ON $H$–TRIVIAL LINE BUNDLES ON TORIC DM STACKS
OF DIM $\geq 3$

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Abstract. We study line bundles on smooth toric DM stacks $\mathbb{P}_\Sigma$ of arbitrary dimension. A sufficient condition is given for when infinitely many line bundles on $\mathbb{P}_\Sigma$ have trivial cohomology. In dimension three, the sufficient condition is also a necessary condition in the case that $\Sigma$ has no more than one pair of collinear rays.

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1. INTRODUCTION

Exceptional collections of line bundles on toric Deligne-Mumford stacks have attracted considerable interest over the years [5, 9–14]. Study of such exceptional collections leads one to consider line bundles with all cohomology groups equal to zero, such as $\mathcal{O}(-1), \ldots, \mathcal{O}(-n)$ on $\mathbb{CP}^n$. We call such line bundles $H$–trivial [17]. Paper [17] gives a combinatorial criterion for when a toric DM stack of dimension two possesses infinitely many such line bundles.

Theorem 1.1. [17] Let $\mathbb{P}_\Sigma$ be a proper smooth dimension two toric Deligne-Mumford stack associated to a complete stacky fan $\Sigma = (\Sigma, \{v_i\}_{i=1}^n)$. Then there are infinitely many $H$–trivial line bundles on $\mathbb{P}_\Sigma$ if and only if there exists $\{i, j\} \subset \{1, 2, \cdots, n\}$ such that $v_i$ and $v_j$ are collinear.

In this paper, we explore line bundles on $\mathbb{P}_\Sigma$ for smooth toric varieties and DM stacks in higher dimension. We associate with each $\Sigma$–piecewise linear function $\psi$ a convex polytope $\Lambda_\psi$ in the lattice of characters, see Definition 2.12. We obtain a sufficient condition for when there exist infinitely many $H$–trivial line bundles on $\mathbb{P}_\Sigma$.

Theorem 2.13. Let $\mathbb{P}_\Sigma$ be a proper smooth dimension $m$ toric DM stack associated to a complete stacky fan $\Sigma = (\Sigma, \{v_i\}_{i=1}^n)$. If there exists
a $\Sigma$–piecewise linear function $\psi$ such that $\dim(\Lambda_\psi) < m$, then there are infinitely many $H$–trivial line bundles on $\mathbb{P}_\Sigma$.

Moreover, we also get a criterion for when there exist infinitely many $H$–trivial line bundles on $\mathbb{P}_\Sigma$ for smooth toric varieties and DM stacks in dimension three when there is no more than one pair of collinear rays in $\Sigma$.

**Theorem 4.10.** Let $\mathbb{P}_\Sigma$ be a proper smooth dimension three toric DM stack associated to a complete stacky fan $\Sigma = (\Sigma, \{v_i\}_{i=1}^n)$. Assume there exists no more than one pair of collinear rays in $\Sigma$. Then there are infinitely many $H$–trivial line bundles on $\mathbb{P}_\Sigma$ if and only if there is a $\Sigma$–piecewise linear function $\psi$ such that $\dim(\Lambda_\psi) < 3$.

The paper is organized as follows. In section 2, we give an overview of smooth toric DM stacks, their Picard groups and the cohomology of line bundles on the stacks. Then we define forbidden cones and forbidden sets and state the first main result Theorem 2.13. Section 3 focuses on the proof of Theorem 2.13. We first exhibit an important way of producing infinitely many $H$–trivial line bundles in Proposition 3.1. Then we relate it to the existence of a $\Sigma$–piecewise linear function $\psi$ such that $\dim(\Lambda_\psi) < 3$. In section 4, we consider the case when $N$ has rank three. We prove a sufficient and necessary condition for the existence of infinitely many $H$–trivial line bundles under the assumption that there is no more than one pair of collinear rays in $\Sigma$. Section 5 proposes two conjectures that would generalize our results.

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2. **Line bundles on toric DM stacks and their cohomology**

In this section, we introduce toric DM stacks $\mathbb{P}_\Sigma$ and their Picard groups $\text{Pic}(\mathbb{P}_\Sigma)$, and describe the cohomology of line bundles on $\mathbb{P}_\Sigma$. We formulate our main results.

In order to refrain from the technicalities of the derived Gale duality of [3], we consider a lattice $N$ which is a free abelian group of finite rank. Let $\Sigma$ be a complete simplicial fan in $N$. We choose a lattice point $v$ in each of the one-dimensional cones of $\Sigma$. If $\Sigma$ has $n$ one-dimensional cones, we get a complete stacky fan $\Sigma = (\Sigma, \{v_i\}_{i=1}^n)$, see [3]. The toric DM stack $\mathbb{P}_\Sigma$ associated to this stacky fan $\Sigma$ is constructed in [3] as a stack version of the homogeneous coordinate ring construction of [4]. The description of

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1The importance of pairs of collinear rays was already observed in the paper [15] in the context of vanishing of cohomology of divisorial sheaves on toric varieties.
line bundles on the DM stacks is analogous to the description of the Picard group that was given in [6,8]. By [16], we know the line bundles on \( \mathbb{P}_\Sigma \) are in bijection with collections of integers, up to global linear functions, as described below.

**Proposition 2.1.** The Picard group of \( \mathbb{P}_\Sigma \) is isomorphic to the quotient of \( \mathbb{Z}^n \) with basis \( \{E_i\}_{i=1}^n \) by the subgroup of elements of the form \( \sum_{i=1}^n (w_i \cdot v_i)E_i \) for all \( w \) in the character lattice \( M = \mathbb{N}^* \).

**Proof.** See [2]. \( \square \)

Now we remind the reader how to calculate the cohomology of a line bundle \( L \) on \( \mathbb{P}_\Sigma \). For each \( r = (r_i)_{i=1}^n \in \mathbb{Z}^n \), we define \( \text{Supp}(r) \) to be the abstract simplicial complex on \( n \) vertices \( \{1, \ldots, n\} \) as follows

\[
\text{Supp}(r) = \{ J \subseteq \{1, \ldots, n\} \mid r_i \geq 0 \text{ for all } i \in J \text{ and there exists a cone of } \Sigma \text{ containing all } v_i, i \in J \}.
\]

The following proposition gives a description of the cohomology of a linear bundle \( L \) on \( \mathbb{P}_\Sigma \) in terms of the reduced simplicial homology spaces of \( \text{Supp}(r) \).

**Proposition 2.2.** [2] Let \( L \in \text{Pic}(\mathbb{P}_\Sigma) \). Then

\[
H^j(\mathbb{P}_\Sigma, L) = \bigoplus_{r \in \mathbb{Z}^n} H^j_{\text{red}}(\text{Supp}(r)),
\]

where the sum is over all \( r = (r_i)_{i=1}^n \in \mathbb{Z}^n \) such that \( \mathcal{O}(\sum_{i=1}^n r_i E_i) \cong L \).

