ON THE GEOMETRY OF A FAKE PROJECTIVE PLANE WITH 21 AUTOMORPHISMS

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Abstract. A fake projective plane is a complex surface with the same Betti numbers as $\mathbb{C}P^2$ but not biholomorphic to it. We study the fake projective plane $\mathbb{P}^2_{\text{fake}} = (a = 7, p = 2, \emptyset, D_3 2_7)$ in the Cartwright-Steger classification. In this paper, we exploit the large symmetries given by $\mathrm{Aut}(\mathbb{P}^2_{\text{fake}}) = C_7 \rtimes C_3$ to construct an embedding of this surface into $\mathbb{C}P^5$ as a system of 56 sextics with coefficients in $\mathbb{Q}(\sqrt{-7})$. For each torsion line bundle $T \in \mathrm{Pic}(\mathbb{P}^2_{\text{fake}})$, we also compute and study the linear systems $|nH + T|$ with small n, where H is an ample generator of the Néron–Severi group.

1. Introduction

A question by Severi asked whether there exists a complex surface homeomorphic to $\mathbb{C}P^2$ but not biholomorphic to it. A classical corollary in Yau's proof of the Calabi-Yau conjecture [Yau77] answered this question in the negative by showing that any complex surface homotopic to $\mathbb{C}P^2$ must also be biholomorphic to $\mathbb{C}P^2$. This prompted Mumford [Mum79] to construct the first example of a complex surface with the same Betti numbers as $\mathbb{C}P^2$ but not biholomorphic to it. These surfaces are now called *fake projective planes* (FPPs for short).

Fake projective planes have ample canonical bundle and hence are algebraic surfaces of general type by Chow's Theorem. They serve as canonical examples for complex surfaces of general type with the smallest Euler characteristics. Studying their geometry and classification is a subject of interest for many algebraic geometers.

For a fake projective plane X, the Hodge Theorem implies that $H^j(X, \mathbb{C}) = \bigoplus_{p+q=j} H^{p,q}(X)$. It then follows from the Hodge symmetry that the Hodge numbers of X must be

$$h^{p,q}(X) = \begin{cases} 1, \text{if } (p,q) \in \{(0,0),(1,1),(2,2)\} \\ 0, \text{otherwise.} \end{cases}$$

Let c_1 and c_2 be the first and second Chern numbers of X respectively, Noether's formula asserts that

$$\frac{c_1^2 + c_2}{12} = \chi(0) = \sum_{j=0}^{2} (-1)^j h^{0,j}(X) = 1$$

Recalling that c_2 is equal to the topological Euler characteristics of X, it then follows that $c_1^2 = 3c_2$ and hence the Bogomolov–Miyaoka–Yau inequality is an equality. A classical result in [Yau77] asserts that this equality is true if and only if X is the quotient of the complex 2-ball $\mathbb{B}^2 := \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 < 1\}$ by a torsion-free co-compact discrete subgroup of $PU(2, 1)$.

Much of the work in classifying all fake projective planes has been done by analyzing these quotients. In the same paper by Mumford [Mum79], he noted that there exist only finitely many fake projective planes up to isomorphism. This is a consequence of Weil's result [Wei60] that discrete co-compact subgroups of $PU(2, 1)$ are rigid. Later on, Prasad and Yeung [PY07] showed that all fake projective planes must fall into one of 28 distinct non-empty classes. Finally, Cartwright and Steger [CS10] obtained a complete classification of 50 conjugate pairs of fake projective planes into the aforementioned 28 different classes. For a comprehensive survey on the history of classifying all fake projective planes, we refer the reader to the expository paper by Yeung in [Yeu08].

2020 *Mathematics Subject Classification.* Primary: 14E25, 14Q10. Secondary: 14C20, 14J29.

In this paper, we will study the fake projective plane $\mathbb{P}^2_{\text{fake}} \coloneqq (a = 7, p = 2, \emptyset, D_3 2_7)$ in the Cartwright-Steger classification. It is one of the only 3 pairs of fake projective planes whose automorphism group has the largest cardinality, being the unique non-abelian semi-direct product $C_7 \rtimes C_3$. The other two pairs are respectively the fake projective plane $(C20, p = 2, \emptyset, D_3 2_7)$ and Keum's fake projective plane [Keu06] $(a = 7, p = 2, 7, D_3 2_7)$.

