# ASYMPTOTICS OF IMMACULATE LINE BUNDLES ON SMOOTH TORIC DELIGNE-MUMFORD STACKS

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ABSTRACT. A line bundle is immaculate if its cohomology vanishes in every dimension. We give a criterion for when a smooth toric Deligne-Mumford stack has infinitely many immaculate line bundles. This answers positively a question of Borisov and Wang.

#### 1. Introduction

Let X be a smooth proper toric Deligne-Mumford stack of dimension d over an algebraically closed field k of characteristic 0. In particular, X could be a smooth proper toric variety. Following [ABKW20], a line bundle  $\mathcal{L}$  on X is called *immaculate* iff  $H^i(X,\mathcal{L}) = 0$  for all  $i \geq 0$ . We denote the set of isomorphism classes of immaculate line bundles by Imm(X). Following work in [Wan23, BW19], we find a criterion for when Imm(X) is infinite.

To describe the criterion, we need to use the notion of forbidden cones, introduced in [BH09]. These are rational polyhedral cones  $F_I$  in the real vector space  $\operatorname{Pic}_{\mathbb{R}}(X) := \operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$  indexed by certain subsets I of the rays in the fan of X; see Section 2 for more details. If the image in  $\operatorname{Pic}_{\mathbb{R}}(X)$  of a line bundle does not lie in any of the forbidden cones, then it is immaculate (although the converse generally fails). Let  $C_I$  denote the translate of the forbidden cone  $F_I$  such that the vertex is at the origin.

In this paper we prove the following statement, which answers positively one of the questions raised in [BW19, Section 5].

**Theorem 1.1.** There exist infinitely many immaculate line bundles on X if and only if there exists a line l passing through the origin of  $\operatorname{Pic}_{\mathbb{R}}(X)$  such that, for every translated forbidden cone  $C_I$  of X, the intersection with the interior  $l \cap C_I^{\circ}$  is empty.

It is easy to see that the union of all translated forbidden cones  $C_I$  is all of  $\operatorname{Pic}_{\mathbb{R}}(X)$ . The theorem above is equivalent to saying that there are finitely many immaculate line bundles if and only if the interiors  $C_I^{\circ}$  of all the translated forbidden cones cover  $\operatorname{Pic}_{\mathbb{R}}(X) \setminus \{0\}$ .

In the course of proving this theorem, we actually establish a finer structural result on the asymptotics of immaculate line bundles, which we describe now.

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We embed  $\operatorname{Pic}_{\mathbb{R}}(X) \cong \mathbb{R}^r$  as an open subset of a real projective space  $\mathbb{P}(\operatorname{Pic}_{\mathbb{R}}(X) \oplus \mathbb{R}) \cong \mathbb{RP}^r$  where r is the rank of  $\operatorname{Pic}(X)$ . Let  $\operatorname{Imm}^{\infty}(X)$  denote the set of accumulation points of the image of  $\operatorname{Imm}(X)$  in  $\mathbb{RP}^r$ . Clearly,  $\operatorname{Imm}^{\infty}(X)$  is a subset of the hyperplane at infinity  $\Pi \cong \mathbb{RP}^{r-1}$ . For each forbidden cone, let  $D_I := \overline{C_I} \setminus C_I$  be the complement of the closure of  $C_I$  in  $\mathbb{RP}^r$ . We see that  $D_I$  is a closed subset of  $\Pi$ .

**Theorem 1.2.** Let  $D_I^{\circ}$  be the relative interior of  $D_I$  in  $\Pi$ . Then

$$\mathrm{Imm}^\infty(X) = \Pi \setminus \bigcup_I D_I^\circ$$

where the union is over all I for which  $C_I$  is a translate of a forbidden cone.

**Remark 1.3.** It is clear that Theorem 1.1 is an immediate consequence of Theorem 1.2. Indeed, the torsion of Pic(X) is finite, so Imm(X) is infinite if and only if its image in  $Pic_{\mathbb{R}}(X)$  is infinite. The latter is infinite if and only if it has an accumulation point in  $\mathbb{RP}^r$ .

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## 2. Forbidden Cones

We review the notion of forbidden cones and their relation to immaculate line bundles first established in [BH09]. Note that our conventions are more in line with [ABKW20].

