it is of interest to consider the behavior as $t \to \infty$ of the ergodic mean $\sigma_t = \frac{1}{t}\int_0^t S(\tau)x\, d\tau$.

In §I we prove that σ_t converges weakly as $t \to \infty$ to a limit σ which can be identified as the unique element in $F \cap \overline{\text{conv}}\ \omega(x)$ (where $\omega(x)$ denotes the weak ω-limit set of $S(t)x$). In addition, σ coincides with the asymptotic center of $S(t)x$ in the sense of Edelstein. In general σ_t does not converge strongly to σ; yet in some special cases of importance, strong convergence holds.

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In §II we consider gradient flows; i.e. $S(t)x$ denotes the solution of

$$\begin{cases} \frac{du}{dt} + \partial\varphi(u) \ni 0 & \text{a.e. on } (0,\infty) \\ u(0) = x \end{cases}$$

where φ is a convex ℓ.s.c. function on H. As we shall see $S(t)x$ converges weakly as $t \to \infty$ to a limit σ such that $\varphi(\sigma) = \text{Min}\ \varphi$. However, strong convergence does <u>not</u> hold in general; this means that the method of steepest descent applied to a (smooth) convex function converges weakly - but not strongly in general.

In §III we consider the asymptotic behavior of the solution of an equation of the form

$$\frac{du}{dt} + \partial\varphi(u) \ni f(t) \text{ a.e. on } (0,\infty),\ u(0) = x$$

where $f(t)$ is a T-periodic forcing term. Here, the system "converges" as $t \to \infty$ to a T-periodic motion.

In §IV we conclude with a simple application.

Many of the results we discuss are due to J. B. Baillon and to R. Bruck; we also refer to the works of Dafermos-Slemrod [17], Brezis-Browder [12], Pazy [19], [20], [21], Reich [22].

I. <u>Convergence of the ergodic mean</u>.

We begin with a simple result

<u>Theorem 1</u> ([8]) σ_t converges weakly as $t \to \infty$ to some $\sigma \in F$ - which depends on x.

<u>Remark 1</u>. In general σ_t does not converge strongly (see §II).

In order to identify the limit σ we need some definitions:

a) $\omega(x)$ denotes the weak ω-limit set of $S(t)x$ i.e. $y \in \omega(x)$ iff there is a sequence $t_n \to \infty$ such that $S(t_n)x$ converges weakly to y.

b) The asymptotic center of a bounded function $u(t):[0,\infty) \to H$ is defined as follows. Given $y \in H$, set $G(y) = \limsup_{t\to\infty} |u(t) - y|^2$ so that G is strictly convex,

continuous, $G(y) \to \infty$ as $|y| \to \infty$. Therefore G achieves its minimum on H at a unique point: $AC\{u(t)\}$, called the asymptotic center - a concept introduced by Edelstein.

Theorem 2 ([15]). Let $\sigma = \text{weak} \lim_{t\to\infty} \sigma_t$. Then

(1) $\sigma = F \cap \overline{\text{conv}}\, \omega(x)$

(2) $\sigma = AC\{S(t)x\}$.

Remark 2. Instead of σ_t consider now a more general averaging process:

$$\sigma_n = \int_0^\infty S(\tau)x\, a_n(\tau)d\tau$$

where $a_n \in L^1(0,\infty)$, $a_n \geq 0$ and $\int_0^\infty a_n(\tau)d\tau = 1$. Assume that the functions a_n are of bounded variation and $\int_0^\infty |da_n| \to 0$. Then σ_n converges weakly to σ as $n \to \infty$, where σ is the same as in Theorem 1 (σ is independent of a_n).

We begin with a simple Lemma

Lemma 1. Assume $p_1, p_2 \in F$, $q_1, q_2 \in \omega(x)$. Then $(p_1 - p_2, q_1 - q_2) = 0$. In particular $F \cap \overline{\text{conv}}\, \omega(x)$ contains at most one point.

Proof of Lemma 1

Set $u(t) = S(t)x$. Given $p \in F$, the function $|u(t) - p|^2$ is nonincreasing and thus converges as $t \to \infty$ to a limit, say $\ell(p)$.

We have

$$|u(t) - p_1|^2$$

$$= |u(t) - p_2|^2 + 2(u(t) - p_2, p_2 - p_1) + |p_2 - p_1|^2 .$$

Choosing $t = t_n$ such that $u(t_n) \longrightarrow q_1$ we find

(3) $\ell(p_1) = \ell(p_2) + 2(q_1 - p_2, p_2 - p_1) + |p_2 - p_1|^2$.

