ON H−TRIVIAL LINE BUNDLES ON TORIC DM STACKS OF DIM ≥ 3

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ABSTRACT. We study line bundles on smooth toric DM stacks $\mathbb{P}_{\mathbf{\Sigma}}$ of arbitrary dimension. A sufficient condition is given for when infinitely many line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$ have trivial cohomology. In dimension three, the sufficient condition is also a necessary condition in the case that $\mathbf{\Sigma}$ has no more than one pair of collinear rays.

CONTENTS

1. INTRODUCTION

Exceptional collections of line bundles on toric Deligne-Mumford stacks have attracted considerable interest over the years [5, 9–14]. Study of such exceptional collections leads one to consider line bundles with all cohomology groups equal to zero, such as $\mathcal{O}(-1), \ldots, \mathcal{O}(-n)$ on $\mathbb{CP}^n$. We call such line bundles H−trivial [17]. Paper [17] gives a combinatorial criterion for when a toric DM stack of dimension two possesses infinitely many such line bundles.

Theorem 1.1. *[17] Let $\mathbb{P}_{\mathbf{\Sigma}}$ be a proper smooth dimension two toric Deligne-Mumford stack associated to a complete stacky fan $\mathbf{\Sigma} = (\Sigma, \{v_i\}_{i=1}^n)$. Then there are infinitely many* H−*trivial line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$ if and only if there exists $\{i, j\} \subset \{1, 2, \cdots, n\}$ such that v_i and v_j are collinear.*

In this paper, we explore line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$ for smooth toric varieties and DM stacks in higher dimension. We associate with each $\mathbf{\Sigma}$−piecewise linear function ψ a convex polytope Λ_ψ in the lattice of characters, see Definition 2.12. We obtain a sufficient condition for when there exist infinitely many H−trivial line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$.

Theorem 2.13. Let $\mathbb{P}_{\mathbf{\Sigma}}$ be a proper smooth dimension m toric DM stack associated to a complete stacky fan $\mathbf{\Sigma} = (\Sigma, \{v_i\}_{i=1}^n)$. If there exists a $\mathbf{\Sigma}-$piecewise linear function ψ such that $\dim(\Lambda_\psi) < m$, then there are infinitely many H$-$trivial line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$.

Moreover, we also get a criterion for when there exist infinitely many H$-$trivial line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$ for smooth toric varieties and DM stacks in dimension three when there is no more than one pair of collinear rays in $\mathbf{\Sigma}$. [1]

Theorem 4.10. Let $\mathbb{P}_{\mathbf{\Sigma}}$ be a proper smooth dimension three toric DM stack associated to a complete stacky fan $\mathbf{\Sigma} = (\Sigma, \{v_i\}_{i=1}^n)$. Assume there exists no more than one pair of collinear rays in $\mathbf{\Sigma}$. Then there are infinitely many H$-$trivial line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$ if and only if there is a $\mathbf{\Sigma}-$piecewise linear function ψ such that $\dim(\Lambda_\psi) < 3$.

The paper is organized as follows. In section 2, we give an overview of smooth toric DM stacks, their Picard groups and the cohomology of line bundles on the stacks. Then we define forbidden cones and forbidden sets and state the first main result Theorem 2.13. Section 3 focuses on the proof of Theorem 2.13. We first exhibit an important way of producing infinitely many H$-$trivial line bundles in Proposition 3.1. Then we relate it to the existence of a $\mathbf{\Sigma}-$piecewise linear function ψ such that $\dim(\Lambda_\psi) < m$. In section 4, we consider the case when N has rank three. We prove a sufficient and necessary condition for the existence of infinitely many H$-$trivial line bundles under the assumption that there is no more than one pair of collinear rays in $\mathbf{\Sigma}$. Section 5 proposes two conjectures that would generalize our results.

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2. Line bundles on toric DM stacks and their cohomology

In this section, we introduce toric DM stacks $\mathbb{P}_{\mathbf{\Sigma}}$ and their Picard groups $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$, and describe the cohomology of line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$. We formulate our main results.

In order to refrain from the technicalities of the derived Gale duality of [3], we consider a lattice N which is a free abelian group of finite rank. Let Σ be a complete simplicial fan in N. We choose a lattice point v in each of the one-dimensional cones of Σ. If Σ has n one-dimensional cones, we get a complete stacky fan $\mathbf{\Sigma} = (\Sigma, \{v_i\}_{i=1}^n)$, see [3]. The toric DM stack $\mathbb{P}_{\mathbf{\Sigma}}$ associated to this stacky fan $\mathbf{\Sigma}$ is constructed in [3] as a stack version of the homogeneous coordinate ring construction of [4]. The description of

[1] The importance of pairs of collinear rays was already observed in the paper [15] in the context of vanishing of cohomology of divisorial sheaves on toric varieties.

line bundles on the DM stacks is analogous to the description of the Picard group that was given in [6, 8]. By [16], we know the line bundles on $\mathbb{P}_{\boldsymbol{\Sigma}}$ are in bijection with collections of integers, up to global linear functions, as described below.

Proposition 2.1. *The Picard group of $\mathbb{P}_{\boldsymbol{\Sigma}}$ is isomorphic to the quotient of $\mathbb{Z}^n$ with basis $\{E_i\}_{i=1}^n$ by the subgroup of elements of the form $\sum_{i=1}^n (w_i \cdot v_i) E_i$ for all w in the character lattice $M = N^*$.*

Proof. See [2]. □

Now we remind the reader how to calculate the cohomology of a line bundle $\mathcal{L}$ on $\mathbb{P}_{\boldsymbol{\Sigma}}$. For each $\mathbf{r} = (r_i)_{i=1}^n \in \mathbb{Z}^n$, we define $Supp(\mathbf{r})$ to be the abstract simplicial complex on n vertices $\{1, \ldots, n\}$ as follows

$$\begin{aligned} Supp(\mathbf{r}) =& \{J \subseteq \{1, \ldots, n\} | r_i \geq 0 \text{ for all } i \in J \\ & \text{and there exists a cone of } \Sigma \text{ containing all } v_i, i \in J\}. \end{aligned}$$

The following proposition gives a description of the cohomology of a linear bundle $\mathcal{L}$ on $\mathbb{P}_{\boldsymbol{\Sigma}}$ in terms of the reduced simplicial homology spaces of $Supp(\mathbf{r})$.

Proposition 2.2. *[2] Let $\mathcal{L} \in \mathrm{Pic}(\mathbb{P}_{\boldsymbol{\Sigma}})$. Then*

$$\mathrm{H}^j(\mathbb{P}_{\boldsymbol{\Sigma}}, \mathcal{L}) = \bigoplus \mathrm{H}^{red}_{rkN-j-1}(Supp(\mathbf{r})),$$

where the sum is over all $\mathbf{r} = (r_i)_{i=1}^n \in \mathbb{Z}^n$ such that $\mathcal{O}(\sum_{i=1}^n r_i E_i) \cong \mathcal{L}$.