Recall from [BCS05] that a smooth projective toric Deligne-Mumford stack X is determined by the combinatorial data of a stacky fan  $\Sigma = (N, \Sigma, \{v_i\})$ . Here N is a finitely generated free abelian group,  $\Sigma$  is a complete simplicial fan in  $N \otimes_{\mathbb{Z}} \mathbb{R}$ , and  $v_1, \ldots, v_n$  are nonzero elements of N, one in each of the rays of  $\Sigma$ . The rank of N is the same as the dimension d of X.

**Remark 2.1.** More generally, one can take N to be any finitely generated abelian group. For simplicity of presentation, we assume that N is free.

The Picard group Pic(X) is isomorphic to the quotient of the group  $\mathbb{Z}^n$  of torus-equivariant divisors by the subgroup of principal torus-equivariant divisors. More precisely, the rays of  $\Sigma$  give rise to a canonical generating set  $E_1, \ldots, E_n$  for Pic(X). The relations on  $\{E_i\}$  are given by

$$\sum_{i=1}^{n} \langle m, v_i \rangle E_i = 0$$

for all  $m \in M = \text{Hom}(N, \mathbb{Z})$ . We denote by  $\bar{E}_i$  the images of  $E_i$  in  $\text{Pic}_{\mathbb{R}}(X)$ . The relations among  $\bar{E}_i$  are given by the same formula with  $m \in M_{\mathbb{R}}$ .

Given a subset  $I \subseteq \{1, ..., n\}$ , we define a subset  $FS_I = q_I + CS_I$  of Pic(X) where

$$q_I = -\sum_{i \in I} E_i,$$

is a point and

$$CS_I = \sum_{i \neq I} \mathbb{Z}_{\geq 0} E_i - \sum_{i \in I} \mathbb{Z}_{\geq 0} E_i,$$

is a monoid. More precisely, we define a function  $\pi_I: \mathbb{Z}^n_{>0} \to \mathrm{Pic}(X)$  via

(2.1) 
$$\pi_I(a_1, \dots, a_n) = \sum_{i \notin I} a_i E_i - \sum_{i \in I} (1 + a_i) E_i$$

and observe that  $FS_I$  is the image of  $\pi_I$ . We also define a convex cone  $F_I := q_I + C_I$  in  $\operatorname{Pic}_{\mathbb{R}}(X)$  where

$$C_I = \sum_{i \notin I} \mathbb{R}_{\geq 0} \bar{E}_i - \sum_{i \in I} \mathbb{R}_{\geq 0} \bar{E}_i.$$

Clearly, the image of  $FS_I$  under  $Pic(X) \to Pic_{\mathbb{R}}(X)$  lies in  $F_I$ .

Recall that the fan  $\Sigma$  gives rise to an abstract simplicial complex on  $\{1,\ldots,n\}$ , which we also denote by  $\Sigma$ . Given a subset  $I\subseteq\{1,\ldots,n\}$ , we may form a new simplicial complex  $\Sigma|_I$  on I where  $\sigma\subseteq I$  is in  $\Sigma|_I$  if and only if  $\Sigma$  contains the cone generated by the rays indexed by  $\sigma$ . Observe that  $\Sigma|_{\{1,\ldots,n\}}$  corresponds to  $\Sigma$  itself, while  $\Sigma|_{\emptyset}$  is the simplicial complex  $\{\emptyset\}$ . Let  $\widetilde{H}_n(\Omega,k)$  denote the reduced homology of the abstract simplicial complex  $\Omega$ . Recall that  $\widetilde{H}_{-1}(\Omega,k)\neq 0$  if and only if  $\Omega=\{\emptyset\}$ .

**Proposition 2.2.** For  $\mathcal{L} \in \text{Pic}(X)$ , we have

$$H^{i}(X,\mathcal{L}) = \bigoplus_{I \subseteq \{1,\dots,n\}} \widetilde{H}_{i-1}(\Sigma|_{I},k)^{\oplus p_{I}}.$$

where  $p_I$  is the cardinality of  $\pi_I^{-1}(\mathcal{L})$ .

*Proof.* Adjusting for conventions, this is [BH09, Proposition 4.1].  $\Box$ 

If the reduced homology  $\widetilde{H}_i(\Sigma|_I, k)$  is not trivial for some  $i \geq -1$ , then we say I is tempting. If I is tempting, then  $FS_I$  is a forbidden set and  $F_I$  is a forbidden cone. As an immediate consequence of the previous proposition we have:

**Proposition 2.3.** A line bundle  $\mathcal{L}$  is immaculate if and only if  $\mathcal{L}$  is not contained in any forbidden set  $FS_I$  for a tempting set I.