Similarly

(4) $\ell(p_1) = \ell(p_2) + 2(q_2 - p_2, p_2 - p_1) + |p_2 - p_1|^2$.

Comparing (3) and (4) leads to the conclusion.

Proofs of Theorem 1, 2 and Remark 2

Let a_n and σ_n be as in Remark 2. Since $\|a_n\|_{L^\infty} \le \int_0^\infty |da_n| \to 0$ it follows that every weak limit point of σ_n lies in $\overline{\text{conv}}\ \omega(x)$.

We recall that $S(t)$ has a generator in the sense of Komura, Kato, Crandall-Pazy (see e.g. [11]); that is, there exists a maximal monotone operator A with $\overline{D(A)} = C$ such that when $x \in D(A)$, $S(t)x$ coincides with the unique solution $u(t)$ of $\frac{du}{dt} + Au \ni 0$, a.e. on $(0,\infty)$, $u(0) = x$. Assume first $x \in D(A)$ and thus

$$(Av + \frac{du}{dt}(t),\ v - u(t)) \ge 0 \quad \forall v \in D(A)$$

or

$$(Av,\ v - u(t)) \ge \frac{1}{2}\frac{d}{dt}|u(t) - v|^2 \quad .$$

Multiplying by $a_n(t)$ and integrating we find

$$(Av,\ v - \sigma_n) \ge -\frac{1}{2}a_n(0)|x - v|^2 - \frac{1}{2}\int_0^\infty |u(t) - v|^2 da_n \quad .$$

Hence $\forall v \in D(A)$

$$(5) \qquad (Av,\ v - \sigma_n) \ge -\|u - v\|^2_{L^\infty}\int_0^\infty |da_n| \quad .$$

By a density argument, (5) holds now even when $x \in \overline{D(A)} = C$.

It follows from (5) that every weak limit point of σ_n lies in F. Indeed let $\sigma_{n_k} \rightharpoonup \sigma$; we have

$$(Av,\ v - \sigma) \ge 0 \quad \forall v \in D(A)$$

and so $0 \in A\sigma$ i.e. $\sigma \in F$.

Consequently every weak limit point of σ_n lies in $F \cap \overline{\text{conv}}\ \omega(x)$ - which consists of a single element.

We prove now that σ coincides with $AC\{S(t)x\}$. Given $y \in H$ we have

$$|u(t) - \sigma|^2 = |u(t) - y|^2 + 2(u(t) - y,\ y - \sigma) + |y - \sigma|^2 \quad .$$

Thus

$$\int_0^\infty |u(t) - \sigma|^2 a_n(t)dt = \int_0^\infty |u(t) - y|^2 a_n(t)dt +$$

$$+ 2(\sigma_n - y,\ y - \sigma) + |y - \sigma|^2 \quad .$$

Since $\sigma \in F$, $|u(t) - \sigma|^2$ converges to a limit as $t \to \infty$, which is simply $G(\sigma)$. Since on the other hand $G(y) = \lim_{t\to\infty} \sup|u(t) - y|^2$ we find

$$G(\sigma) \leq G(y) - |y - \sigma|^2 \quad \forall y \in H \quad .$$

Consequently $\sigma = AC\{S(t)x\}$.

Remark 3. Let T be a contraction in H having at least one fixed point. Baillon [2] has proved the following (see also [19]):

Theorem 3. The Cesaro mean

$$\sigma_n = \frac{1}{n}(x + Tx + \ldots T^{n-1}x)$$

converges weakly as $n \to \infty$ to a fixed point of T.

Theorem 3 can be viewed as a nonlinear version of the classical ergodic theory of von Neumann, Kakutani, Yosida. Theorem 1 can be derived from Theorem 3 by using a device due to Konishi (see [6]). In general σ_n does not converge strongly.

Theorem 3 has been extended by Baillon to L^p spaces, $1 < p < \infty$, in [6] and subsequently by Bruck [16] to more general spaces. The proofs are very tricky. In view of Konishi's device - which is valid in general Banach spaces - Theorem 1 holds true in L^p spaces $1 < p < \infty$.

Strong convergence

In some special cases σ_t converges strongly as $t \to \infty$. For example when the orbit $\bigcup_{t>0} S(t)x$ is relatively compact - this is the setting of Dafermos-Slemrod [17].

Another surprising condition which implies strong convergence of σ_t is the oddness of $S(t)$ - the fact that oddness has an impact on strong convergence was first observed by Bruck [14] in the case of gradient flows.