Proof. See [2]. □

Remark 2.3. *We have $\mathrm{H}^0(\mathcal{L}) \neq 0$ if and only if there exists $\mathbf{r} \in \mathbb{Z}^n_{\geq 0}$ such that $\mathcal{O}(\sum_{i=1}^n r_i E_i) \cong \mathcal{L}$. Another extreme case is that $\mathrm{H}^{rk(N)}(\mathcal{L})$ only appears when the simplicial complex $Supp(\mathbf{r}) = \{\emptyset\}$, i.e. when $\mathcal{O}(\sum_{i=1}^n r_i E_i) \cong \mathcal{L}$ with all $r_i \leq -1$.*

Remark 2.4. *Let $\mathcal{L} \cong \mathcal{O}(\sum_{i=1}^n a_i E_i)$ be a line bundle in $\mathrm{Pic}(\mathbb{P}_{\boldsymbol{\Sigma}})$. Assume there is another expression $\mathcal{L} \cong \mathcal{O}(\sum_{i=1}^n r_i E_i)$. Then by Proposition 2.1, there exists an element $f \in M$ such that $r_i = a_i + f(v_i)$ for $i = 1, \ldots, n$, where $f(v_i) = (f.v_i)$. Thus the cohomology of $\mathcal{L}$ can also be written as following:*

$$\mathrm{H}^j(\mathbb{P}_{\boldsymbol{\Sigma}}, \mathcal{L}) = \bigoplus_{f \in N^*} \mathrm{H}^{red}_{rkN-j-1}(Supp(\mathbf{r}_f)),$$

where $\mathbf{r}_f = (a_i + f(v_i))_{i=1}^n$.

In this paper, our primary objects of interest are H−trivial line bundles which we define below.

Definition 2.5. *Let $\mathcal{L}$ be a line bundle in $\mathrm{Pic}(\mathbb{P}_{\boldsymbol{\Sigma}})$. We say that $\mathcal{L}$ is* H−*trivial iff $\mathrm{H}^j(\mathbb{P}_{\boldsymbol{\Sigma}}, \mathcal{L}) = 0$ for all $j \geq 0$.*

A combinatorial criterion for H−triviality is given in terms of forbidden sets introduced below, see [2, 7].

Definition 2.6. *For every subset $I \subseteq \{1, \ldots, n\}$, we denote C_I to be the simplicial complex $Supp(\mathbf{r})$ where $r_i = -1$ for $i \notin I$ and $r_i = 0$ for $i \in I$. Let $\Delta = \{I \subseteq \{1, \ldots, n\} | C_I$ has nontrivial reduced homology$\}$. By Remark 2.3, Δ contains $\{1, \ldots, n\}$ and $\emptyset$. For each $I \in \Delta$, the forbidden set associated to I is defined by*

$$FS_I := \{\mathcal{O}(\sum_{i \notin I}(-1 - r_i)E_i + \sum_{i \in I} r_i E_i) | r_i \in \mathbb{Z}_{\geq 0} \text{ for all } i\}.$$

Proposition 2.7. *Let $\mathcal{L}$ be a line bundle on $\mathbb{P}_{\mathbf{\Sigma}}$. Then $\mathcal{L}$ is* H−*trivial if and only if $\mathcal{L}$ does not lie in FS_I for any $I \in \Delta$.*

Proof. This follows immediately from Proposition 2.2. □

We introduce $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}}) = \mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}}) \otimes \mathbb{R}$ which can be regarded as a quotient of $\mathbb{R}^n$ with basis elements given by E_i. We know $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ is a vector space with dimension equal to the rank of $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$.

Definition 2.8. *For each $I \in \Delta$, we define the forbidden point by*

$$q_I = -\sum_{i \notin I} E_i \in \mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}}).$$

We define a cone associated to I with vertex at the origin to be

$$Z_I = \sum_{i \in I} \mathbb{R}_{\geq 0} E_i - \sum_{i \notin I} \mathbb{R}_{\geq 0} E_i.$$

We define the forbidden cone $FC_I \subseteq \mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ by

$$FC_I = q_I + Z_I.$$

Remark 2.9. *By definition, we have $FS_I \subseteq FC_I$ for any $I \in \Delta$.*

In dimension two, the set Δ is especially simple.

Example 2.10. *Let Σ be a complete simplicial fan Σ in N with n one-dimensional cones and n lattice points $\{v_i\}_{i=1}^n$ chosen in each of the one-dimensional cones of Σ. In the case that $N = \mathbb{Z}^2$, the maximal cones of Σ are $\mathbb{R}_{\geq 0} v_1 + \mathbb{R}_{\geq 0} v_2, \mathbb{R}_{\geq 0} v_2 + \mathbb{R}_{\geq 0} v_3, \ldots, \mathbb{R}_{\geq 0} v_n + \mathbb{R}_{\geq 0} v_1$, see Figure 1. We describe $\Delta = \{\emptyset, \{1, \ldots, n\}\} \cup \{I \subset \{1, \ldots, n\} | C_I$ is disconnected$\}$. For example, we have $\{1, 3\} \in \Delta$ if $n > 3$, $\{n, 2, 3\} \in \Delta$ if $n > 4$, but $\{1, 2\} \notin \Delta$, $\{n, 1, 2\} \notin \Delta$ for all $n > 2$, see Figure 1.*

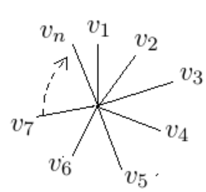

FIGURE 1.

In dimension three we can describe Δ as follows.

Example 2.11. *In the case that* $N = \mathbb{Z}^3$*, we describe* $\Delta = \{\emptyset, \{1,\ldots,n\}\} \cup \{I \subset \{1,\ldots,n\} | C_I \text{ is disconnected}\} \cup \{I \subset \{1,\ldots,n\} | C_{\{1,\ldots,n\}\setminus I} \text{ is disconnected}\}$*. Indeed* $\mathrm{H}^2(\mathcal{L})$ *is nontrivial iff* C_I *is disconnected and* $\mathrm{H}^1(\mathcal{L})$ *is nontrivial iff* $C_{\{1,\ldots,n\}\setminus I}$ *is disconnected.*

In order to state the main results of this paper, we need to associate to any $\mathbf{\Sigma}-$piecewise linear function ψ on $N_{\mathbb{R}}$ a convex polytope Λ_ψ in the character space $M_{\mathbb{R}} = M \otimes \mathbb{R}$.

Definition 2.12. *For each maximal dimensional cone* $\sigma \in \mathbf{\Sigma}(m)$*, let* ψ_σ *be the linear function on* $N_{\mathbb{R}}$ *such that* $\psi_\sigma = \psi$ *in cone* σ*. We define* $\Lambda_\psi \subset M_{\mathbb{R}}$ *to be the convex hull of the set* $\{\psi_\sigma | \sigma \in \mathbf{\Sigma}(m)\}$*.*

Our first main result whose proof is given in the next section is a combinatorial condition for toric DM stacks in any dimension to have infinitely many H−trivial line bundles.