**Proof.** See [2]. \( \square \)

**Remark 2.3.** We have \( H^0(L) \neq 0 \) if and only if there exists \( r \in \mathbb{Z}^n_{\geq 0} \) such that \( \mathcal{O}(\sum_{i=1}^n r_i E_i) \cong L \). Another extreme case is that \( H^k(N)(L) \) only appears when the simplicial complex \( \text{Supp}(r) = \{\emptyset\} \), i.e. when \( \mathcal{O}(\sum_{i=1}^n r_i E_i) \cong L \) with all \( r_i \leq -1 \).

**Remark 2.4.** Let \( L \cong \mathcal{O}(\sum_{i=1}^n a_i E_i) \) be a line bundle in \( \text{Pic}(\mathbb{P}_\Sigma) \). Assume there is another expression \( L \cong \mathcal{O}(\sum_{i=1}^n r_i E_i) \). Then by Proposition [2.1] there exists an element \( f \in M \) such that \( r_i = a_i + f(v_i) \) for \( i = 1, \ldots, n \), where \( f(v_i) = (f.v_i) \). Thus the cohomology of \( L \) can also be written as following:

\[
H^j(\mathbb{P}_\Sigma, L) = \bigoplus_{f \in \mathbb{N}^*} H^j_{\text{red}}(\text{Supp}(r_f)),
\]

where \( r_f = (a_i + f(v_i))_{i=1}^n \).

In this paper, our primary objects of interest are \( H \)-trivial line bundles which we define below.

**Definition 2.5.** Let \( L \) be a line bundle in \( \text{Pic}(\mathbb{P}_\Sigma) \). We say that \( L \) is \( H \)-trivial iff \( H^j(\mathbb{P}_\Sigma, L) = 0 \) for all \( j \geq 0 \).

A combinatorial criterion for \( H \)-triviality is given in terms of forbidden sets introduced below, see [2,7].
Definition 2.6. For every subset $I \subseteq \{1, \ldots, n\}$, we denote $C_I$ to be the simplicial complex $\text{Supp}(r)$ where $r_i = -1$ for $i \notin I$ and $r_i = 0$ for $i \in I$. Let $\Delta = \{ I \subseteq \{1, \ldots, n\} | C_I \text{ has nontrivial reduced homology} \}$. By Remark 2.3, $\Delta$ contains $\{1, \ldots, n\}$ and $\emptyset$. For each $I \in \Delta$, the forbidden set associated to $I$ is defined by

$$FS_I := \{ O(\sum_{i \notin I} (-1 - r_i)E_i + \sum_{i \in I} r_i E_i) | r_i \in \mathbb{Z}_{\geq 0} \text{ for all } i \}.$$

Proposition 2.7. Let $\mathcal{L}$ be a line bundle on $\mathbb{P}_\Sigma$. Then $\mathcal{L}$ is $H$-trivial if and only if $\mathcal{L}$ does not lie in $FS_I$ for any $I \in \Delta$.

Proof. This follows immediately from Proposition 2.2. □

We introduce $\text{Pic}_R(\mathbb{P}_\Sigma) = \text{Pic}(\mathbb{P}_\Sigma) \otimes R$ which can be regarded as a quotient of $\mathbb{R}^n$ with basis elements given by $E_i$. We know $\text{Pic}_R(\mathbb{P}_\Sigma)$ is a vector space with dimension equal to the rank of $\text{Pic}(\mathbb{P}_\Sigma)$.

Definition 2.8. For each $I \in \Delta$, we define the forbidden point by

$$q_I = -\sum_{i \notin I} E_i \in \text{Pic}_R(\mathbb{P}_\Sigma).$$

We define a cone associated to $I$ with vertex at the origin to be

$$Z_I = \sum_{i \in I} \mathbb{R}_{\geq 0} E_i - \sum_{i \notin I} \mathbb{R}_{\geq 0} E_i.$$

We define the forbidden cone $FC_I \subseteq \text{Pic}_R(\mathbb{P}_\Sigma)$ by

$$FC_I = q_I + Z_I.$$ 

Remark 2.9. By definition, we have $FS_I \subseteq FC_I$ for any $I \in \Delta$.

In dimension two, the set $\Delta$ is especially simple.

Example 2.10. Let $\Sigma$ be a complete simplicial fan $\Sigma$ in $N$ with $n$ one-dimensional cones and $n$ lattice points $\{v_i\}_{i=1}^n$ chosen in each of the one-dimensional cones of $\Sigma$. In the case that $N = \mathbb{Z}^2$, the maximal cones of $\Sigma$ are $\mathbb{R}_{\geq 0} v_1 + \mathbb{R}_{\geq 0} v_2, \mathbb{R}_{\geq 0} v_2 + \mathbb{R}_{\geq 0} v_3, \ldots, \mathbb{R}_{\geq 0} v_n + \mathbb{R}_{\geq 0} v_1$, see Figure 7. We describe $\Delta = \{ \emptyset, \{1, \ldots, n\} \} \cup \{ I \subseteq \{1, \ldots, n\} | C_I \text{ is disconnected} \}$. For example, we have $\{1, 3\} \in \Delta$ if $n > 3$, $\{n, 2, 3\} \in \Delta$ if $n > 4$, but $\{1, 2\} \notin \Delta$, $\{n, 1, 2\} \notin \Delta$ for all $n > 2$, see Figure 7.
Example 2.11. In the case that $N = \mathbb{Z}^3$, we describe $\Delta = \emptyset \cup \{I \in \{1, \ldots, n\} | C_I \text{ is disconnected} \} \cup \{I \in \{1, \ldots, n\} | C_{\{1, \ldots, n\}\setminus I} \text{ is disconnected} \}$. Indeed $H^2(\mathcal{L})$ is nontrivial iff $C_I$ is disconnected and $H^1(\mathcal{L})$ is nontrivial iff $C_{\{1, \ldots, n\}\setminus I}$ is disconnected.

In order to state the main results of this paper, we need to associate to any $\Sigma$–piecewise linear function $\psi$ on $N_\mathbb{R}$ a convex polytope $\Lambda_{\psi}$ in the character space $M_\mathbb{R} = M \otimes \mathbb{R}$.

Definition 2.12. For each maximal dimensional cone $\sigma \in \Sigma(m)$, let $\psi_\sigma$ be the linear function on $N_\mathbb{R}$ such that $\psi_\sigma = \psi$ in cone $\sigma$. We define $\Lambda_{\psi} \subset M_\mathbb{R}$ to be the convex hull of the set $\{ \psi_\sigma | \sigma \in \Sigma(m) \}$.

Our first main result whose proof is given in the next section is a combinatorial condition for toric DM stacks in any dimension to have infinitely many $H$–trivial line bundles.