Theorem 4 ([4]). Assume C is symmetric and $S(t)$ is odd i.e. $S(t)(-u) = -S(t)u \ \forall u \in C, \forall t > 0$. Then σ_t converges strongly as $t \to \infty$. More generally $\sigma_n = \int_0^\infty S(t)x\, a_n(t)dt$ converges strongly as $n \to \infty$ provided a_n satisfies the assumptions in Remark 2.

Remark 4. It is not known whether Theorem 4 holds in L^p spaces $1 < p < \infty$.

Remark 5. Assume T is an odd contraction on H. Then Baillon [3] has proved that

$$\sigma_n = \frac{1}{n}(x + Tx + \ldots T^{n-1}x)$$

converges strongly to a fixed point of T. Such a result, combined with Konishi's device could be used to prove Theorem 4.

A crucial ingredient in the proof of Theorem 4 is the following:

Lemma 2 ([15], [12], [22]). Let $u(t)$ be a function defined on $(0,\infty)$ with values into H such that: for all $h \geq 0$, $(u(t), u(t+h))$ converges as $t \to \infty$ to a limit, say $\ell(h)$, uniformly in h. Then $\sigma_n = \int_0^\infty u(\tau)\, a_n(\tau)d\tau$ converges strongly as $n \to \infty$ to $AC\{u(t)\}$ provided a_n satisfies the conditions of Remark 2.

For the proof of Lemma 2 we refer to [15], [12] and [22]. Note that a special case of Lemma 2 asserts that if $\{x_n\}$ is a sequence such that (x_n, x_{n+i}) converges as $n \to \infty$ to $\ell(i)$, uniformly in i, then $\sigma_n = \frac{1}{n}(x_0 + x_1 + \ldots x_{n-1})$ converges strongly.

Proof of Theorem 4. Let $t \geq s$ and $x,y \in C$. We have

$$|S(t+h)x - S(t)y| \leq |S(s+h)x - S(s)y|$$

and so

$$|S(t+h)x|^2 - 2(S(t+h)x, S(t)y) + |S(t)y|^2 \leq$$

$$|S(s+h)x|^2 - 2(S(s+h)x, S(s)y) + |S(s)y|^2 \quad .$$

Hence we find

$$2[(S(s+h)x, S(s)y) - (S(t+h)x, S(t)y)]$$

$$\leq |S(s+h)x|^2 + |S(s)y|^2 - |S(t+h)x|^2 - |S(t)y|^2 \quad .$$

Choosing $y = \pm x$ and using the oddness of $S(t)$ leads to

$$2|(S(s+h)x, S(s)x) - (S(t+h)x, S(t)x|$$

$$\leq 2|S(s)x|^2 - 2|S(t+h)x|^2 \quad .$$

But $|S(t)x|$ converges as $t \to \infty$. It follows that $(S(t)x, S(t+h)x)$ is Cauchy as $t \to \infty$, uniformly in h. Therefore Lemma 2 can be applied.

II. Gradient flows

For the class of gradient flows corresponding to convex functions it is not necessary to consider averages of orbits: the orbit itself converges as $t \to \infty$. More precisely let φ be a ℓ.s.c. convex function on H such that $\operatorname{Min} \varphi$ is achieved. Set $F = \{p;\ \varphi(p) = \operatorname{Min} \varphi\}$ and set $A = \partial\varphi$. We denote by $S(t)x$ the semigroup generated by $-A$ i.e. $S(t)x = u(t)$ is the unique solution of

$$\frac{du}{dt} + Au \ni 0 \quad \text{a.e. on} \quad (0,\infty),\ u(0) = x \quad .$$

Theorem 5 (Bruck [14]). For each $x \in \overline{D(A)}$ $S(t)x$ converges weakly as $t \to \infty$ to some $p \in F$.

Theorem 5 is an obvious consequence of Theorem 1 and the following

Lemma 3. We have for every $x \in \overline{D(A)}$

$$\lim_{t\to\infty} \left|S(t)x - \frac{1}{t}\int_0^t S(\tau)x\,d\tau\right| = 0 \quad . \tag{6}$$

Proof. We have, setting $u(t) = S(t)x$

$$\left|u(t) - \frac{1}{t}\int_0^t u(\tau)\,d\tau\right| = \frac{1}{t}\left|\int_0^t u'(\tau)\ \tau\,d\tau\right| \quad .$$