Theorem 2.13. *Let* $\mathbb{P}_{\mathbf{\Sigma}}$ *be a proper smooth dimension* m *toric DM stack associated to a complete stacky fan* $\mathbf{\Sigma} = (\Sigma, \{v_i\}_{i=1}^n)$*. If there is a* $\mathbf{\Sigma}-$*piecewise linear function* ψ *such that* $\dim(\Lambda_\psi) < m$*, then there are infinitely many* H−*trivial line bundles on* $\mathbb{P}_{\mathbf{\Sigma}}$*.*

3. Proof of the first main result

In this section, we give the proof of Theorem 2.13. We start by describing a key method of constructing infinitely many H−trivial line bundles.

Proposition 3.1. *Let* $\mathbb{P}_{\mathbf{\Sigma}}$ *be a proper smooth dimension* m *toric DM stack associated to a complete stacky fan* $\mathbf{\Sigma} = (\Sigma, \{v_i\}_{i=1}^n)$*. If there is a non-linear* $\mathbf{\Sigma}-$*piecewise linear function* ψ *and a vertex* $v \in \mathbf{\Sigma}(1)$ *such that* ψ *is constant on all lines parallel to* v*, then there are infinitely many* H−*trivial line bundles on* $\mathbb{P}_{\mathbf{\Sigma}}$*.*

In order to prove Proposition 3.1, we need several lemmas. Let $\sum_{i\in I} a_i E_i = \sum_{i\in I} a_i E_i + \sum_{i\in I^c} a_i E_i$ be an element in $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$, where $I = \{i | a_i \geq 0\}$ and $I^c = \{1,\ldots,n\} \setminus I = \{i | a_i < 0\}$. Let φ be a $\mathbf{\Sigma}-$piecewise linear function on $\mathbb{R}^m$ such that $\varphi(v_i) = a_i$.

Lemma 3.2. *Let* $S_{\varphi<0} = \{v \in \big(\mathrm{N}_{\mathbb{R}} \setminus \{\mathbf{0}\}\big) / \mathbb{R}_{>0} | \varphi(v) < 0\}$ *and* $I^c = \{i | \varphi(v_i) < 0\}$*, then* C_{I^c} *is homotopic to* $S_{\varphi<0}$*. Here we consider* C_{I^c} *to be the geometric realization of an abstract simplicial complex [1].*

Proof. We think of C_{I^c} as a subspace of topological space $S_{\varphi<0}$ with the inclusion $C_{I^c} \hookrightarrow S_{\varphi<0}$ given by

$$\sum_{\substack{i\in J\in C_{I^c},\\ \sum\lambda_i=1}} \lambda_i v_i \mapsto \mathbb{R}_{>0}\Big(\sum_{\substack{i\in J\in C_{I^c},\\ \sum\lambda_i=1}} \lambda_i v_i\Big).$$

Now we consider a map $F : S_{\varphi<0} \times [0,1] \to S_{\varphi<0}$ which is defined as follows. For any point $a \in S_{\varphi<0}$, let σ be a cone of $\mathbf{\Sigma}$ that contains a. We can write $a = \sum_{i\in I_\sigma} \lambda_i v_i$, where $I_\sigma = \{i | v_i \in \sigma\}$ and all $\lambda_i \geq 0$. Then we define

$$F(a,t) = \sum_{\substack{i\in I_\sigma \\ \varphi(v_i)<0}} \lambda_i v_i + \sum_{\substack{i\in I_\sigma \\ \varphi(v_i)\geq 0}} (1-t)\lambda_i v_i.$$

Assume we choose another cone σ' containing a and write $a = \sum_{i\in I_{\sigma'}} \lambda'_i v_i$, where $I_{\sigma'} = \{i | v_i \in \sigma'\}$ and all $\lambda'_i \geq 0$. Then $\lambda_i = 0 = \lambda'_i$ if $v_i \notin \sigma \cap \sigma'$ and $\lambda_i = \lambda'_i$ if $v_i \in \sigma \cap \sigma'$. Thus the map F is well defined. Also the F is continuous since λ_i change continuously when the point a moves from one cone to another. We immediately see that $F(a,0) = a$, $F(a,1) \in S_{\varphi<0}$ for any $a \in S_{\varphi<0}$ by definition of F. Moreover $F(a,t) = a$ for any $a \in C_{I^c}$ and any $t \in [0,1]$ since $\varphi(v_i) < 0$ for all $i \in I_\sigma$ if $a \in C_{I^c}$. Therefore F is a strong deformation retraction of topological space $S_{\varphi<0}$ onto subspace C_{I^c}. □

Lemma 3.3. *Let* $S_{\varphi\geq 0} = \{v \in \big(\mathrm{N}_\mathbb{R} \setminus \{\mathbf{0}\}\big)/\mathbb{R}_{>0} | \varphi(v) \geq 0\}$ *and* $I = \{i | \varphi(v_i) \geq 0\}$*, then* C_I *is homotopic to* $S_{\varphi\geq 0}$.

Proof. The proof is the same as for Lemma 3.2. □

Lemma 3.4. *Let* $I \in \{1, 2, \ldots, n\}$*. Then we have* $H^{red}_{j-1}(C_I) = \big(H^{red}_{rkN-j-1}(C_{I^c})\big)^*$*. This implies* C_I *has nontrivial reduced homology if and only if* C_{I^c} *has nontrivial reduced homology.*

Proof. Since the sphere S^{rkN-1} is homeomorphic to $\big(\mathrm{N}_\mathbb{R} \setminus \{\mathbf{0}\}\big)/\mathbb{R}_{>0}$, we have $S^{rkN-1} = S_{\varphi\geq 0} \sqcup S_{\varphi<0}$. By Alexander duality [1], we have an isomorphism of reduced homology and reduced cohomology $\mathrm{H}^{red}_{j-1}(S_{\varphi\geq 0}) \cong \mathrm{H}_{red}^{rkN-1-j}(S_{\varphi<0})$. Using the Universal Coefficient Theorem, we get $\mathrm{H}_{red}^{rkN-1-j}(S_{\varphi<0}) = \big(\mathrm{H}^{red}_{rkN-1-j}(S_{\varphi<0})\big)^*$. Since C_I is homotopic to $S_{\varphi\geq 0}$ by Lemma 3.3 and C_{I^c} is homotopic to $S_{\varphi<0}$ by Lemma 3.2, we obtain $H^{red}_{j-1}(C_I) = \big(H^{red}_{rkN-j-1}(C_{I^c})\big)^*$. □

We have the following corollary.

Corollary 3.5. *The topological spaces* $\{v \in \mathrm{N}_\mathbb{R} | \varphi(v) < 0\}$ *and* C_{I^c} *are homotopic.*

Proof. Since $\{v \in \mathrm{N}_\mathbb{R} | \varphi(v) < 0\} = \{v \in \mathrm{N}_\mathbb{R} \setminus \{\mathbf{0}\} | \varphi(v) < 0\}$ is a fibration with fibre $\mathbb{R}_{>0}$ over $\{v \in \big(\mathrm{N}_\mathbb{R} \setminus \{\mathbf{0}\}\big)/\mathbb{R}_{>0} | \varphi(v) < 0\}$, these spaces are homotopic to each other. Then we use Lemma 3.2. □

For some fixed $v \in \mathbf{\Sigma}(1)$, we consider all lines parallel to v. The parametric equation of such a line is $l(t) := l(0) + tv$ for some $l(0) \in N_\mathbb{R}$, where $t \in \mathbb{R}$.

