arXiv:alg-geom/9703023v2 2 Apr 1997

On Betti numbers and Chern classes of varieties with trivial odd cohomology groups

Lev A. Borisov
Mathematical Sciences
Research Institute
Berkeley, CA 94720
e-mail: lborisov@msri.org

Abstract

It was observed in a very recent preprint (hep-th/9703086) by T. Eguchi, K. Hori, and Ch.-Sh. Xiong that a curious identity between Betti numbers and Chern classes holds for many examples of Fano varieties. The goal of this paper is to prove that for varieties with trivial odd cohomology groups this identity is equivalent to having zero Hodge numbers $h^{p,q}$ for $p \neq q$.

1 Introduction

Let X be a smooth complex projective variety of dimension n whose odd cohomology groups $H^{2k+1}(\mathbf{C})$ are zero. It was noticed in [2] that a curious identity

$$\frac{1}{4}\sum_k h^{2k}(k-\frac{n-1}{2})(1-k+\frac{n-1}{2}) = \frac{1}{24}\left(\frac{3-n}{2}\chi(X) - \int_X c_1(X)\wedge c_{n-1}(X)\right)$$

holds in many examples. The authors of [2] were primarily concerned with Fano varieties X and the above identity was a prerequisite for the conjectural construction of Virasoro operators that control quantum cohomology of X. The above condition could be rewritten as

$$\sum_{k=0}^{n} h^{2k}(k-\frac{n}{2})^2 = \frac{1}{6}c_1c_{n-1} + \frac{n}{12}c_n$$

where c_l is the l-th Chern class of X. The goal of this paper is to prove the following result.

Proposition 1.1 *If X is a smooth complex projective variety of dimension n with $H^{odd}(X, \mathbf{C}) = 0$ then*

$$\sum_{k=0}^{n} h^{2k}(k - \frac{n}{2})^2 \leq \frac{1}{6}c_1 c_{n-1} + \frac{n}{12}c_n$$

and equality holds if and only if

$$h^{p,q} = 0 \text{ for all } p \neq q.$$

In addition, we give an application of this result to the combinatorics of reflexive polytopes that describe smooth toric Fano varieties.

After this preprint was submitted to the archive, the author was informed by Anatoly Libgober that the proof of the crucial Proposition 2.2 is contained in [1]. In addition, Victor Batyrev presented to the author a combinatorial proof of Corollary 2.3 for arbitrary smooth toric varieties.

2 Proof of the main result

Let X be a smooth complex projective variety of dimension n. We introduce the E-polynomial

$$E(u, v) = \sum_{p,q} (-1)^{p+q} h^{p,q} u^p v^q$$

where $h^{p,q} = \dim H^q(X, \Lambda^p T^* X)$ are Hodge numbers of X. We also introduce

$$\chi_p = (-1)^p \chi(\Lambda^p T^* X) = \sum_q (-1)^{p+q} h^{p,q}$$

and the polynomial

$$\hat{E}(t) = \sum_p \chi_p t^p = E(t, 1).$$

Remark 2.1 *If $h^{p,q} = 0$ for all $p \neq q$ then $\chi_p = h^{p,p}$.*

Proposition 2.2 *In the above notations*

$$\sum_{p=0}^{n} \chi_p (p - \frac{n}{2})^2 = \frac{1}{6}c_1 c_{n-1} + \frac{n}{12}c_n.$$

Proof. This result comes as an easy application of Hirzebruch-Riemann-Roch theorem [3]. We start by rewriting the left hand side in terms of the polynomial $\hat{E}$.

$$\sum_p \chi_p (p-\frac{n}{2})^2 = \sum_p \chi_p p(p-1) + \sum_p \chi_p (1-n)(p-\frac{n}{2}) + (\sum_p \chi_p)(\frac{n}{2}-\frac{n^2}{4})$$

$$= \frac{d^2}{dt^2}\hat{E}(t)|_{t=1} + \hat{E}(1)(\frac{n}{2}-\frac{n^2}{4}).$$

We have used $\chi_p = \chi_{n-p}$ to get rid of the second sum. By Hirzebruch-Riemann-Roch theorem,

$$\hat{E}(t) = \sum_p t^p (-1)^p \chi(\Lambda^p T^* X) = \int_X Td(X) \sum_p (-t)^p ch(\Lambda^p T^* X).$$

As usual, we introduce Chern roots α_i such that $c(TX)(w) = \prod_i (1+\alpha_i w)$, see for example [4]. Then

$$\hat{E}(t) = \int_X (\prod_i \frac{\alpha_i}{1-\mathrm{e}^{-\alpha_i}}) \sum_p (-t)^p \sum_{i_1<\ldots<i_p} \mathrm{e}^{-\alpha_{i_1}-\ldots-\alpha_{i_p}}$$

$$= \int_X \prod_i \alpha_i (1+(1-t)\frac{e^{-\alpha_i}}{1-\mathrm{e}^{-\alpha_i}}).$$

This shows, of course, that $\hat{E}(1) = \chi(X) = c_n$. Besides we can calculate

$$\frac{d^2}{dt^2}\hat{E}(t)|_{t=1} = 2\sum_{i<j}\int_X (\prod_{k\neq i,j}\alpha_k)(1-\frac{1}{2}\alpha_i+\frac{1}{12}\alpha_i^2)(1-\frac{1}{2}\alpha_j+\frac{1}{12}\alpha_j^2)$$

$$= \frac{1}{6}c_1 c_{n-1} + (\frac{n^2}{4}-\frac{5n}{12})c_n$$

and the rest is straightforward. □

We combine Proposition 2.2 with Remark 2.1 to get the following corollary.