Therefore it suffices to prove that $\lim_{t\to\infty} t|u'(t)| = 0$. We know (see [10] Theorem 22) that

$$\int_0^\infty \left|\frac{du}{dt}(t)\right|^2 t\ dt \le \frac{1}{2}|\operatorname{dist}\ (x, F)|^2 \quad .$$

Since the function $t \mapsto \left|\frac{du}{dt}(t)\right|$ is non increasing it follows that

$$\frac{3}{8}\left|\frac{du}{dt}(t)\right|^2 t^2 \le \int_{t/2}^t \left|\frac{du}{dt}(\tau)\right|^2 \tau\ dt \to 0 \quad \text{as} \quad t \to \infty \quad .$$

Remark 6. Baillon [5] has constructed an example of a convex ℓ.s.c. function φ such that $S(t)x$ does not converge strongly as $t \to \infty$. The existence of such an example had been suggested earlier by Kōmura. In fact such a φ can be

chosen to be C^1 with grad φ lipschitzian (see [7]). In view of (6), $\sigma_t = \frac{1}{t}\int_0^t S(\tau)x$ does not converge strongly as $t \to \infty$, thus providing a counterexample to the strong convergence of the ergodic mean.

Strong convergence of $S(t)x$ holds in the following cases:

a) For every M the set $\{u \in H, |u| \leq M$ and $\varphi(u) \leq M\}$ is compact (see [10]).

b) The function φ is even (see [14]); here A is odd and we conclude by Theorem 4 and (6) that $S(t)x$ converges strongly.

It would be of interest to find other conditions implying the strong convergence of $S(t)x$. For example, assume φ is even and let $f \in H$ be such that $\operatorname{Min}_{z\in H}\{\varphi(z) - (f,z)\}$ is achieved.

Let $u(t)$ be the solution of $\frac{du}{dt} + \partial\varphi(u) \ni f, u(0) = x$. Does $u(t)$ converge strongly as $t \to \infty$?

Remark 7. A "discrete" version of Theorem 5 has been proved in [13]: let $A = \partial\varphi$ and let u_n be the sequence defined by the "implicit" scheme

$$\frac{u_{n+1} - u_n}{\tau_n} + Au_{n+1} \ni 0 \qquad \tau_n > 0 \quad .$$

Then $u_n \rightharpoonup p \in F$ provided $\sum_1^\infty \tau_n = \infty$.

III. Periodic forcing

We turn now to the following question. Consider a system governed by a convex potential φ with a periodic forcing term $f(t)$:

$$(7) \qquad \begin{cases} \frac{du}{dt}(t) + \partial\varphi(u(t)) \ni f(t) & \text{a.e. on } (0,\infty) \\ u(0) = u_0 \quad . \end{cases}$$

What happens as $t \to \infty$? The answer is given by the following

Theorem 6 (Baillon-Haraux [9])

Assume $f \in L^2_{\ell oc}(0,\infty; H)$ is T-periodic and assume $u(t)$ remains bounded as $t \to \infty$. Then there exists a T-periodic function $\bar{u}(t)$ satisfying

$$\frac{d\bar{u}}{dt} + \partial\varphi(\bar{u}) \ni f \quad \text{a.e. on} \quad (0,\infty)$$

such that $\underset{t\to\infty}{\text{w-lim}}(u(t) - \bar{u}(t)) = 0$.

In addition

$$\lim_{n\to\infty} \int_0^T \left|\frac{du}{dt}(nT + t) - \frac{d\bar{u}}{dt}(t)\right|^2 dt = 0 \quad .$$

In other words the system evolves "slowly" to a periodic motion. For the proof of Theorem 6 we refer to [9]. The proof involves an interesting Lemma which we describe in a special case:

<u>Lemma 4</u>. Let $f \in L^2(0,T;\, H)$ and let $A = \partial\varphi$. Assume u_n and u are the solutions of

$$\frac{du_n}{dt} + Au_n \ni f \quad \text{a.e. on} \quad (0,T),\ u_n(0) = u_{0n}$$

$$\frac{du}{dt} + Au \ni f \quad \text{a.e. on} \quad (0,T),\ u(0) = u_0$$

with $u_{0n} \to u_0$. Then $u_n \to u$ in $C([0,T];\, H)$ and

$$\int_0^T \left|\frac{du_n}{dt} - \frac{du}{dt}\right|^2 t\, dt \to 0 \ .$$

<u>Proof</u>. We know (see [10]) that $u_n \to u$ in $C([0,T];\, H)$, $\varphi(u_n),\ \varphi(u) \in L^1(0,T)$ and $\sqrt{t}\,\frac{du_n}{dt},\ \sqrt{t}\,\frac{du}{dt} \in L^2(0,T;\, H)$. We have

$$(8)\quad \varphi(u(t)) - \varphi(u_n(t)) \geq \left(f(t) - \frac{du_n(t)}{dt},\ u(t) - u_n(t)\right)$$

and for fixed $v \in D(\varphi)$

$$(9)\quad \varphi(v) - \varphi(u_n(t)) \geq \left(f(t) - \frac{du_n(t)}{dt},\ v - u_n(t)\right).$$