Theorem 2.13. Let $\mathbb{P}_\Sigma$ be a proper smooth dimension $m$ toric DM stack associated to a complete stacky fan $\Sigma = (\Sigma, \{v_i\}_{i=1}^n)$. If there is a $\Sigma$–piecewise linear function $\psi$ such that $\dim(\Lambda_{\psi}) < m$, then there are infinitely many $H$–trivial line bundles on $\mathbb{P}_\Sigma$.

3. Proof of the first main result

In this section, we give the proof of Theorem 2.13. We start by describing a key method of constructing infinitely many $H$–trivial line bundles.

Proposition 3.1. Let $\mathbb{P}_\Sigma$ be a proper smooth dimension $m$ toric DM stack associated to a complete stacky fan $\Sigma = (\Sigma, \{v_i\}_{i=1}^n)$. If there is a non-linear $\Sigma$–piecewise linear function $\psi$ and a vertex $v \in \Sigma(1)$ such that $\psi$ is constant on all lines parallel to $v$, then there are infinitely many $H$–trivial line bundles on $\mathbb{P}_\Sigma$.

In order to prove Proposition 3.1 we need several lemmas. Let $\sum_{i \in I} a_i E_i = \sum_{i \in I} a_i E_i + \sum_{i \in I^c} a_i E_i$ be an element in $\text{Pic}(\mathbb{P}_\Sigma)$, where $I = \{i | a_i \geq 0\}$ and $I^c = \{1, \ldots, n\} \setminus I = \{i | a_i < 0\}$. Let $\varphi$ be a $\Sigma$–piecewise linear function on $\mathbb{R}^m$ such that $\varphi(v_i) = a_i$.

Lemma 3.2. Let $S_{\varphi<0} = \{v \in (N_\mathbb{R} \setminus \{0\})/\mathbb{R}_{>0}|\varphi(v) < 0\}$ and $I^c = \{i | \varphi(v_i) < 0\}$, then $C_{I^c}$ is homotopic to $S_{\varphi<0}$. Here we consider $C_{I^c}$ to be the geometric realization of an abstract simplicial complex $[\mathcal{I}]$.

Proof. We think of $C_{I^c}$ as a subspace of topological space $S_{\varphi<0}$ with the inclusion $C_{I^c} \hookrightarrow S_{\varphi<0}$ given by

$$\sum_{i \in J \subseteq C_{I^c}} \sum_{\lambda_i = 1} \lambda_i v_i \mapsto \mathbb{R}_{>0} (\sum_{i \in J \subseteq C_{I^c}} \sum_{\lambda_i = 1} \lambda_i v_i ).$$
Now we consider a map \( F : S_{\varphi<0} \times [0,1] \to S_{\varphi<0} \) which is defined as follows. For any point \( a \in S_{\varphi<0} \), let \( \sigma \) be a cone of \( \Sigma \) that contains \( a \). We can write \( a = \sum_{i \in I_{\sigma}} \lambda_i v_i \), where \( I_{\sigma} = \{ i| v_i \in \sigma \} \) and all \( \lambda_i \geq 0 \). Then we define
\[
F(a,t) = \sum_{i \in I_{\sigma}, \varphi(v_i) < 0} \lambda_i v_i + \sum_{i \in I_{\sigma}, \varphi(v_i) \geq 0} (1-t) \lambda_i v_i.
\]
Assume we choose another cone \( \sigma' \) containing \( a \) and write \( a = \sum_{i \in I_{\sigma'}} \lambda'_i v_i \), where \( I_{\sigma'} = \{ i| v_i \in \sigma' \} \) and all \( \lambda'_i \geq 0 \). Then \( \lambda_i = 0 = \lambda'_i \) if \( v_i \notin \sigma \cap \sigma' \) and \( \lambda_i = \lambda'_i \) if \( v_i \in \sigma \cap \sigma' \). Thus the map \( F \) is well defined. Also the \( F \) is continuous since \( \lambda_i \) change continuously when the point \( a \) moves from one cone to another. We immediately see that \( F(a,0) = a \), \( F(a,1) \in S_{\varphi<0} \) for any \( a \in S_{\varphi<0} \) by definition of \( F \). Moreover \( F(a,t) = a \) for any \( a \in C_{I_0} \) and any \( t \in [0,1] \) since \( \varphi(v_i) < 0 \) for all \( i \in I_0 \) if \( a \in C_{I_0} \). Therefore \( F \) is a strong deformation retraction of topological space \( S_{\varphi<0} \) onto subspace \( C_{I_0} \).

**Lemma 3.3.** Let \( S_{\varphi \geq 0} = \{ v \in (N_R \setminus \{ 0 \})/\mathbb{R}_>0| \varphi(v) \geq 0 \} \) and \( I = \{ i| \varphi(v_i) \geq 0 \} \), then \( C_I \) is homotopic to \( S_{\varphi \geq 0} \).

**Proof.** The proof is the same as for Lemma 3.2. \( \square \)

**Lemma 3.4.** Let \( I \in \{ 1,2,\ldots,n \} \). Then we have \( H^\text{red}_{j-1}(C_I) = (H^\text{red}_{rkN-j-1}(C_{I'}))^\ast \). This implies \( C_I \) has nontrivial reduced homology if and only if \( C_{I'} \) has nontrivial reduced homology.

**Proof.** Since the sphere \( S^{rkN-1} \) is homeomorphic to \( (N_R \setminus \{ 0 \})/\mathbb{R}_>0 \), we have \( S^{rkN-1} = S_{\varphi \geq 0} \sqcup S_{\varphi < 0} \). By Alexander duality \([1]\), we have an isomorphism of reduced homology and reduced cohomology \( H^\text{red}_{j-1}(S_{\varphi \geq 0}) \cong H^\text{red}_{rkN-1-j}(S_{\varphi < 0}) \). Using the Universal Coefficient Theorem, we get \( H^\text{red}_{rkN-1-j}(S_{\varphi < 0}) = (H^\text{red}_{rkN-1-j}(S_{\varphi < 0}))^\ast \). Since \( C_I \) is homotopic to \( S_{\varphi \geq 0} \) by Lemma 3.3 and \( C_{I'} \) is homotopic to \( S_{\varphi < 0} \) by Lemma 3.2, we obtain \( H^\text{red}_{j-1}(C_I) = (H^\text{red}_{rkN-j-1}(C_{I'}))^\ast \). \( \square \)

We have the following corollary.

**Corollary 3.5.** The topological spaces \( \{ v \in N_R| \varphi(v) < 0 \} \) and \( C_{I'} \) are homotopic.

**Proof.** Since \( \{ v \in N_R| \varphi(v) < 0 \} = \{ v \in N_R \setminus \{ 0 \}| \varphi(v) < 0 \} \) is a fibration with fibre \( \mathbb{R}_>0 \) over \( \{ v \in (N_R \setminus \{ 0 \})/\mathbb{R}_>0| \varphi(v) < 0 \} \), these spaces are homotopic to each other. Then we use Lemma 3.2. \( \square \)

For some fixed \( v \in \Sigma(1) \), we consider all lines parallel to \( v \). The parametric equation of such a line is \( l(t) := l(0) + tv \) for some \( l(0) \in N_R \), where \( t \in \mathbb{R} \).