Lemma 3.6. *Let* φ *be a* $\mathbf{\Sigma}-$*piecewise linear function on* $N_\mathbb{R}$ *and* $l(t) := l(0) + tv$ *be the parametric equation of the line parallel to* v*. Then for any point in the interior of the region corresponding to* $\sigma \in \mathbf{\Sigma}(m)$*, the derivative of the function* $\psi(l(t))$ *equals* $\varphi_\sigma(v)$.

Proof. Left to the reader. □

Corollary 3.7. *For a nonzero $v \in N_{\mathbb{R}}$, we have $\psi_\sigma(v) = 0$ for all cones $\sigma \in \mathbf{\Sigma}(m)$ if and only if ψ is constant on all lines parallel to v.*

Proof of Proposition 3.1. Without loss of generality, we assume $v = v_1$. We claim that $\mathcal{L} = \mathcal{O}(\sum_{i=2}^n \psi(v_i)E_i - E_1)$ is H−trivial.

Let a_i be the coefficient of E_i in $\sum_{i=2}^n \psi(v_i)E_i - E_1$. By Remark 2.4, we have

$$\mathrm{H}^j(\mathbb{P}_{\mathbf{\Sigma}}, \mathcal{L}) = \bigoplus_{f \in M} \mathrm{H}^{red}_{rkN-j-1}(Supp(\mathbf{r}_f)),$$

where $\mathbf{r}_f = (a_i + f(v_i))_{i=1}^n$. In order to show $\mathrm{H}^*(\mathbb{P}_{\mathbf{\Sigma}}, \mathcal{L}) = 0$, it is sufficient to show $Supp(\mathbf{r}_f)$ is contractible for each $f \in M$. Let $\sum_i r_i E_i = \sum_{i=2}^n \psi(v_i)E_i - E_1 + f(v_i)E_i$. We know that $r_i = (f+\psi)(v_i)$ for $i = 2, \dots, n$ and $r_1 = f(v_1) - 1$. Let $I = \{i | r_i \geq 0\}$ and $I^c = \{1, \dots, n\} \setminus I$.

Case $r_1 \neq -1$. We have $f(v_1) \neq 0$. We regard ≥ 0 and < 0 as different signs. Since $f(v_1)$ is an integer, $f(v_1)-1$ is non-negative only when $f(v_1) \geq 1$ and $f(v_1) - 1$ is negative only when $f(v_1) < 0$. Thus the sign of r_i is the same as the sign of $(f + \psi)(v_i)$ for all $i \in \{1, 2, \dots, n\}$. Let φ be the $\mathbf{\Sigma}$−piecewise linear function φ such that $\varphi(v_i) = (\psi + f)(v_i)$ for all $i \in \{1, 2, \dots, n\}$. For any line $l(t) = l(0) + tv_1$ parallel to v_1, we have $\varphi(l(t)) = f(l(0) + tv_1) + \psi(l(t)) = f(v_1)t + c_l$ for some constant value c_l since ψ is constant on all lines parallel to v_1. So for any line $l(t)$ parallel to v_1, if $f(v_1) < 0$, $\varphi(l(t))$ is negative when $t > 0$ is sufficiently large. If $f(v_1) > 0$, $\varphi(l(t))$ is negative when $t < 0$ and $|t|$ is sufficiently large. This implies $\{v \in \mathrm{N}_{\mathbb{R}} | \varphi(v) < 0\}$ is contractible. So C_{I^c} is contractible by Lemma 3.5. Thus $Supp(\mathbf{r}_f) = C_I$ has trivial reduced homology by Lemma 3.4.

Case $r_1 = -1$. We have $f(v_1) = 0$. Let $\overline{\varphi}$ be the $\mathbf{\Sigma}$−piecewise linear function such that $\overline{\varphi}(v_i) = (\psi+f)(v_i)$ for $i \in \{2, \dots, n\}$ and $\overline{\varphi}(v_1) = -1$. Let $l(t) = l(0)+tv_1$ be any line parallel to v_1. There exists t_0 such that $l(t)$ lies in interior of a cone σ_0 with the ray v_1 if and only if $t > t_0$. By Lemma 3.6, the derivative of $\psi(l(t))$ equals 0 since ψ is constant on all lines parallel to v_1. For any point in the interior of the region corresponding to $\sigma \neq \sigma_0$, the derivative of the function $\overline{\varphi}(l(t))$ equals $\overline{\varphi}_\sigma(v_1) = \psi_\sigma(v_1) + f(v_1) = \psi_\sigma(v_1) = 0$. For any point in the interior of the region corresponding to σ_0, the derivative of the function $\overline{\varphi}(l(t))$ equals $\overline{\varphi}_{\sigma_0}(v_1) = -1$. Thus φ is constant on $l(t)$ for $t \leq t_0$ and the derivative of the function at $l(t)$ is $\overline{\varphi}_{\sigma_0}(v_1) = -1$ for $t > t_0$. So $\overline{\varphi}(l(t))$ is negative when $t > t_0$ is sufficiently large, or for all t. Thus $\{v \in \mathrm{N}_{\mathbb{R}} | \overline{\varphi}(v) < 0\}$ is contractible. Since $r_i = \overline{\varphi}(v_i)$, so $I^c = \{i | \overline{\varphi}(v_i) < 0\}$. Then C_{I^c} is contractible by Lemma 3.5. Thus $Supp(\mathbf{r}_f) = C_I$ has trivial reduced homology by Lemma 3.4.

Since the same argument applies to $r\psi$ instead of ψ, we have for any $r \in \mathbb{Z}$ $\mathcal{L} = \mathcal{O}(\sum_{i=2}^n r\psi(v_i)E_i - E_1)$ is H−trivial. These line bundles are different from one another because ψ is non-linear. Thus there are infinitely many H−trivial line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$. □

In order to prove Theorem 2.13, it suffices to show that $\dim\Lambda_\psi < m$ implies that there exists $v \in \mathbf{\Sigma}(1)$ such that $\langle \Lambda_\psi, v\rangle = c$ for a constant c.

Proposition 3.8. *$\dim\Lambda_\psi < m$ if and only if there exists $v \in \mathbf{\Sigma}(1)$ such that $\langle \Lambda_\psi, v\rangle = c$ for a constant c.*

Proof. We have that $\dim\Lambda_\psi < m$ if and only if Λ_ψ is inside an affine hyperplane in $M_\mathbb{R}$. That is equivalent to say there exists $v \in N_\mathbb{R}$ such that $\langle \Lambda_\psi, v\rangle = c$, where c is a constant. The essence of this proposition is that v may be chosen in $\mathbf{\Sigma}(1)$. We adjust ψ by adding a linear function in M such that $\psi_\sigma(v) = 0$ for all cones $\sigma \in \mathbf{\Sigma}(m)$ and $\psi = 0$ on a cone τ which contains v in its interior.