**Remark 2.4.** Proposition 2.3 implies that if the image of  $\mathcal{L}$  in  $\operatorname{Pic}_{\mathbb{R}}(X)$  is not contained in any forbidden cone, then  $\mathcal{L}$  is immaculate. However, the converse does not generally hold.

Our conventions ensure that  $\emptyset$  is tempting and  $F_{\emptyset}$  is just the effective cone. Observe that  $\Sigma|_{\{1,\ldots,n\}}$  corresponds to  $\Sigma$  itself, which is a simplicial (d-1)-sphere; thus  $\widetilde{H}_{d-1}(\Sigma) \neq 0$  and  $\{1,\ldots,n\}$  is tempting. The forbidden set  $FS_{\{1,\ldots,n\}}$  corresponds to line bundles  $\mathcal{L}$  for which  $H^d(X,\mathcal{L}) \neq 0$ .

**Remark 2.5.** Let  $s: Pic(X) \to Pic(X)$  be the Serre duality map  $s(D) = K_X - D$ . The canonical divisor of X is given by

$$K_X = q_{\{1,\dots,n\}} = -\sum_{i=1}^n E_i.$$

Thus  $s(FS_I) = FS_{I^c}$  for all subsets I where  $I^c := \{1, \ldots, n\} \setminus I$  denotes the complement. Consequently, I is tempting if and only if  $I^c$  is tempting. (See also [ABKW20, Remark 5.4].)

## 3. The Thomsen Zonotope

For every positive integer m, the m-th  $toric\ Frobenus$  is the endomorphism  $F_m: X \to X$  obtained by extending the power map  $x \mapsto x^m$  on the torus to all of X. In [Tho00], Thomsen shows that the direct image  $(F_m)_* \mathcal{O}_X$  of the structure sheaf is a direct sum of line bundles and gave a precise description of the line bundles that result (see [Ach15] for a shorter proof). The  $Thomsen\ collection$  is the (finite) set of line bundles that occur as direct summands of

$$\bigoplus_{m=1}^{\infty} (F_m)_* \mathcal{O}_X.$$

It is well known that all non-trivial line bundles in the Thomsen collection are immaculate.

We define the *Thomsen half-open zonotope* as the set

$$Z^h := \left\{ \sum_{i=1}^n \gamma_i \bar{E}_i \mid \gamma_i \in (-1, 0] \right\},\,$$

in  $\operatorname{Pic}_{\mathbb{R}}(X)$ . It is a convex set, and the line bundles in the Thomsen collection map to  $Z^h$ . We denote the closure of  $Z^h$  by Z and the interior by  $Z^\circ$ . The polytope Z is a zonotope — a Minkowski sum of finitely many line segments  $[-1,0]\bar{E}_i$ . Similarly,  $Z^\circ = \sum_{i=1}^n (-1,0)\bar{E}_i$ .

**Proposition 3.1.** If  $F_I = q_I + C_I$  is a forbidden cone, then Z is a subset of  $q_I - C_I$ . Moreover,  $Z \cap U = (q_I - C_I) \cap U$  in a neighborhood U of  $q_I$ .

*Proof.* This follows almost immediately from the definitions. The set Z consists of points that can be written as

$$\sum_{i=1}^{n} a_i \bar{E}_I$$

where  $a_i \in [-1,0]$  for all indices. The set  $q_I - C_I$  consists of such points where  $a_i \leq 0$  if  $i \notin I$  and  $a_i \geq -1$  if  $i \in I$ . The points in Z are all of this

form. Conversely, for a sufficiently small  $\varepsilon$ -ball U around  $q_I$ , there exists a  $0 < \delta < 1$ , such that all points of  $(q_I - C_I) \cap U$  can be written as above with  $a_i \in (-\delta, 0]$  for  $i \notin I$  and  $a_i \in [-1, -1 + \delta)$  for  $i \in I$ . These points lie inside Z, thus the second statement follows.

**Lemma 3.2.** There is a line bundle mapping to  $Z^{\circ}$ .

*Proof.* Pick an element  $m \in \text{Hom}(N, \mathbb{R})$  such that  $\langle m, v_i \rangle$  is noninteger for all i. Then

$$\sum_{i=1}^{n} \lfloor \langle m, v_i \rangle \rfloor \bar{E}_i = -\sum_{i=1}^{n} \{ \langle m, v_i \rangle \} \bar{E}_i \in Z^{\circ}$$

so the line bundle  $\sum_{i=1}^{n} \lfloor \langle m, v_i \rangle \rfloor E_i$  has the desired property.