**Lemma 3.6.** Let \( \varphi \) be a \( \Sigma \)-piecewise linear function on \( N_R \) and \( l(t) := l(0) + tv \) be the parametric equation of the line parallel to \( v \). Then for any point in the interior of the region corresponding to \( \sigma \in \Sigma(m) \), the derivative of the function \( \psi(l(t)) \) equals \( \varphi_{|\sigma}(v) \).
Proof. Left to the reader. □

Corollary 3.7. For a nonzero $v \in N_\mathbb{R}$, we have $\psi_\sigma(v) = 0$ for all cones $\sigma \in \Sigma(m)$ if and only if $\psi$ is constant on all lines parallel to $v$.

Proof of Proposition 3.1. Without loss of generality, we assume $v = v_1$.

We claim that $\mathcal{L} = \mathcal{O}(\sum_{i=2}^{n} \psi(v_i)E_i - E_1)$ is $H$–trivial.

Let $a_i$ be the coefficient of $E_i$ in $\sum_{i=2}^{n} \psi(v_i)E_i - E_1$. By Remark 2.4, we have

$$H^1(\mathbb{P}_\Sigma, \mathcal{L}) = \bigoplus_{f \in M} H^\text{red}_{kN-j-1}(\text{Supp}(r_f)),$$

where $r_f = (a_i + f(v_i))_{i=1}^{n}$. In order to show $H^*(\mathbb{P}_\Sigma, \mathcal{L}) = 0$, it is sufficient to show $\text{Supp}(r_f)$ is contractible for each $f \in M$. Let $\sum_{i=1}^{n} r_i E_i = \sum_{i=2}^{n} \psi(v_i)E_i - E_1 + f(v_1)E_1$. We know that $r_i = (f + \psi)(v_i)$ for $i = 2, \ldots, n$ and $r_1 = f(v_1) - 1$. Let $I = \{ i | r_i \geq 0 \}$ and $I^c = \{ 1, \ldots, n \} \setminus I$.

Case $r_1 \neq -1$. We have $f(v_1) \neq 0$. We regard $\geq 0$ and $< 0$ as different signs. Since $f(v_1)$ is an integer, $f(v_1) - 1$ is non-negative only when $f(v_1) \geq 1$ and $f(v_1) - 1$ is negative only when $f(v_1) < 0$. Thus the sign of $r_i$ is the same as the sign of $(f + \psi)(v_i)$ for all $i \in \{ 1, 2, \ldots, n \}$. Let $\varphi$ be the $\Sigma$–piecewise linear function $\varphi$ such that $\varphi(v_i) = (f + \psi)(v_i)$ for all $i \in \{ 1, 2, \ldots, n \}$. For any line $l(t) = l(0) + tv_1$ parallel to $v_1$, we have $\varphi(l(t)) = f(l(0) + tv_1) + \psi(l(t)) = f(v_1)t + c_l$ for some constant value $c_l$ since $\psi$ is constant on all lines parallel to $v_1$. So for any line $l(t)$ parallel to $v_1$, if $f(v_1) < 0$, $\varphi(l(t))$ is negative when $t > 0$ is sufficiently large. If $f(v_1) > 0$, $\varphi(l(t))$ is negative when $t < 0$ and $|t|$ is sufficiently large. This implies $\{ v \in N_\mathbb{R} | \varphi(v) < 0 \}$ is contractible. So $C_{I^c}$ is contractible by Lemma 3.5. Thus $\text{Supp}(r_f) = C_I$ has trivial reduced homology by Lemma 3.4.

Case $r_1 = -1$. We have $f(v_1) = 0$. Let $\overline{\varphi}$ be the $\Sigma$–piecewise linear function such that $\overline{\varphi}(v_i) = (f + \psi)(v_i)$ for $i \in \{ 2, \ldots, n \}$ and $\overline{\varphi}(v_1) = -1$. Let $l(t) = l(0) + tv_1$ be any line parallel to $v_1$. There exists $t_0$ such that $l(t)$ lies in interior of a cone $\sigma_0$ with the ray $v_1$ if and only if $t > t_0$. By Lemma 3.6, the derivative of $\psi(l(t))$ equals 0 since $\psi$ is constant on all lines parallel to $v_1$. For any point in the interior of the region corresponding to $\sigma \neq \sigma_0$, the derivative of the function $\overline{\varphi}(v)$ equals $\overline{\varphi}_\sigma(v) = \psi_\sigma(v) + f(v) = \psi_\sigma(v) = 0$. For any point in the interior of the region corresponding to $\sigma_0$, the derivative of the function $\overline{\varphi}(l(t))$ equals $\overline{\varphi}_{\sigma_0}(v) = -1$. Thus $\varphi$ is constant on $l(t)$ for $t \leq t_0$ and the derivative of the function at $l(t)$ is $\overline{\varphi}_{\sigma_0}(v) = -1$ for $t > t_0$. So $\overline{\varphi}(l(t))$ is negative when $t > t_0$ is sufficiently large, or for all $t$. Thus $\{ v \in N_\mathbb{R} | \overline{\varphi}(v) < 0 \}$ is contractible. Since $r_i = \overline{\varphi}(v_i)$, so $I^c = \{ i | \overline{\varphi}(v_i) < 0 \}$. Then $C_{I^c}$ is contractible by Lemma 3.5. Thus $\text{Supp}(r_f) = C_I$ has trivial reduced homology by Lemma 3.4

Since the same argument applies to $r\psi$ instead of $\psi$, we have for any $r \in \mathbb{Z}$ $\mathcal{L} = \mathcal{O}(\sum_{i=2}^{n} r\psi(v_i)E_i - E_1)$ is $H$–trivial. These line bundles are different from one another because $\psi$ is non-linear. Thus there are infinitely many $H$–trivial line bundles on $\mathbb{P}_\Sigma$. □
In order to prove Theorem 2.13 it suffices to show that $\dim \Lambda_\psi < m$ implies that there exists $v \in \Sigma(1)$ such that $\langle \Lambda_\psi, v \rangle = c$ for a constant $c$.

**Proposition 3.8.** $\dim \Lambda_\psi < m$ if and only if there exists $v \in \Sigma(1)$ such that $\langle \Lambda_\psi, v \rangle = c$ for a constant $c$.