First, we claim that for every $\sigma \in \mathbf{\Sigma}(m)$, there is a $\widetilde{\sigma} \in \mathrm{Star}(\tau)(m) = \{\sigma \in \mathbf{\Sigma}(m) | \tau \subseteq \sigma\}$ such that $\psi_\sigma = \psi_{\widetilde{\sigma}}$. We consider the projection $pr : N_\mathbb{R} \to N_\mathbb{R}/\mathbb{R}v$ and choose a maximum-dimensional cone $\widetilde{\sigma} \in \mathrm{Star}(\tau)(m)$ such that $pr(\sigma)$ and $pr(\widetilde{\sigma})$ have overlapping interiors. Let $D = pr(\sigma) \cap pr(\widetilde{\sigma})$, see Figure 2. There are linear functions $\varphi_{pr(\sigma)}$ and $\varphi_{pr(\widetilde{\sigma})}$ on $N_\mathbb{R}/\mathbb{R}v$ such that $\psi_\sigma = \varphi_{pr(\sigma)} \circ pr$ and $\psi_{\widetilde{\sigma}} = \varphi_{pr(\widetilde{\sigma})} \circ pr$.

For any $p \in D$, we pick a point q_1 in σ such that $pr(q_1) = p$ and a point q_2 in $\widetilde{\sigma}$ such that $pr(q_2) = p$. Since q_1 and q_2 are on a line parallel to v, by Lemma 3.7, we get $\psi(q_1) = \psi(q_2)$. Since $\psi_\sigma(q_1) = \psi(q_1)$ and $\psi_{\widetilde{\sigma}}(q_2) = \psi(q_2)$, we have

$$\varphi_{pr(\sigma)}(p) = \psi_\sigma(q_1) = \psi_{\widetilde{\sigma}}(q_2) = \varphi_{pr(\widetilde{\sigma})}(p).$$

So we get $\varphi_{pr(\sigma)} = \varphi_{pr(\widetilde{\sigma})}$ on D which is a full-dimensional set. This implies $\varphi_{pr(\sigma)} = \varphi_{pr(\widetilde{\sigma})}$ on $N_\mathbb{R}/\mathbb{R}v$. Thus $\psi_\sigma = \psi_{\widetilde{\sigma}}$.

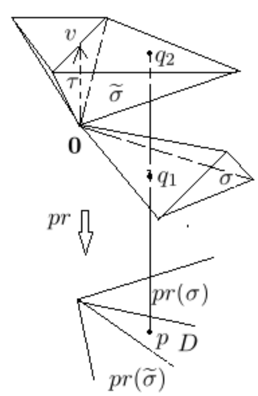

Figure 2.

We choose a vertex w of the cone τ. For any maximum-dimensional cone $\widetilde{\sigma}$ in $\mathrm{Star}(\tau)$, we have $\psi_{\widetilde{\sigma}}(w) = \psi(w) = 0$ since $w \in \tau \subset \widetilde{\sigma}$. Then for all cones $\sigma \in \mathbf{\Sigma}(m)$, we obtain $\psi_\sigma(w) = \psi_{\widetilde{\sigma}}(w) = 0$. □

Now we are ready to prove Theorem 2.13.

Proof of Theorem 2.13. Proposition 3.8, Corollary 3.7 and Proposition 3.1 imply the result. □

4. Dimension three case

In this section we focus on proper smooth dimension three toric DM stacks $\mathbb{P}_{\mathbf{\Sigma}}$ associated to a complete stacky fan $\mathbf{\Sigma} = (\Sigma, \{v_i\}_{i=1}^n)$ in N of rank three. We give a criterion for when there exist infinitely many H−trivial line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$ for smooth toric varieties and DM stacks in dimension three under the assumption that there is no more than one pair of collinear rays in $\mathbf{\Sigma}$.

The following lemma highlights the importance of diagonals in $\mathbf{\Sigma}$, i.e., pairs (s,t) such that v_s and v_t are collinear.

Lemma 4.1. *Let $\mathcal{L} = \sum_{i=1}^n f(v_i)E_i$ be an element in $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$, where f is a $\mathbf{\Sigma}-$piecewise linear function on $N_{\mathbb{R}}$. Let $s \in \{1, 2, \ldots, n\}$. There exists a $\mathbf{\Sigma}-$piecewise linear function g on $N_{\mathbb{R}}$ such that*

- $\mathcal{L} = \sum_{i=1}^n g(v_i)E_i$,
- $g(v_s) = 0$,
- $g(v_i) \neq 0$ *for all* $v_i \notin \mathbb{R}v_s$,
- *If* $v_t \in \mathbb{R}v_s$, $t \neq s$, *then* $g(v_t) = 0$ *iff* f *is linear on* $\mathbb{R}v_s$.

Proof. We consider $g(v) = f(v) - m \cdot v$, where m is generic element in the affine plane $\{m \in M_{\mathbb{R}} | m \cdot v_s = f(v_s)\}$. We immediately get $\mathcal{L} = \sum_{i=1}^n g(v_i)E_i$ and $g(v_s) = 0$.

The fact that m is chosen generic means that $m \cdot v_i \neq f(v_i)$ for all i except when $v_i \in \mathbb{R}v_s$. If $v_t \in \mathbb{R}v_s$, then g is linear on $\mathbb{R}v_s$ iff $g(v_t) = 0$, and g is linear on $\mathbb{R}v_s$ iff f is linear on $\mathbb{R}v_s$. □

Definition 4.2. *For a point $v_i \in \{v_i\}_{i=1}^n$, we denote the neighborhood of v_i to be*

$$B_i = \{v_j \in \{v_1, \cdots, v_n\} | v_j \text{ and } v_i \text{ span a two-dimensional cone of } \mathbf{\Sigma}\}.$$

Let $v_{j_1}, v_{j_2}, \ldots, v_{j_l}$ be all the vectors in B_i which are ordered clockwise. Let g be a $\mathbf{\Sigma}-$piecewise linear function on $N_{\mathbb{R}}$. We regard ≥ 0 and < 0 as different signs. We count the number of pairs of vectors $\{v_{j_k}, v_{j_{k+1}}\} \subset \{v_{j_1}, \ldots, v_{j_l}\}$ such that $f(v_{j_k})$ and $f(v_{j_{k+1}})$ have different signs. We call it the number of sign changes of f among $v_{j_1}, \ldots, v_{j_l}$. For example, there are exactly two sign changes in B_i in Figure 3.

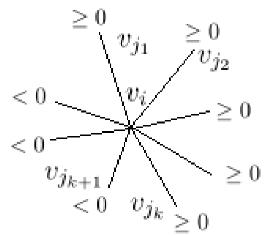

FIGURE 3.

Recall that in Definition 2.8, the cones Z_I are shifts of the forbidden cones FC_I.