It is known that there exists a unique ample divisor class H on $\mathbb{P}^2_{\text{fake}}$ such that its canonical class K is equal to $3H$, and the Picard group of $\mathbb{P}^2_{\text{fake}}$ is given by $\text{Pic}(\mathbb{P}^2_{\text{fake}}) \cong \mathbb{Z}H \oplus (\mathbb{Z}/2\mathbb{Z})^4$. In [BK20], Borisov and Keum produced an explicit construction of the $\mathbb{P}^2_{\text{fake}}$ in $\mathbb{C}P^9$ as 84 cubic equations in 10 variables using the 10 global sections of $H^0(\mathbb{P}^2_{\text{fake}}, 6H)$.

Our first goal is to study how the automorphism group $\text{Aut}(\mathbb{P}^2_{\text{fake}}) \cong C_7 \rtimes C_3$ of $\mathbb{P}^2_{\text{fake}}$ acts on $\text{Pic}(\mathbb{P}^2_{\text{fake}})$. Clearly any automorphism has to fix H, so the question amounts to investigating what happens on the torsion subgroup. For each non-trivial torsion element $T \in \text{Pic}(\mathbb{P}^2_{\text{fake}})$, we will compute explicit representatives of $3H + T$ as non-reduced cuts in $6H$ in Section 2. We focus on three torsion classes - $D, D_1, D + D_1$ - which are representatives of the C_7 orbits of the automorphism action.

Another question we are interested in is how the linear systems $|nH + T|$ behave for $n \leq 5$ and torsion line bundles $T \in \text{Pic}(\mathbb{P}^2_{\text{fake}})$. This is because as n increases, the expansion in dimensions tends to trivialize phenomena in lower dimensions (for example, the Kodaira Vanishing Theorem becomes applicable for $n \geq 4$). Specifically, we show that

Theorem 1.1. *On the fake projective plane $\mathbb{P}^2_{\text{fake}}$ and for torsion line bundle $T \in \text{Pic}(\mathbb{P}^2_{\text{fake}})$,*

(1) $h^0(\mathbb{P}^2_{\text{fake}}, 2H + T) = 0$.

(2) $4H + T$ *is basepoint free if and only if* T *is non-trivial.*

(3) $5H + D$ *is very ample, but* $5H$ *is not very ample.*

Note that the case of $T \in \{0, D\}$ in Theorem 1.1(1) is a consequence of Theorem 5.3 in [GKS23]. Our contribution is for the other 14 torsion line bundles. As a consequence (or rather in the proof of) Theorem 1.1(3), we also produce an explicit embedding of $\mathbb{P}^2_{\text{fake}}$ as 56 sextics in $\mathbb{C}P^5$ with coefficients in $\mathbb{Q}(\sqrt{-7})$.

For all fake projective planes X with known explicit equations (see [BK20], [Bor23], [BF20], and [BBF22]), their embeddings were constructed in $6H_X$, where H_X is the unique ample divisor class on X such that $3H_X = K_X$ the canonical divisor class. In [BL23], the authors were able to further embed Keum's fake projective plane with $5H_X$. Whether or not $6H_X$ is very ample for all fake projective planes X is still an open question, but Theorem 1.1(3) shows that a similar conjecture is false for the case of $5H_X$.

A common feature in the research on fake projective planes is liberal use of mathematical computing. The proof of Theorem 1.1 depends heavily on the use of the computer algebra systems `Mathematica` [Inc], `Magma` [BCP97], and `Macaulay2` [GS]. Our code repository [BJLM] accompanying this paper is available on GitHub and on our webpage.

1.1. **Outline.** In Section 2, we first describe a method of computing $3H + D, 3H + D_1, 3H + D + D_1$ and determining the group relations of the torsion divisors. We then find the explicit quadratic polynomials vanishing on $3H + T$ for each torsion divisor T. In Section 3, we compute explicit representatives of $4H$ and prove the "only if" of Theorem 1.1(2). In Section 4, we use the explicit representatives of $|4H|$ to compute explicit maps given by sections of $5H$ and $5H + D$ into $\mathbb{C}P^5$ to prove Theorem 1.1(3), and we also compute explicit equations for the zero locus of the C_7-equivariant sections of $5H$. In Section 5, we use the sections of $5H$ to compute explicit equations for the zero locus of the sections $4H + T$ for $T \neq 0$ a torsion line bundle. This proves the "if" direction of Theorem 1.1(2). In the same section, we also use this result to prove Theorem 1.1(1).