Corollary 2.3 *If $h^{p,q}=0$ for all $p\neq q$ then*

$$\sum_{p=0}^{n} h^{2p}(p-\frac{n}{2})^2 = \frac{1}{6}c_1 c_{n-1} + \frac{n}{12}c_n.$$

This corollary gives a sufficient condition for the identity of [2]. Now we will see that this condition is also necessary, that is we will prove Proposition 1.1.

Proof of Proposition 1.1. Because of $h^{odd}=0$, we have $\chi_p = \sum_q h^{p,q}$ and

$$\sum_{p=0}^{n} h^{2p}(p-\frac{n}{2})^2 = \sum_{p,q} h^{p,q}(\frac{p+q}{2}-\frac{n}{2})^2$$

$$= \sum_p (\sum_q h^{p,q})(p - \frac{n}{2})^2 + \sum_{p,q} h^{p,q}(\frac{q-p}{2})(\frac{3p+q}{2} - n)$$

$$= \sum_p \chi_p (p - \frac{n}{2})^2 - \sum_{p,q} h^{p,q}(\frac{q-p}{2})^2 + \sum_{p,q} h^{p,q}(\frac{q-p}{2})(p+q-n).$$

Because of $h^{p,q} = h^{q,p}$, the last sum is zero. Together with the result of Proposition 2.2, we get

$$\sum_{p=0}^{n} h^{2p}(p - \frac{n}{2})^2 = \frac{1}{6}c_1 c_{n-1} + \frac{1}{12}c_n - \sum_{p,q} h^{p,q}(\frac{q-p}{2})^2$$

which proves the proposition. □

3 Application to toric Fano varieties

The goal of this section is to give a combinatorial equivalent of Corollary 2.3 for the case of smooth toric Fano varieties.

Recall (see for example [5]) that a smooth toric Fano variety can be defined in terms of the polytope $\Delta \in M$ that supports the sections of the anticanonical line bundle.

To calculate the left hand side, notice that X is a disjoint union of algebraic tori T_θ. Using the additivity of the cohomology with compact support, we get

$$\hat{E}(t) = \sum_{\theta \subseteq \Delta} (t-1)^{\dim(\theta)}$$

and

$$\frac{d^2}{dt^2}\hat{E}(t)|_{t=1} = 2\#\{\theta \subseteq \Delta, \dim(\theta) = 2\}.$$

Analogously,

$$\hat{E}(1) = \#\{\theta \subseteq \Delta, \dim(\theta) = 0\}.$$

To calculate the right hand side of the identity, notice that

$$c(TX)(w) = \prod_{\theta, \dim(\theta) = n-1} (1 + D_\theta w),$$

where D_θ is the closure of the strata that corresponds to θ, and that $\sum D_\theta$ is a divisor with normal crossings. This gives

$$c_1 = \sum_{\theta, \dim(\theta) = n-1} D_\theta$$

$$c_{n-1} = \sum_{\theta, \dim(\theta) = 1} l_\theta$$

where in the second identity l_θ is the closure of T_θ. One can show that $c_1 l_\theta$ equals the number of interior points of θ plus one. Now an easy calculation shows that the identity of Corollary 2.3 becomes

$$\#\{\theta \subseteq \Delta, \dim(\theta) = 2\} = \frac{1}{12} \sum_{\theta, \dim(\theta)=1} \#\{P \in M,\ P \in \mathrm{interior}(\theta)\}$$

$$+(\frac{n^2}{8} - \frac{n}{6})\#\{\theta \subseteq \Delta, \dim(\theta) = 0\}.$$

References

[1] A. Libgober, J. Wood, *Uniqueness of the complex structure on Kähler manifolds of certain homolgy types*, J. Diff. Geom., vol. 32 (1990), 139-154

[2] T. Eguchi, K. Hori, Ch.-Sh. Xiong, *Quantum cohomology and Virasoro algebra*, preprint hep-th/9703086

[3] F. Hirzebruch, *Topological methods in algebraic geometry*, 1956, Grundlehren der math. Wissenschaften, vol. 131, Third enlarged edition, Springer-Verlag, 1966

[4] W. Fulton, *Intersection theory*, Springer-Verlag, 1984

[5] V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, J. Algebraic Geom., 3 (1994), 493-535