Lemma 4.3. *Let $\mathcal{L}=\sum_{i=1}^n g(v_i)E_i$ be a nonzero element in* $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$ *which is not contained in the interior of Z_I for any $I\in\Delta$, where g is a $\mathbf{\Sigma}-$piecewise linear function on $N_{\mathbb{R}}$. Assume there is some $s\in\{1,2,\cdots,n\}$ such that $g(v_s)=0$ and $g(v_i)\neq 0$ for all $i\neq s$. Then there exist exactly two sign changes in B_s, see Figure 3.*

Proof. The main idea is that one can perturb g by a linear function to achieve $g(v_s)>0$ or $g(v_s)<0$ without changing signs of other $g(v_i)$. Let $C_{>0}$ be the simplicial complex $C_{\{i|g(v_i)>0\}}$ and $C_{\geq 0}$ be the simplicial complex $C_{\{i|g(v_i)\geq 0\}}$. The reduced simplicial homology complex $W(C_{>0})$ is as follows:

$$0\to\bigoplus_{\substack{J\in C_{>0}\\|J|=3}}\mathbb{C}\to\bigoplus_{\substack{J\in C_{>0}\\|J|=2}}\mathbb{C}\to\bigoplus_{\substack{J\in C_{>0}\\|J|=1}}\mathbb{C}\to\mathbb{C}\to 0$$

The reduced simplicial homology complex $W(C_{\geq 0})$ is as follows:

$$0\to\bigoplus_{\substack{J\in C_{\geq 0}\\|J|=3}}\mathbb{C}\to\bigoplus_{\substack{J\in C_{\geq 0}\\|J|=2}}\mathbb{C}\to\bigoplus_{\substack{J\in C_{\geq 0}\\|J|=1}}\mathbb{C}\to\mathbb{C}\to 0$$

We use inclusions $\bigoplus_{\substack{J\in C_{>0}\\|J|=k}}\mathbb{C}\hookrightarrow\bigoplus_{\substack{J\in C_{\geq 0}\\|J|=k}}\mathbb{C}$ for each k to obtain an exact sequence of complexes:

$$0\to W(C_{>0})\to W(C_{\geq 0})\to W(C_{\geq 0})/W(C_{>0})\to 0 \tag{4.1}$$

We can take a linear function f such that $f(v_s)>0$ and $|f(v_i)|<|g(v_i)|$ for all $i\neq s$. Let $g'=g+f$, we have $g'(v_i)=(g+f)(v_i)\neq 0$ for $i\in\{1,2,\ldots,n\}$. Thus

$$\mathcal{L}=\sum_{i\in I_{>0}}\alpha_iE_i+\sum_{i\notin I_{>0}}\alpha_iE_i$$

in $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$, where $\alpha_i=g'(v_i)\neq 0$ for all i and $I_{>0}=\{i|g'(v)>0\}$. Also we have $C_{\geq 0}=C_{I_{>0}}$. Since $\mathcal{L}$ is not contained in the interior of Z_I for any $I\in\Delta$, we get $I_{>0}\notin\Delta$. This implies $C_{\geq 0}=C_{I_{>0}}$ has trivial reduced homology.

Similarly, we can take a linear function f' such that $f'(v_s) < 0$ and $|f'(v_i)| < |g(v_i)|$ for all $i \neq s$ to show that $C_{>0}$ has trivial reduced homology.

Since $W(C_{\geq 0})$ and $W(C_{>0})$ have trivial homology, by snake lemma and the exact sequence (4.1), the complex $W(C_{\geq 0})/W(C_{>0})$ has trivial homology. Thus the Euler characteristic $\chi(W(C_{\geq 0})/W(C_{>0})) = 0$. If $g(v_i) > 0 (< 0)$ for all $v_i \in B_s$, the Euler characteristic $\chi(W(C_{\geq 0})/W(C_{>0})) = 1$ by computing the alternating sum of numbers of $k-$cells, which leads to contradiction. Thus there exists at least two sign changes in B_s. By computing the alternating sum of numbers of $k-$cells, the Euler characteristic $\chi(W(C_{\geq 0})/W(C_{>0}))$ equals

$$1 - \text{ the number of connect components of positive sign in } B_s.$$

Thus there exists only one component of positive sign in B_s. This implies that there are exactly two sign changes in B_s. □

Let $\mathcal{L} = \sum_{i=1}^{n} g(v_i)E_i$ be an element in $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$ which is not contained in the interior of Z_I for any $I \in \Delta$, where g is a $\mathbf{\Sigma}-$piecewise linear function on $N_{\mathbb{R}}$. Assume there is some $s \in \{1, 2 \cdots, n\}$ such that $g(v_s) = 0$. Then we can consider the projection $\pi : N_{\mathbb{R}} \to \mathbb{R}^2 = N_{\mathbb{R}}/\langle v_s \rangle$ along the line $\mathbb{R}v_s$.

Under the projection, the neighborhood B_s gives a two-dimensional complete stacky fan $\mathbf{\Sigma}' = (\Sigma', \{w_i\}_{i \in J})$, where $w_i = \pi(v_i)$, $v_i \in B_s$ and $J = \{i \in \{1, 2, \ldots, n\} | v_i \in B_s\}$. We can think of g as a $\mathbf{\Sigma}'-$piecewise linear function on $\mathbb{R}^2 = N_{\mathbb{R}}/\langle v_s \rangle$ since $g(v_s) = 0$. Let $\{E'_i\}_{i \in J}$ be the generators of $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}'})$. Let $\Delta' = \{\emptyset, J\} \cup \{I' \subset J | C'_{I'} \text{ is disconnected}\}$. And for any $I' \in \Delta'$, let $Z'_{I'} = \sum_{i \in I'} \mathbb{R}_{\geq 0} E'_i - \sum_{i \notin I'} \mathbb{R}_{\geq 0} E'_i$ be the cone associated to I' with vertex at the origin which is obtained by shifting the forbidden cone $FC'_I \subseteq \mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}'})$.

Lemma 4.4. *The point $\mathcal{D} = \sum_{i \in J} g(v_i)E'_i \in \mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}'})$ is not contained in the interior of $Z'_{I'}$ for any $I' \in \Delta'$.*

Proof. Assume $\mathcal{D}$ is contained in $Z'_{I'}$ for some $I' \in \Delta'$. This implies that there exists a linear function f on $\mathbb{R}^2 = \mathbb{R}^3/\langle v_s \rangle$ such that $f(w_i) + g(v_i) \neq 0$ for any $i \in J$ and $I' = \{i \in J | f(w_i) + g(v_i)\}$ has non-trivial reduced homology. This would mean that the number of sign changes in B_s is not two, which contradicts Lemma 4.3. □

Lemma 4.5. *Let $\mathcal{L} = \sum_{i=1}^{n} g(v_i)E_i$ be an element in $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$ which is not contained in the interior of Z_I for any $I \in \Delta$, where g is a $\mathbf{\Sigma}-$piecewise linear function on $N_{\mathbb{R}}$. Assume v_s is such that there is no v_t with $t \neq s$, $\mathbb{R}v_s = \mathbb{R}v_t$; or g is not linear on $\mathbb{R}v_s$, then*

- *either g is linear on $\mathrm{Star}(v_s) = \{\sigma \in \mathbf{\Sigma} | v_s \subseteq \sigma\}$*
- *or $g|_{\mathrm{Star}(v_s)}$ has two half spaces of linearity, i.e., there exists $v_p, v_q \in B_s$ such that $v_p, v_s, v_q, \mathbf{0}$ are coplanar and $g|_{\mathrm{Star}(v_s)}$ is linear on either side of this plane, see Figure 4.*

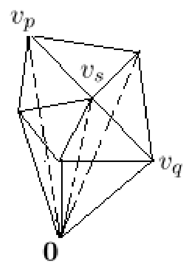

FIGURE 4.