The next result is not new but we provide a proof for the reader's benefit. A similar statement can be found in [ABKW20, Lemma 5.24(ii)].

**Lemma 3.3.** If I is tempting, then  $q_I$  is a vertex of Z. Equivalently,  $C_I$  is strongly convex with vertex at the origin.

*Proof.* From Proposition 3.1, we see that  $q_I$  is a vertex of Z if and only if  $C_I \cap (-C_I) \cap U = \{0\}$  for some neighborhood U of the origin.

Suppose  $q_I$  is not a vertex of Z. Then there is a non-zero vector v in  $C_I \cap (-C_I)$ . Since  $C_I$  is rational polyhedral we can assume v can be written as a  $\mathbb{Q}$ -linear combinations of the generators of both cones. By taking the difference and scaling, we obtain a non-trivial relation  $\sum_{i \notin I} a_i \bar{E}_i - \sum_{i \in I} a_i \bar{E}_i = 0$  in  $\mathrm{Pic}_{\mathbb{R}}(X)$ , where integers  $a_1, \ldots, a_n$  are nonnegative and not all zero. We can scale further by an appropriate integer to ensure that

$$\sum_{i \notin I} a_i E_i - \sum_{i \in I} a_i E_i = 0$$

holds in Pic(X). For any nonnegative integer k we write

$$-\sum_{i \in I} E_i = \sum_{i \in I} (-1 - ka_i) E_i + \sum_{i \notin I} (ka_i) E_i$$

so  $\pi_I^{-1}(-\sum_{i\in I} E_i)$  is infinite. Since I is tempting, by Proposition 2.2 we get  $H^*(-\sum_{i\in I} E_i)$  is infinite-dimensional, in contradiction to X being proper.

**Remark 3.4.** There are typically vertices of Z which do not correspond to any tempting sets I.

# 4. Proof of the Main Theorem

We will use the following well known result, which is often considered in the context of higher-dimensional analogs of the Frobenius Coin Problem; see, for example, [RA05, Theorem 6.5.1].

**Lemma 4.1.** Let L be a finitely generated abelian group and let  $\pi: L \to L_{\mathbb{R}}$  be the natural map. Let M be a finitely generated submonoid of L which generates L as a subgroup. Then there exists an element  $m \in M$  such that

$$\pi^{-1}(m + \mathbb{R}_{\geq 0}\pi(M)) \subseteq M.$$

We now proceed to the proof of our main theorem.

*Proof of Theorem 1.2.* The argument for the inclusion  $\subseteq$  in Theorem 1.2 is essentially in the proof of [Wan23, Proposition 3.8]. For the reader's convenience, we sketch it here.

Suppose I is tempting. Observe that  $\Pi \cap \overline{v + C_I} = \Pi \cap \overline{v' + C_I}$  for any  $v, v' \in \operatorname{Pic}_{\mathbb{R}}(X)$  and  $\Pi \cap \overline{C_I} = \Pi \cap \overline{C_{I^c}}$ . In view of Proposition 2.3 and Lemma 4.1 applied to the group  $\operatorname{Pic}(X)$  and monoids  $CS_I$  and  $CS_{I^c}$ , there exist  $v, v' \in \operatorname{Pic}(X)$  such that  $v + C_I$  and  $v' + C_{I^c}$  contain no images of immaculate line bundles under the map  $\operatorname{Pic}(X) \to \operatorname{Pic}_{\mathbb{R}}(X)$ . It remains to observe that for any  $L \in D_I^\circ$  the set  $A = \overline{(v + C_I) \cup (v' + C_{I^c})}$  contains an open neighborhood of L and no images of immaculate line bundles. Thus L is not in  $\operatorname{Imm}^\infty(X)$ .

We now establish the inclusion  $\supseteq$ , which is new.