**Proof.** We have that $\dim \Lambda_\psi < m$ if and only if $\Lambda_\psi$ is inside an affine hyperplane in $M_\mathbb{R}$. That is equivalent to say there exists $v \in N_\mathbb{R}$ such that $\langle \Lambda_\psi, v \rangle = c$, where $c$ is a constant. The essence of this proposition is that $v$ may be chosen in $\Sigma(1)$. We adjust $\psi$ by adding a linear function in $M$ such that $\psi_\sigma(v) = 0$ for all cones $\sigma \in \Sigma(m)$ and $\psi = 0$ on a cone $\tau$ which contains $v$ in its interior.

First, we claim that for every $\sigma \in \Sigma(m)$, there is a $\tilde{\sigma} \in \text{Star}(\tau)(m) = \{ \sigma \in \Sigma(m) | \tau \subseteq \sigma \}$ such that $\psi_\sigma = \psi_{\tilde{\sigma}}$. We consider the projection $pr : N_\mathbb{R} \to N_\mathbb{R}/\mathbb{R}v$ and choose a maximum-dimensional cone $\tilde{\sigma} \in \text{Star}(\tau)(m)$ such that $pr(\sigma)$ and $pr(\tilde{\sigma})$ have overlapping interiors. Let $D = pr(\sigma) \cap pr(\tilde{\sigma})$, see Figure 2. There are linear functions $\varphi_{pr(\sigma)}$ and $\varphi_{pr(\tilde{\sigma})}$ on $N_\mathbb{R}/\mathbb{R}v$ such that $\psi_\sigma = \varphi_{pr(\sigma)} \circ pr$ and $\psi_{\tilde{\sigma}} = \varphi_{pr(\tilde{\sigma})} \circ pr$.

For any $p \in D$, we pick a point $q_1$ in $\sigma$ such that $pr(q_1) = p$ and a point $q_2$ in $\tilde{\sigma}$ such that $pr(q_2) = p$. Since $q_1$ and $q_2$ are on a line parallel to $v$, by Lemma 3.7 we get $\psi(q_1) = \psi(q_2)$. Since $\psi_\sigma(q_1) = \psi(q_1)$ and $\psi_{\tilde{\sigma}}(q_2) = \psi(q_2)$, we have $\varphi_{pr(\sigma)}(p) = \psi_\sigma(q_1) = \psi_{\tilde{\sigma}}(q_2) = \varphi_{pr(\tilde{\sigma})}(p)$.

So we get $\varphi_{pr(\sigma)} = \varphi_{pr(\tilde{\sigma})}$ on $D$ which is a full-dimensional set. This implies $\varphi_{pr(\sigma)} = \varphi_{pr(\tilde{\sigma})}$ on $N_\mathbb{R}/\mathbb{R}v$. Thus $\psi_\sigma = \psi_{\tilde{\sigma}}$.

![Figure 2](image_url)

We choose a vertex $w$ of the cone $\tau$. For any maximum-dimensional cone $\tilde{\sigma}$ in $\text{Star}(\tau)$, we have $\psi_{\tilde{\sigma}}(w) = \psi(w) = 0$ since $w \in \tau \subset \tilde{\sigma}$. Then for all cones $\sigma \in \Sigma(m)$, we obtain $\psi_\sigma(w) = \psi_{\tilde{\sigma}}(w) = 0$. □

Now we are ready to prove Theorem 2.13.
4. Dimension three case

In this section we focus on proper smooth dimension three toric DM stacks \( \mathbb{P}_\Sigma \) associated to a complete stacky fan \( \Sigma = (\Sigma, \{v_i\}_{i=1}^n) \) in \( N \) of rank three. We give a criterion for when there exist infinitely many \( H \)-trivial line bundles on \( \mathbb{P}_\Sigma \) for smooth toric varieties and DM stacks in dimension three under the assumption that there is no more than one pair of collinear rays in \( \Sigma \).

The following lemma highlights the importance of diagonals in \( \Sigma \), i.e., pairs \((s, t)\) such that \( v_s \) and \( v_t \) are collinear.

**Lemma 4.1.** Let \( \mathcal{L} = \sum_{i=1}^n f(v_i)E_i \) be an element in \( \text{Pic}_\mathbb{R}(\mathbb{P}_\Sigma) \), where \( f \) is a \( \Sigma \)-piecewise linear function on \( N_R \). Let \( s \in \{1, 2, \ldots, n\} \). There exists a \( \Sigma \)-piecewise linear function \( g \) on \( N_R \) such that

- \( \mathcal{L} = \sum_{i=1}^n g(v_i)E_i \),
- \( g(v_s) = 0 \),
- \( g(v_i) \neq 0 \) for all \( v_i \notin \mathbb{R}v_s \),
- If \( v_t \in \mathbb{R}v_s, t \neq s \), then \( g(v_t) = 0 \) if and only if \( f \) is linear on \( \mathbb{R}v_s \).

**Proof.** We consider \( g(v) = f(v) - m \cdot v \), where \( m \) is generic element in the affine plane \( \{m \in M_\mathbb{R} | m \cdot v_s = f(v_s)\} \). We immediately get \( \mathcal{L} = \sum_{i=1}^n g(v_i)E_i \) and \( g(v_s) = 0 \).

The fact that \( m \) is chosen generic means that \( m \cdot v_i \neq f(v_i) \) for all \( i \) except when \( v_i \in \mathbb{R}v_s \). If \( v_t \in \mathbb{R}v_s \), then \( g \) is linear on \( \mathbb{R}v_s \) iff \( g(v_t) = 0 \), and \( g \) is linear on \( \mathbb{R}v_s \) iff \( f \) is linear on \( \mathbb{R}v_s \). \( \square \)

**Definition 4.2.** For a point \( v_i \in \{v_i\}_{i=1}^n \), we denote the neighborhood of \( v_i \) to be

\[ B_i = \{v_j \in \{v_1, \cdots, v_n\} \mid v_j \text{ and } v_i \text{ span a two-dimensional cone of } \Sigma \}. \]

Let \( v_{j_1}, v_{j_2}, \ldots, v_{j_l} \) be all the vectors in \( B_i \) which are ordered clockwise. Let \( g \) be a \( \Sigma \)-piecewise linear function on \( N_R \). We regard \( \geq 0 \) and \( < 0 \) as different signs. We count the number of pairs of vectors \( \{v_{j_k}, v_{j_{k+1}}\} \subset \{v_{j_1}, \ldots, v_{j_l}\} \) such that \( f(v_{j_k}) \) and \( f(v_{j_{k+1}}) \) have different signs. We call it the number of sign changes of \( f \) among \( v_{j_1}, \ldots, v_{j_l} \). For example, there are exactly two sign changes in \( B_i \) in Figure 3.
Recall that in Definition 2.8 the cones $Z_I$ are shifts of the forbidden cones $FC_I$.