Acknowledgements. This research was conducted while the second author was participating during the 2023 DIMACS REU program at the Rutgers University Department of Mathematics, mentored by the first author. We are grateful to Lazaros Gallos, Kristen Hendricks, and the Rutgers University DIMACS for organizing the REU program and making this project possible.

2. Computing torsion divisors with $3H + T$

In this section, we construct explicit non-reduced cuts representing $3H + T$ for each non-trivial torsion line bundle $T \in \mathrm{Pic}(\mathbb{P}^2_{\mathrm{fake}})$. This is done in two steps. We first find the cuts in a finite field with the code `3H-Reduction.txt` in our code repository [BJLM]. We then lift the solutions back to $\mathbb{Q}(\sqrt{-7})$ in the first part of the file `3H-Torsion.nb`.

In addition, we also compute the group relations between the explicit representatives we found. Finally, we also compute quadratic polynomials vanishing on each curve in the class of $3H + T$ in the embedding of $\mathbb{P}^2_{\mathrm{fake}}$ in $\mathbb{C}P^9$. These calculations are done in the remainder of `3H-Torsion.nb`.

2.1. **Construction of the non-reduced cuts.** We would like to construct the torsion classes of the Picard group. Suppose T is a non-trivial torsion class, then Theorem 2.2 of [GKS23] asserts that $H^0(\mathbb{P}^2_{\mathrm{fake}}, 3H + T)$ is one-dimensional. Suppose we take generator $\alpha \in H^0(\mathbb{P}^2_{\mathrm{fake}}, 3H + T)$, then since all non-trivial torsions of the Picard group have order 2, we have $\alpha^{\otimes 2} \in H^0(\mathbb{P}^2_{\mathrm{fake}}, 6H)$. Thus, we would like to construct $3H + T$ from elements in $H^0(\mathbb{P}^2_{\mathrm{fake}}, 6H)$, whose zero is an nonreduced curve.

We can do this by examining how the automorphism group of $\mathbb{P}^2_{\mathrm{fake}}$ interacts with the torsion classes. In [BK20], the automorphism group acts on the homogeneous coordinates of $\mathbb{C}P^9$ as follows:

$$g_3(U_0 : U_1 : U_2 : U_3 : U_4 : U_5 : U_6 : U_7 : U_8 : U_9) = (U_0 : U_2 : U_3 : U_1 : U_5 : U_6 : U_4 : U_8 : U_9 : U_7) \tag{1}$$

$$\begin{aligned} g_7(U_0 : U_1 : U_2 : U_3 : U_4 : U_5 : U_6 : U_7 : U_8 : U_9)& \\ = (U_0 : \zeta^6 U_1 : \zeta^5 U_2 : \zeta^3 U_3 : \zeta^1 U_4 : \zeta^2 U_5 &: \zeta^4 U_6 : \zeta^1 U_7 : \zeta^2 U_8 : \zeta^4 U_9), \end{aligned} \tag{2}$$

where g_3 and g_7 are generators of C_3 and C_7 respectively and ζ is the 7-th root of unity $\exp(2\pi i/7)$. We first observe that

Lemma 2.1. *There are at least* 3 *non-trivial torsion classes of* $\mathrm{Pic}(\mathbb{P}^2_{\mathrm{fake}})$ *that are invariant under the* C_3 *action.*

Proof of Lemma 2.1. Up to scaling, clearly U_0 in $6H$ is fixed by the entire automorphism group, so there must exist some non-trivial torsion element D such that $3H + D$ corresponds to U_0 as a non-reduced curve in $6H$. By Maschke's Theorem, the one dimension given by D splits off and we are left with a three dimensional representation of the automorphism group. There are 7 non-trivial elements in the three dimensional representation, so there must exist some element here fixed by the C_3 action. $\square$

Lemma 2.1 suggests that we should be searching for C_3-invariant curves of the form

$$U_0 + a_1(U_1 + U_2 + U_3) + a_2(U_4 + U_5 + U_6) + a_3(U_7 + U_8 + U_9) \tag{3}$$