Suppose  $L \in \Pi$  is a point such that  $L \notin D_I^{\circ}$  for any tempting I. We first assume that L is rational, so that the corresponding line  $\ell \subseteq \operatorname{Pic}_{\mathbb{R}}(X)$  has infinitely many images of elements of  $\operatorname{Pic}(X)$  on it. Pick a point  $z_0$  in  $Z^{\circ}$  which is an image of a line bundle, its existence guaranteed by Lemma 3.2. Since  $\ell$  is rational, there are infinitely many elements  $\bar{E}$  in the shifted line  $z_0 + \ell$  which are images of elements of  $\operatorname{Pic}(X)$ . These have  $L \in \Pi$  as their (unique) accumulation point, and we will show that each  $\bar{E}$  is an image of an immaculate line bundle, thus establishing that  $L \in \operatorname{Imm}^{\infty}(X)$ . By Proposition 2.3 it suffices to show that  $z_0 + \ell$  is disjoint from all  $F_I$  for tempting I.

By Lemma 3.3,  $C_I$  is a strongly convex rational polyhedral cone with vertex at the origin. Since  $\ell$  is disjoint from the interior of  $C_I$ , there exists a nonzero linear functional  $h: \operatorname{Pic}_{\mathbb{R}}(X) \to \mathbb{R}$  such that  $h(C_I) \geq 0$  but  $h(\ell) = 0$ . By Proposition 3.1, we see that  $z_0 - q_I \in -C_I^{\circ}$ , thus  $h(z - q_I) < 0$ . So for any  $v \in \ell$  we have  $h(z_0 + v) < h(q_I)$ , while we have  $h(F_I) = h(q_I) + h(C_I) \geq h(q_I)$ .

We have thus established that the rational points of

$$\Pi \setminus \bigcup_{I \text{ tempting}} D_I^{\circ}$$

lie in  $\operatorname{Imm}^{\infty}(X)$ . However, there are finitely many inequalities with rational coefficients defining each  $C_I$ . Moreover, there are only finitely many tempting sets I. Thus, the rational points of the set  $\Pi \setminus \bigcup_{I \text{ tempting }} D_I^{\circ}$  are dense. Since all rational points lie in  $\operatorname{Imm}^{\infty}(X)$  and both sets are closed, we have the desired inclusion.

**Remark 4.2.** Our main theorem is phrased using the compactification of  $\operatorname{Pic}_{\mathbb{R}}(X)$  with boundary  $\Pi \cong \mathbb{RP}^{r-1}$ . Instead, we could have used the spherical boundary  $(\operatorname{Pic}_{\mathbb{R}}(X) \setminus \{0\}) / \mathbb{R}_{>0} \cong S^{r-1}$ . However, in view of Remark 2.5, there is no substantial difference between these two perspectives.

**Remark 4.3.** From the proof of our main theorem, we see that if there are infinitely many immaculate line bundles, then there are infinitely many outside the forbidden cones. However, there may still be infinitely many line bundles inside the forbidden cones as well. For example, consider the stack  $X = \mathbb{P}(2:3) \times \mathbb{P}^1$  where  $\mathbb{P}(2:3)$  is a weighted projective line. The line bundles  $\mathcal{O}_X(1,n)$  are immaculate for every positive n, but they all lie in the interior of the effective cone.

Remark 4.4. In [Per11], Perling explores conditions for vanishing of cohomology groups in the closely related context of divisorial sheaves on toric varieties. In particular, he finds a sufficient condition for the existence of infinitely many immaculate divisorial sheaves in terms of the Iitaka dimensions of divisors on the boundary of the nef cone. There, the study of discriminantal arrangements plays a similar role as our use of the Thomsen Zonotope above.

**Remark 4.5.** Finally, we point out that while the criterion from Theorem 1.2 is computable in theory, finding all the tempting subsets and forbidden cones is complicated in practice. Another condition that can be checked directly on the fan is conjectured in [BW19, Section 5].

An element of  $\operatorname{Pic}_{\mathbb{R}}(X)$  can be represented as a function  $\psi$  on  $N_{\mathbb{R}}$  which is linear on every cone of  $\Sigma$ , which is well-defined up to a global linear function. The maximal cones  $\{\sigma\}$  of  $\Sigma$  then correspond to a finite set of points  $\{\psi_{\sigma}\}$  in the dual space  $M_{\mathbb{R}}$ . If the convex hull  $\Lambda_{\psi}$  of these points is not full-dimensional, then we obtain a point of  $\operatorname{Imm}^{\infty}(X)$  by [BW19, Theorem 2.14].

While this is a sufficient condition for having infinitely many immaculate line bundles, it is not known whether it is necessary — even for toric varieties of dimension 3.

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