**Lemma 4.3.** Let $\mathcal{L} = \sum_{i=1}^{n} g(v_i)E_i$ be a nonzero element in $\text{Pic}(\mathbb{P}_\Sigma)$ which is not contained in the interior of $Z_I$ for any $I \in \Delta$, where $g$ is a $\Sigma$-piecewise linear function on $N_\mathbb{R}$. Assume there is some $s \in \{1, 2, \ldots, n\}$ such that $g(v_s) = 0$ and $g(v_i) \neq 0$ for all $i \neq s$. Then there exist exactly two sign changes in $B_s$, see Figure 3.

**Proof.** The main idea is that one can perturb $g$ by a linear function to achieve $g(v_s) > 0$ or $g(v_s) < 0$ without changing signs of other $g(v_i)$. Let $C > 0$ be the simplicial complex $C \{i|g(v_i) > 0\}$ and $C \geq 0$ be the simplicial complex $C \{i|g(v_i) \geq 0\}$. The reduced simplicial homology complex $W(C > 0)$ is as follows:

![Diagram](image)

The reduced simplicial homology complex $W(C \geq 0)$ is as follows:

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We use inclusions $\bigoplus_{|J|=k} C \hookrightarrow \bigoplus_{|J|=k} C$ for each $k$ to obtain an exact sequence of complexes:

$$0 \to \bigoplus_{|J|=3} C \to \bigoplus_{|J|=2} C \to \bigoplus_{|J|=1} C \to C \to 0$$

$$0 \to \bigoplus_{|J|=3} C \to \bigoplus_{|J|=2} C \to \bigoplus_{|J|=1} C \to C \to 0$$

We use inclusions $\bigoplus_{|J|=k} C \hookrightarrow \bigoplus_{|J|=k} C$ for each $k$ to obtain an exact sequence of complexes:

$$0 \to W(C \geq 0) \to W(C \geq 0) \to W(C \geq 0)/W(C > 0) \to 0$$

We can take a linear function $f$ such that $f(v_s) > 0$ and $|f(v_i)| < |g(v_i)|$ for all $i \neq s$. Let $g' = g + f$, we have $g'(v_i) = (g + f)(v_i) \neq 0$ for $i \in \{1, 2, \ldots, n\}$. Thus

$$\mathcal{L} = \sum_{i \in I > 0} \alpha_i E_i + \sum_{i \notin I > 0} \alpha_i E_i$$

in $\text{Pic}_\mathbb{R}(\mathbb{P}_\Sigma)$, where $\alpha_i = g'(v_i) \neq 0$ for all $i$ and $I > 0 = \{i|g'(v) > 0\}$. Also we have $C \geq 0 = C_{I > 0}$. Since $\mathcal{L}$ is not contained in the interior of $Z_I$ for any $I \in \Delta$, we get $I > 0 \notin \Delta$. This implies $C \geq 0 = C_{I > 0}$ has trivial reduced homology.
Similarly, we can take a linear function $f'$ such that $f'(v_s) < 0$ and $|f'(v_i)| < |g(v_i)|$ for all $i \neq s$ to show that $C_{>0}$ has trivial reduced homology.

Since $W(C_{>0})$ and $W(C_{>0})$ have trivial homology, by snake lemma and the exact sequence (4.1), the complex $W(C_{>0})/W(C_{>0})$ has trivial homology. Thus the Euler characteristic $\chi(W(C_{>0})/W(C_{>0})) = 0$. If $g(v_i) > 0(0 < 0)$ for all $v_i \in B_s$, the Euler characteristic $\chi(W(C_{>0})/W(C_{>0})) = 1$ by computing the alternating sum of numbers of $k$–cells, which leads to contradiction. Thus there exists at least two sign changes in $B_s$. By computing the alternating sum of numbers of $k$–cells, the Euler characteristic $\chi(W(C_{>0})/W(C_{>0}))$ equals

$$1 - \text{ the number of connect components of positive sign in } B_s.$$ 

Thus there exists only one component of positive sign in $B_s$. This implies that there are exactly two sign changes in $B_s$. \hfill \square

Let $\mathcal{L} = \sum_{i=1}^n g(v_i)E_i$ be an element in Pic$(\mathbb{P}_\Sigma)$ which is not contained in the interior of $Z_I$ for any $I \in \Delta$, where $g$ is a $\Sigma$–piecewise linear function on $N_\mathbb{R}$. Assume there is some $s \in \{1, 2, \ldots, n\}$ such that $g(v_s) = 0$. Then we can consider the projection $\pi : N_\mathbb{R} \to \mathbb{R}^2 = N_\mathbb{R}/\langle v_s \rangle$ along the line $\mathbb{R}v_s$.

Under the projection, the neighborhood $B_s$ gives a two-dimensional complete stacky fan $\Sigma' = (\Sigma', \{w_i\}_{i \in J})$, where $w_i = \pi(v_i)$, $v_i \in B_s$ and $J = \{i \in \{1, 2, \ldots, n\}|v_i \in B_s\}$. We can think of $g$ as a $\Sigma'$–piecewise linear function on $\mathbb{R}^2 = N_\mathbb{R}/\langle v_s \rangle$ since $g(v_s) = 0$. Let $\{E'_i\}_{i \in J}$ be the generators of Pic$(\mathbb{P}_{\Sigma'})$. Let $\Delta' = \{\emptyset, J\} \cup \{I' \subset J|C'_{I'} \text{ is disconnected}\}$. And for any $I' \in \Delta'$, let $Z'_{I'} = \sum_{i \in I'} \mathbb{R}_{>0}E'_i - \sum_{i \in J'} \mathbb{R}_{>0}E'_i$ be the cone associated to $I'$ with vertex at the origin which is obtained by shifting the forbidden cone $FC'_I \subseteq \mathbb{P}_{\Sigma'}$.

**Lemma 4.4.** The point $\mathcal{D} = \sum_{i \in J} g(v_i)E'_i \in \text{Pic}(\mathbb{P}_{\Sigma'})$ is not contained in the interior of $Z'_{I'}$ for any $I' \in \Delta'$.