Proof. By Lemma 4.1, we can assume $g(v_s) = 0$. Then by Lemma 4.4, we have $\mathcal{D} = \sum_{i \in J} g(v_i) E'_i \in \mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}'})$ not in the interior of $Z'_{I'}$ for any $I \in \Delta'$. If $\mathcal{D} = 0$ in $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}'})$, then g is linear on $\mathrm{Star}(v_s)$. Otherwise, by Lemma 4.1 in paper [17] $g|_{\mathrm{Star}(v_s)}$ is a pullback of a piecewise linear function on $N_{\mathbb{R}}/\mathbb{R}v_s$ with two half plane regions of linearity. Thus $g|_{\mathrm{Star}(v_s)}$ has two half spaces of linearity, see Figure 4. □

Proposition 4.6. *Assume that v_i and v_j are not collinear for any $\{i, j\} \subset \{1, 2, \cdots, n\}$ and there are infinitely many* H−*trivial line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$. Then there exists a plane which does not intersect with the interior of any maximal cone of $\mathbf{\Sigma}$.*

Proof. Since there are infinitely many H−trivial line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$, by the argument of proof in Proposition 3.8 in [17], there exists a non-zero element $\mathcal{L} = \sum_{i=1}^{n} g(v_i) E_i \in \mathbb{P}_{\mathbf{\Sigma}}$ which is not contained in any Z_I for $I \in \Delta$, where g is a $\mathbf{\Sigma}-$piecewise linear function on $N_{\mathbb{R}}$.

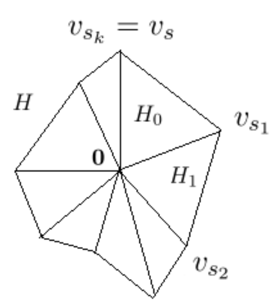

FIGURE 5.

Since $\mathcal{L} \neq 0$, there exists a vertex v_s such that g is not linear in $\mathrm{Star}(v_s)$. By Lemma 4.5, g has two half planes of linearity in $\mathrm{Star}(v_s)$. The line breaking the linearity corresponds to a plane in $\mathbb{R}^3$ which we denote by H_0. We know that H_0 passes through the vector v_s, the origin $\mathbf{0}$ and a point in B_s which we denote by v_{s_1}. Also, g is linear on each side of plane H_0 in $\mathrm{Star}(v_s)$. Thus H_0 does not intersect with interior of any maximal cones of $\mathbf{\Sigma}$ with v_s as a ray. By assumption, we know v_{s_1} and v_i is not collinear for any $i \neq s_1$. By Lemma 4.1 and the same argument for v_s, we get a plane H_1 passing through v_{s_1} such that g is linear on each side of plane H_1 in $\mathrm{Star}(v_s)$. Thus H_1 equals H_0 and does not intersect with interior of any maximal cones of $\mathbf{\Sigma}$ with v_{s_1} as ray. We continue the process, there exits a positive number k such that $v_{s_k} = v_s$. We get a plane $H = H_i$ for $i = 0, \cdots, k$ which does not intersect with interior of any maximal cones of $\mathbf{\Sigma}$. See Figure 5. □

Theorem 4.7. *Let $\mathbb{P}_{\mathbf{\Sigma}}$ be a proper smooth dimension three toric DM stack associated to a complete stacky fan $\mathbf{\Sigma} = (\Sigma, \{v_i\}_{i=1}^n)$. Assume there exist no collinear pairs of rays in $\mathbf{\Sigma}$ and there are infinitely many* H−*trivial line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$. Then there is a $\mathbf{\Sigma}-$piecewise linear function ψ such that* $\dim(\Lambda_\psi) < 3$.

Proof. By Proposition 4.6, there is a plane H which does not intersect with the interior of any maximal cone of $\mathbf{\Sigma}$ and contains v_s for some $s \in \{1, \ldots, n\}$. Then we have a projection $\pi : N_{\mathbb{R}} \to N_{\mathbb{R}}/\mathbb{R}H \cong \mathbb{R}^1$ along this plane H. We pick a function f which is linear along two half lines of $\mathbb{R}^1 = N_{\mathbb{R}}/\mathbb{R}H$ and $f(\mathbf{0}) = 0$. We get $\pi^*(f) = f \circ pi$ is a $\mathbf{\Sigma}-$piecewise linear function on $\mathbb{R}^3 = N_{\mathbb{R}}$ since the preimage of origin in $N_{\mathbb{R}}/\mathbb{R}H \cong \mathbb{R}^1$ is H which does not intersect with interior of any maximal cones of $\mathbf{\Sigma}$. Let $\psi = \pi^*(f)$. We have $\psi = \pi^*(f)$ is zero on H since $\pi(H) = \mathbf{0}$. Also $\psi = \pi^*(f)$ is constant on any plane parallel H' to H since image of H' under π is one point in $N_{\mathbb{R}}/\mathbb{R}H$. For any line parallel to $\mathbb{R}v_s$ is on a plane parallel to H, so ψ is constant on any line parallel to $\mathbb{R}v_s$. Then by Proposition 3.8 and Corollary 3.7, we know $\dim(\Lambda_\psi) < 3$. In fact, in this case $\dim(\Lambda_\psi) = 1$. □

Now we consider the case when there exists exactly one collinear pair in $\{v_1, \ldots, v_n\}$.

Lemma 4.8. *Assume that v_s and v_t are the only collinear pair and there are infinitely many* H−*trivial line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$. Then either there is a plane which does not intersect with the interior of any maximal cones of $\mathbf{\Sigma}$ or there exist at least three half planes which do not intersect with the interior of any maximal cones of $\mathbf{\Sigma}$ passing through $\mathbb{R}v_s$ and three vectors in B_s respectively.*

Proof. Since there are infinitely many H−trivial line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$, by the argument of proof of Proposition 3.8 in [17], there exists a non-zero element $\mathcal{L} = \sum_{i=1}^n g(v_i)E_i \in \mathbb{P}_{\mathbf{\Sigma}}$ which is not contained in any Z_I for $I \in \Delta$, where g is a $\mathbf{\Sigma}-$piecewise linear function on $\mathbb{R}^3$.

If g is not linear along the line $\mathbb{R}v_s$, then by Lemma 4.1, it is the same case as the one without a collinear pair. By same argument as in Proposition 4.6, there is a plane which does not intersect with interior of any maximal cones of $\mathbf{\Sigma}$.