From (9) we deduce

$$(10)\quad \int_0^\varepsilon \varphi(u_n)dt \leq \varepsilon\varphi(v) - \int_0^\varepsilon (f,\ v - u_n)dt - \frac{1}{2}|v - u_n(\varepsilon)|^2 + \frac{1}{2}|v - u_{0n}|^2$$

and from (8)

$$(11)\quad \int_\varepsilon^T \varphi(u_n)\,dt \le \int_\varepsilon^T \varphi(u)\,dt + \int_\varepsilon^T (f,\ u_n - u)\,dt$$
$$+ \int_\varepsilon^T \left(\frac{du}{dt},\ u - u_n\right)dt - \frac{1}{2}|u_n(T) - u(T)|^2 + \frac{1}{2}|u_n(\varepsilon) - u(\varepsilon)|^2 .$$

Adding (10) and (11) and using the fact that $u_n \to u$ in $C([0,T];\ H)$ we find

$$\limsup_{n\to\infty} \int_0^T \varphi(u_n)\,dt \le \varepsilon\varphi(v) - \int_0^\varepsilon (f,\ v - u)\,dt - \frac{1}{2}|v - u(\varepsilon)|^2$$
$$+ \frac{1}{2}|v - u_0|^2 + \int_\varepsilon^T \varphi(u)\,dt \quad .$$

As $\varepsilon \to 0$ we obtain

$$(12)\quad \limsup_{n\to\infty} \int_0^T \varphi(u_n)\,dt \le \int_0^T \varphi(u)\,dt \quad .$$

On the other hand we have

$$(13)\quad \int_0^T \left|\frac{du_n}{dt}\right|^2 t\,dt + T\varphi(u_n(T)) - \int_0^T \varphi(u_n)\,dt = \int_0^T \left(f,\ \frac{du_n}{dt}\right) t\,dt$$

$$(14)\quad \int_0^T \left|\frac{du}{dt}\right|^2 t\,dt + T\varphi(u(T)) - \int_0^T \varphi(u)\,dt = \int_0^T \left(f,\ \frac{du}{dt}\right) t\,dt \quad .$$

Combining (12, (13), (14) and the fact that $\sqrt{t}\,\frac{du_n}{dt}$ converges weakly in $L^2(0,\ T;\ H)$ to $\sqrt{t}\,\frac{du}{dt}$ we see that

$$\limsup_{n\to\infty} \int_0^T \left|\frac{du_n}{dt}\right|^2 t\,dt \le \int_0^T \left|\frac{du}{dt}\right|^2 t\,dt, \quad \text{and therefore}$$

$$\lim_{n\to\infty} \int_0^T \left|\frac{du_n}{dt} - \frac{du}{dt}\right|^2 t\,dt = 0 \ .$$

<u>Remark 8</u>. Using a similar device, H. Attouch [1] has proved the following. Let $A_n = \partial\varphi_n$ and $A = \partial\varphi$ be such that $A_n \to A$ in the sense of graphs i.e. $(I + A_n)^{-1} \to (I + A)^{-1}$. Assume u_n and u are the solutions of

$$\frac{du_n}{dt} + A_n u_n \ni f \quad \text{a.e. on} \quad (0,T),\ u_n(0) = u_{0n}$$

$$\frac{du}{dt} + Au \ni f \qquad \text{a.e. on} \quad (0,T),\ u(0) = u_{0n}$$

with $u_{0n} \to u_0$.