**Proof.** Assume $\mathcal{D}$ is contained in $Z'_{I'}$, for some $I' \in \Delta'$. This implies that there exists a linear function $f$ on $\mathbb{R}^2 = \mathbb{R}^3/\langle v_s \rangle$ such that $f(w_i) + g(v_i) \neq 0$ for any $i \in J$ and $I' = \{i \in J|f(w_i) + g(v_i)| \}$ has non-trivial reduced homology. This would mean that the number of sign changes in $B_s$ is not two, which contradicts Lemma 4.3 \hfill \square

**Lemma 4.5.** Let $\mathcal{L} = \sum_{i=1}^n g(v_i)E_i$ be an element in Pic$(\mathbb{P}_\Sigma)$ which is not contained in the interior of $Z_I$ for any $I \in \Delta$, where $g$ is a $\Sigma$–piecewise linear function on $N_\mathbb{R}$. Assume $v_s$ is such that there is no $v_t$ with $t \neq s$, $\mathbb{R}v_s = \mathbb{R}v_t$; or $g$ is not linear on $\mathbb{R}v_s$, then

- either $g$ is linear on $\text{Star}(v_s) = \{\sigma \in \Sigma|v_s \subseteq \sigma\}$
- or $g|_{\text{Star}(v_s)}$ has two half spaces of linearity, i.e., there exists $v_p, v_q \in B_s$ such that $v_p, v_s, v_q, 0$ are coplanar and $g|_{\text{Star}(v_s)}$ is linear on either side of this plane, see Figure 4.
**Proof.** By Lemma 4.1 we can assume \( g(v_s) = 0 \). Then by Lemma 4.4 we have \( D = \sum_{i \in J} g(v_i)E'_i \in \text{Pic}(\mathbb{P}_\Sigma) \) not in the interior of \( Z'_I \) for any \( I \in \Delta' \). If \( D = 0 \) in \( \text{Pic}_R(\mathbb{P}_\Sigma) \), then \( g \) is linear on \( \text{Star}(v_s) \). Otherwise, by Lemma 4.1 in paper [17] \( g|_{\text{Star}(v_s)} \) is a pullback of a piecewise linear function on \( N_R/\mathbb{R}v_s \) with two half plane regions of linearity. Thus \( g|_{\text{Star}(v_s)} \) has two half spaces of linearity, see Figure 4. \qed

**Proposition 4.6.** Assume that \( v_i \) and \( v_j \) are not collinear for any \( \{i, j\} \subset \{1, 2, \ldots, n\} \) and there are infinitely many \( H \)-trivial line bundles on \( \mathbb{P}_\Sigma \). Then there exists a plane which does not intersect with the interior of any maximal cone of \( \Sigma \).

**Proof.** Since there are infinitely many \( H \)-trivial line bundles on \( \mathbb{P}_\Sigma \), by the argument of proof in Proposition 3.8 in [17], there exists a non-zero element \( \mathcal{L} = \sum_{i=1}^n g(v_i)E_i \in \mathbb{P}_\Sigma \) which is not contained in any \( Z_I \) for \( I \in \Delta \), where \( g \) is a \( \Sigma \)-piecewise linear function on \( N_R \).

Since \( \mathcal{L} \neq 0 \), there exists a vertex \( v_s \) such that \( g \) is not linear in \( \text{Star}(v_s) \). By Lemma 4.5 \( g \) has two half planes of linearity in \( \text{Star}(v_s) \). The line breaking the linearity corresponds to a plane in \( \mathbb{R}^3 \) which we denote by \( H_0 \). We know that \( H_0 \) passes through the vector \( v_s \), the origin \( 0 \) and a point in \( B_s \) which we denote by \( v_{s_1} \). Also, \( g \) is linear on each side of plane \( H_0 \) in \( \text{Star}(v_s) \). Thus \( H_0 \) does not intersect with interior of any maximal cones of \( \Sigma \) with \( v_s \) as a ray. By assumption, we know \( v_{s_1} \) and \( v_i \) is not collinear.
for any $i \neq s_1$. By Lemma 4.7 and the same argument for $v_s$, we get a plane $H_1$ passing through $v_{s_1}$ such that $g$ is linear on each side of plane $H_1$ in Star($v_s$). Thus $H_1$ equals $H_0$ and does not intersect with interior of any maximal cones of $\Sigma$ with $v_{s_1}$ as ray. We continue the process, there exits a positive number $k$ such that $v_{s_k} = v_s$. We get a plane $H = H_i$ for $i = 0, \cdots, k$ which does not intersect with interior of any maximal cones of $\Sigma$. See Figure 5.

**Theorem 4.7.** Let $\mathbb{P}_\Sigma$ be a proper smooth dimension three toric DM stack associated to a complete stacky fan $\Sigma = (\Sigma, \{v_i\}_{i=1}^n)$. Assume there exist no collinear pairs of rays in $\Sigma$ and there are infinitely many $H$–trivial line bundles on $\mathbb{P}_\Sigma$. Then there is a $\Sigma$–piecewise linear function $\psi$ such that $\dim(\Lambda_\psi) < 3$.

**Proof.** By Proposition 4.6, there is a plane $H$ which does not intersect with the interior of any maximal cone of $\Sigma$ and contains $v_s$ for some $s \in \{1, \ldots, n\}$. Then we have a projection $\pi : N_\mathbb{R} \to N_\mathbb{R}/R H \cong \mathbb{R}^1$ along this plane $H$. We pick a function $f$ which is linear along two half lines of $\mathbb{R}^1 = N_\mathbb{R}/R H$ and $f(0) = 0$. We get $\pi^*(f) = f \circ \pi$ is a $\Sigma$–piecewise linear function on $\mathbb{R}^3 = N_\mathbb{R}$ since the preimage of origin in $N_\mathbb{R}/R H \cong \mathbb{R}^1$ is $H$ which does not intersect with interior of any maximal cones of $\Sigma$. Let $\psi = \pi^*(f)$. We have $\psi = \pi^*(f)$ is zero on $H$ since $\pi(H) = 0$. Also $\psi = \pi^*(f)$ is constant on any plane parallel $H'$ to $H$ since image of origin in $H'$ under $\pi$ is one point in $N_\mathbb{R}/R H$. For any line parallel to $Rv_s$ is on a plane parallel to $H$, so $\psi$ is constant on any line parallel to $Rv_s$. Then by Proposition 3.8 and Corollary 3.7, we know $\dim(\Lambda_\psi) < 3$. In fact, in this case $\dim(\Lambda_\psi) = 1$. □

Now we consider the case when there exists exactly one collinear pair in $\{v_1, \ldots, v_n\}$.

**Lemma 4.8.** Assume that $v_s$ and $v_t$ are the only collinear pair and there are infinitely many $H$–trivial line bundles on $\mathbb{P}_\Sigma$. Then either there is a plane which does not intersect with the interior of any maximal cones of $\Sigma$ or there exist at least three half planes which do not intersect with the interior of any maximal cones of $\Sigma$ passing through $Rv_s$ and three vectors in $B_s$ respectively.

**Proof.** Since there are infinitely many $H$–trivial line bundles on $\mathbb{P}_\Sigma$, by the argument of proof of Proposition 3.8 in [17], there exists a non-zero element $L = \sum_{i=1}^n g(v_i)E_i \in \mathbb{P}_\Sigma$ which is not contained in any $Z_I$ for $I \in \Delta$, where $g$ is a $\Sigma$–piecewise linear function on $\mathbb{R}^3$.

If $g$ is not linear along the line $Rv_s$, then by Lemma 4.1 it is the same case as the one without a collinear pair. By same argument as in Proposition 4.6, there is a plane which does not intersect with interior of any maximal cones of $\Sigma$.