Now we consider the case g is linear along the line $\mathbb{R}v_s = \mathbb{R}v_t$. If g has two half planes of linearity in $\mathrm{Star}(v_s)$ and $\mathrm{Star}(v_t)$, with same argument in the case without collinear pairs in Proposition 4.6, we get that there is a plane which does not intersect with the interior of any maximal cones of $\mathbf{\Sigma}$ with v_s as a ray. Thus without loss of generality, we assume g has at least three regions of linearity in $\mathrm{Star}(v_s)$. Thus at least three vectors in B_s break the linearity. Thus in B_s, we can pick three v_p, v_q and v_l such that g is linear in the cone span by $\{v_s, v_p, v_q\}$ and linear in the cone span by $\{v_s, v_q, v_l\}$. This implies the two-dimensional cone C_{sq} spanned by $\{v_s, v_q\}$ break the linearity.

Now we consider v_p. By assumption, we know v_p and v_i is not collinear for any $i \neq p$. By Lemma 4.1 and the same argument for in Proposition 4.6, we get a plane H_0 passing through v_p such that ψ is linear on each side of plane H_0 near v_p. Thus H_0 is on the same plane as the cone C_{sp} spanned by $\{v_s, v_p\}$ and does not intersect with interior of any maximal cones of $\mathbf{\Sigma}$ with v_p as ray. We continue the process, there exits a positive number k such that $v_{p_k} = v_t$. We get a half plane H_p which does not intersect with interior of any maximal cones of $\mathbf{\Sigma}$ and passes through v_s.

Similarly, we get half plane H_q and H_l which do not intersect with interior of any maximal cones of $\mathbf{\Sigma}$ and passes through v_s. See Figure 6. □

Theorem 4.9. *Let $\mathbb{P}_{\mathbf{\Sigma}}$ be a proper smooth dimension three toric DM stack associated to a complete stacky fan $\mathbf{\Sigma} = (\Sigma, \{v_i\}_{i=1}^n)$. Assume there exist only one collinear pair of rays in $\mathbf{\Sigma}$ and there are infinitely many* H*−trivial line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$. Then there is a $\mathbf{\Sigma}$−piecewise linear function ψ such that* $\dim(\Lambda_\psi) < 3$.

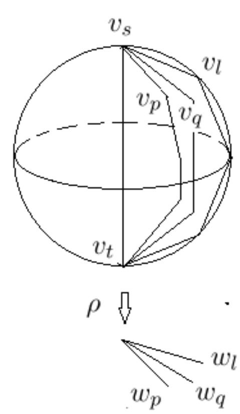

FIGURE 6.

Proof. Let v_s and v_t be the only collinear pair. By assumption and Proposition 4.8, either there exists a projection $\pi : \mathbb{R}^3 \to \mathbb{R}^1$ along a plane which does not intersect with the interior of any maximal cones of $\mathbf{\Sigma}$ or there are three vectors $v_p, v_q, v_l \in B_s$ and a projection $\rho : \mathbb{R}^3 \to \mathbb{R}^2$ alone the line $\mathbb{R}v_s$ such the $H_i = \rho^{-1}\rho(v_i)$ is a half plane which does not intersect with interior of any maximal cones of $\mathbf{\Sigma}$ for each $i = p, q, l$. In former case, with the same argument in Theorem 4.7, there is a $\mathbf{\Sigma}-$piecewise linear function which is constant on all lines parallel to $\mathbb{R}v_s$.

Let $w_i = \rho(v_i)$ for each $i = p, q, l$. In latter case, we pick a function f on $\mathbb{R}^2$ which is linear respectively on the cone C_{pq} spanned by $\{w_p, w_q\}$, the cone C_{ql} spanned by $\{w_q, w_l\}$ and the cone C_{lp} spanned by $\{w_l, w_p\}$. Let $\psi = \rho^*(f)$, see Figure 6. We get $\psi = \rho^*(f)$ is a $\mathbf{\Sigma}-$piecewise linear function on $\mathbb{R}^3$ since the preimage of $\rho(v_i)$ is H_i which does not intersect with interior of any maximal cones of $\mathbf{\Sigma}$ for each $i = p, q, l$. Also, ψ is a $\mathbf{\Sigma}-$piecewise linear function which is constant on any line parallel to v_s since the image of any line under ρ is a point. Then Corollary 3.7 and Proposition 3.8 imply the result. □

Putting all of it together, we get our second main result.

Theorem 4.10. *Let $\mathbb{P}_{\mathbf{\Sigma}}$ be a proper smooth dimension three toric DM stack associated to a complete stacky fan $\mathbf{\Sigma} = (\Sigma, \{v_i\}_{i=1}^n)$. Assume there exists no more than one collinear pair of rays in $\mathbf{\Sigma}$. Then there are infinitely many* H−*trivial line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$ if and only if there is a $\mathbf{\Sigma}-$piecewise linear function ψ such that* $\dim(\Lambda_\psi) < 3$.

Proof. Theorem 4.7, Theorem 4.9 and Theorem 2.13 imply the result. □

5. Comments

In this section, we state the conjecture in full generality, i.e., without a limit of number of collinear pairs of rays in $\mathbf{\Sigma}$ and for arbitrary dimension.

For a proper smooth dimension m toric DM stack $\mathbb{P}_{\mathbf{\Sigma}}$, we have the following three statements:

(1) There is a $\mathbf{\Sigma}-$piecewise linear function ψ such that $\dim(\Lambda_\psi) < m$.
(2) There are infinitely many H−trivial line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$.
(3) There is a nonzero element $\mathcal{L} \in \mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$ which is not contained in the interior of Z_I for any $I \in \Delta$.

By Theorem 2.13, (1) implies (2) without a limit of dimension m and the number of collinear pairs of rays in $\mathbf{\Sigma}$. By the argument of [17], (2) also implies (3) in full generality. By Theorem 4.10, (2) implies (1) when $m = 3$ and there exist no more than one collinear pair of rays in $\mathbf{\Sigma}$. Thus we are inspired to explore the the following two conjectures which are stated in full generality.

Conjecture 5.1. *For $m \in \mathbb{N}$, let $\mathbb{P}_{\mathbf{\Sigma}}$ be a proper smooth dimension m toric DM stack associated to a complete stacky fan $\mathbf{\Sigma} = (\Sigma, \{v_i\}_{i=1}^n)$. Then there are infinitely many* H−*trivial line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$ if and only if there is a $\mathbf{\Sigma}-$piecewise linear function ψ such that* $\dim(\Lambda_\psi) < m$.

This conjecture means that all these three statements (1), (2), (3) are equivalent to each other. The following conjecture is a weaker version which means (2) is equivalent to (3).

Conjecture 5.2. *For $m \in \mathbb{N}$, let $\mathbb{P}_{\mathbf{\Sigma}}$ be a proper smooth dimension m toric DM stack associated to a complete stacky fan $\mathbf{\Sigma} = (\Sigma, \{v_i\}_{i=1}^n)$. Then there are infinitely many $\mathrm{H}-$trivial line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$ if and only if there exists a nonzero element $\mathcal{L} \in \mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$ which is not contained in the interior of Z_I for any $I \in \Delta$.*

We hope that the methods and the approach of this paper will be useful for settling these two conjectures.

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