Then $u_n \to u$ in $C([0,T]; H)$ (this is just the nonlinear version of the Trotter-Kato theorem) and also

$$\int_0^T \left|\frac{du_n}{dt} - \frac{du}{dt}\right|^2 t\, dt \to 0.$$

Remark 9. Let $f \in L^2_{\ell oc}(0,\infty ; H)$, let $A = \partial\varphi$ and let u, v be the solutions of

$$\frac{du}{dt} + Au \ni f \quad \text{a.e. on} \quad (0,\infty),\ u(0) = u_0$$

$$\frac{dv}{dt} + Av \ni f \quad \text{a.e. on} \quad (0,\infty),\ v(0) = v_0 \quad .$$

Does $u(t) - v(t)$ converge weakly to a limit as $t \to \infty$? Note that the answer is positive in case f is T-periodic and $u(t)$ remains bounded. Indeed there are two T-periodic solutions $\bar{u}(t)$ and $\bar{v}(t)$ such that $\text{w-}\lim_{t\to\infty}(u(t) - \bar{u}(t)) = 0$, $\text{w-}\lim_{t\to\infty}(v(t) - \bar{v}(t)) = 0$. Since the difference of two T-periodic solutions is constant (see [9]) it follows that $\text{w-}\lim_{t\to\infty}(u(t) - v(t))$ exists.

IV. An example

Let Ω be a smooth bounded domain in $\mathbb{R}^N$. Given $\psi(x) \in L^2(\Omega)$ with $\psi \geq 0$ on Ω and $u_0 \in H_0^1(\Omega)$, find a function $u(x,t)$ satisfying

$$(15) \qquad \begin{cases} u_t = \min\{\Delta u, \psi\} & \text{on} \quad \Omega \times (0,\infty) \\ u = 0 & \text{on} \quad \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) & \text{on} \quad \Omega \end{cases}$$

(related questions occur in heat control, see [18] Chap. 2).

Theorem 7. There exists a unique solution of (15). In addition $u(x,t)$ converges weakly in H_0^1, as $t \to \infty$, to a function $u_\infty(x)$ satisfying

$$(16) \qquad \min\{\Delta u_\infty, \psi\} = 0 \quad .$$

Remark 10. It is clear that any function $u_\infty \in H_0^1$ satisfying (16) is an equilibrium and that there are many such functions. We don't know how to identify $\lim_{t\to\infty} u(x,t)$ (in terms

of u_0) among all the equilibria. Also we don't know whether $u(x,t)$ converges strongly in H_0^1.

Proof. Let $K = \{v \in H_0^1(\Omega),\ v \leq \psi \text{ a.e.}\}$. On $H = H_0^1(\Omega)$ we consider the scalar product

$$a(u,v) = \int_\Omega \text{grad } u \text{ grad } v \, dx \quad .$$

Problem (15) can be expressed in a weak form as:

$$u_t \in K$$

$$\int_\Omega u_t(v - u_t)dx + a(u,\ v - u_t) \geq 0 \quad \forall v \in K,\ \forall t > 0$$

or equivalently

$$(17) \quad a(u,\ v - u_t) + \varphi(v) - \varphi(u_t) \geq 0 \quad \forall v \in H_0^1,\ \forall t > 0$$

where φ is a convex ℓ.s.c. function defined on H by

$$\varphi(v) = \begin{cases} \frac{1}{2}\int_\Omega |v|^2 dx & \text{if } v \in K \\ +\infty & \text{if } v \notin K \end{cases} .$$

But (17) can be written as $-u \in \partial\varphi(u_t)$ or $u_t - \partial\varphi^*(-u) \ni 0$.

Now we are reduced to the abstract setting and we can use Theorem 5. Note that u_∞ is an equilibrium provided $-u_\infty \in \partial\varphi(0)$ i.e.

$$\frac{1}{2}\int_\Omega v^2 \, dx \geq -a(u_\infty,\ v) \qquad \forall v \in K$$

or equivalently $\int_\Omega \text{grad } u_\infty \text{ grad } v \, dx \geq 0 \quad \forall v \in K$.

Remark 11. Instead of (15) consider now a problem with two side-constraints on u_t: find $u(x,t)$ satisfying

$$(18) \quad \begin{cases} |u_t| \leq \psi & \text{on} \quad \Omega \times (0,\infty) \\ u_t - \Delta u = 0 & \text{on} \quad [|u_t| < \psi] \\ u_t - \Delta u \leq 0 & \text{on} \quad [u_t = \psi] \\ u_t - \Delta u \geq 0 & \text{on} \quad [u_t = -\psi] \\ u = 0 & \text{on} \quad \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) & \text{on} \quad \Omega \ . \end{cases}$$

The same kind of proof as above shows that there is a unique solution u of (18) and that $u(x,t)$ converges strongly in H_0^1 to a function u_∞ satisfying $\Delta u_\infty = 0$ on $[\psi > 0]$. Here strong convergence holds since φ is even.

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Department of Mathématiques
Université Paris VI
75230 Paris, France