Now we consider the case $g$ is linear along the line $Rv_s = Rv_t$. If $g$ has two half planes of linearity in Star($v_s$) and Star($v_t$), with same argument in the case without collinear pairs in Proposition 4.6, we get that there is
a plane which does not intersect with the interior of any maximal cones of \( \Sigma \) with \( v_s \) as a ray. Thus without loss of generality, we assume \( g \) has at least three regions of linearity in \( \text{Star}(v_s) \). Thus at least three vectors in \( B_s \) break the linearity. Thus in \( B_s \), we can pick three \( v_p, v_q \) and \( v_l \) such that \( g \) is linear in the cone span by \( \{v_s, v_p, v_q\} \) and linear in the cone span by \( \{v_s, v_q, v_l\} \). This implies the two-dimensional cone \( C_{sq} \) spanned by \( \{v_s, v_q\} \) break the linearity.

Now we consider \( v_p \). By assumption, we know \( v_p \) and \( v_i \) is not collinear for any \( i \neq p \). By Lemma 4.1 and the same argument for in Proposition 4.6 we get a plane \( H_0 \) passing through \( v_p \) such that \( \psi \) is linear on each side of plane \( H_0 \) near \( v_p \). Thus \( H_0 \) is on the same plane as the cone \( C_{sp} \) spanned by \( \{v_s, v_p\} \) and does not intersect with interior of any maximal cones of \( \Sigma \) with \( v_p \) as ray. We continue the process, there exits a positive number \( k \) such that \( v_{pk} = v_l \). We get a half plane \( H_p \) which does not intersect with interior of any maximal cones of \( \Sigma \) and passes through \( v_s \).

Similarly, we get half plane \( H_q \) and \( H_l \) which do not intersect with interior of any maximal cones of \( \Sigma \) and passes through \( v_s \). See Figure 6.

**Theorem 4.9.** Let \( \mathbb{P}_\Sigma \) be a proper smooth dimension three toric DM stack associated to a complete stacky fan \( \Sigma = (\Sigma, \{v_i\}_{i=1}^n) \). Assume there exist only one collinear pair of rays in \( \Sigma \) and there are infinitely many \( \mathbb{H} \)-trivial line bundles on \( \mathbb{P}_\Sigma \). Then there is a \( \Sigma \)-piecewise linear function \( \psi \) such that \( \dim(\Lambda_\psi) < 3 \).

**Proof.** Let \( v_s \) and \( v_l \) be the only collinear pair. By assumption and Proposition 4.8 either there exists a projection \( \pi : \mathbb{R}^3 \to \mathbb{R}^1 \) along a plane which does not intersect with the interior of any maximal cones of \( \Sigma \) or there are three vectors \( v_p, v_q, v_l \in B_s \) and a projection \( \rho : \mathbb{R}^3 \to \mathbb{R}^2 \) along the line \( \mathbb{R}v_s \) such the \( H_i = \rho^{-1}(\rho(v_i)) \) is a half plane which does not intersect with interior of any maximal cones of \( \Sigma \) for each \( i = p, q, l \). In former case, with the same
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argument in Theorem 4.7, there is a $\Sigma$−piecewise linear function which is constant on all lines parallel to $\mathbb{R}v_s$.

Let $w_i = \rho(v_i)$ for each $i = p, q, l$. In latter case, we pick a function $f$ on $\mathbb{R}^2$ which is linear respectively on the cone $C_{pq}$ spanned by $\{w_p, w_q\}$ and the cone $C_{lp}$ spanned by $\{w_l, w_p\}$. Let $\psi = \rho^*(f)$, see Figure 6. We get $\psi = \rho^*(f)$ is a $\Sigma$−piecewise linear function on $\mathbb{R}^3$ since the preimage of $\rho(v_i)$ is $H_i$ which does not intersect with interior of any maximal cones of $\Sigma$ for each $i = p, q, l$. Also, $\psi$ is a $\Sigma$−piecewise linear function which is constant on any line parallel to $v_s$ since the image of any line under $\rho$ is a point. Then Corollary 3.7 and Proposition 3.8 imply the result.

Putting all of it together, we get our second main result.

**Theorem 4.10.** Let $\mathbb{P}_\Sigma$ be a proper smooth dimension three toric DM stack associated to a complete stacky fan $\Sigma = (\Sigma, \{v_i\}_{i=1}^n)$. Assume there exists no more than one collinear pair of rays in $\Sigma$. Then there are infinitely many $H$−trivial line bundles on $\mathbb{P}_\Sigma$ if and only if there is a $\Sigma$−piecewise linear function $\psi$ such that $\dim(\Lambda_\psi) < 3$.

**Proof.** Theorem 4.7, Theorem 4.9 and Theorem 2.13 imply the result. □

5. Comments

In this section, we state the conjecture in full generality, i.e., without a limit of number of collinear pairs of rays in $\Sigma$ and for arbitrary dimension.

For a proper smooth dimension $m$ toric DM stack $\mathbb{P}_\Sigma$, we have the following three statements:

1. There is a $\Sigma$−piecewise linear function $\psi$ such that $\dim(\Lambda_\psi) < m$.
2. There are infinitely many $H$−trivial line bundles on $\mathbb{P}_\Sigma$.
3. There is a nonzero element $L \in \text{Pic}(\mathbb{P}_\Sigma)$ which is not contained in the interior of $Z_I$ for any $I \in \Delta$.

By Theorem 2.13, (1) implies (2) without a limit of dimension $m$ and the number of collinear pairs of rays in $\Sigma$. By the argument of [17], (2) also implies (3) in full generality. By Theorem 4.10, (2) implies (1) when $m = 3$ and there exist no more than one collinear pair of rays in $\Sigma$. Thus we are inspired to explore the the following two conjectures which are stated in full generality.

**Conjecture 5.1.** For $m \in \mathbb{N}$, let $\mathbb{P}_\Sigma$ be a proper smooth dimension $m$ toric DM stack associated to a complete stacky fan $\Sigma = (\Sigma, \{v_i\}_{i=1}^n)$. Then there are infinitely many $H$−trivial line bundles on $\mathbb{P}_\Sigma$ if and only if there is a $\Sigma$−piecewise linear function $\psi$ such that $\dim(\Lambda_\psi) < m$.

This conjecture means that all these three statements (1), (2), (3) are equivalent to each other. The following conjecture is a weaker version which means (2) is equivalent to (3).
Conjecture 5.2. For $m \in \mathbb{N}$, let $P_{\Sigma}$ be a proper smooth dimension $m$ toric DM stack associated to a complete stacky fan $\Sigma = (\Sigma, \{v_i\}_{i=1}^n)$. Then there are infinitely many $H$–trivial line bundles on $P_{\Sigma}$ if and only if there exists a nonzero element $L \in \text{Pic}(P_{\Sigma})$ which is not contained in the interior of $Z_I$ for any $I \in \Delta$.

We hope that the methods and the approach of this paper will be useful for settling these two conjectures.

References