Subsequently, we obtained the coefficients first in finite fields in the file `3H-Reduction.txt`. We searched for primes p for which -7 is a square in $\mathbb{F}_p$, and then found using `Magma` that the smallest such prime for which the reduction of the scheme preserves the Hilbert polynomial is 11. Then, we shuffled through all possible a_i's in $\mathbb{F}_{11}$ to solve for the triples that produce unreduced schemes. We got three solutions, $(0, 0, 0), (7, 0, 0)$ and $(8, 7, 7)$. The first one corresponds to $3H + D$ as it is fixed by the action of $\mathrm{Aut}(\mathbb{P}^2_{\mathrm{fake}})$.

Next, we lifted the coefficients back to $\mathbb{Q}(\sqrt{-7})$ in `3H-Torsion.nb` through the following procedure. We obtained several points on the last two curves. Suppose one of them is $x = [x_0 : ... : x_9]$. We can set without loss of generality $x_0 = 1$. Then we consider 2 tangent vectors of the form $v = (v_0, v_1, ..., v_9)$. We set similarly $v_0 = 0$, and then $v_1 = 1, v_2 = 0$ for one and $v_1 = 0, v_2 = 1$ for the other to make them orthogonal. Then we needed to make sure the tangent vectors stay in the $T_x\mathbb{P}^2_{\mathrm{fake}}$, so we substituted in $x_i + v_i t$'s for U_i's for each of the 84 equations and the linear curve we obtained, and solved for conditions such that the coefficient of t is zero, which means that the vector is orthogonal to the gradient of all these equations.

We then lifted the coefficients to $\mod 11^2$. We substituted $\sqrt{-7}$ with appropriate values, and adjusted the linear cuts by adding $11(a_1'(U_1+U_2+U_3)+a_2'(U_4+U_5+U_6)+a_3'(U_7+U_8+U_9))$, for some unknown (a_1', a_2', a_3'), which ensures that it agrees with the original cuts $\mod 11$, and similarly the points on these cuts and the tangent vectors at these points. Then we solved for the same equations modulo 11 to obtain the unknown coefficients modulo 11^2. It is clear that we can go through the same process with the new data to consecutively lift them to modulo any 11^d. In the end we obtained the coefficients for the cuts modulo 11^{21}. From these we recover the original coefficients in $\mathbb{Q}(\sqrt{-7})$ as follows. If an element $a \in \mathbb{Q}(\sqrt{-7})$ satisfies an integer relation $c_1 + c_2\sqrt{-7} + c_3 a = 0$, it leads to $c_1 + c_2[\sqrt{-7}] + c_3[a] = 0$ for the corresponding 11-adic numbers. If we have approximations $[a]_{11^{21}}$ and $[\sqrt{-7}]_{11^{21}}$ of $[a]$ and $[\sqrt{-7}]$ respectively, then we get an integer relation

$$c_1 + c_2[\sqrt{-7}]_{11^{21}} + c_3[a]_{11^{21}} + c_4 11^{21} = 0.$$

If c_1, c_2 and c_3 are small, then the above relation can be recovered by using the lattice reduction algorithm to find equations of small norm in the lattice of integer relations on $1, [\sqrt{-7}]_{11^{21}}, [a]_{11^{21}}$ and 11^{21}.

In this way, we obtained the two cuts to be

$$\begin{aligned} 3H + D_1 &\coloneqq U_0 + \frac{1}{2}(1+\sqrt{-7})(U_1+U_2+U_3) \\ 3H + D_8 &\coloneqq U_0 + (-5+\sqrt{-7})(U_1+U_2+U_3) + (4-4\sqrt{-7})(U_4+U_5+U_6) - 4(U_7+U_8+U_9) \end{aligned} \tag{4}$$

We computed points on these cuts to see that they are nonreduced curves, so they are indeed the cuts we were looking for. Notice that $D_8 = D + D_1$, and the three torsion classes D, D_1, and D_8 are the orbit representatives of the C_7 action on the non-trivial torsion classes of $\mathrm{Pic}(\mathbb{P}^2_{\text{fake}})$. Let's also define $D_2, D_3, ..., D_7$ and $D_9, D_{10}, ..., D_{14}$ as the successive C_7-translates of D_1 and D_8 respectively using Equation (2).

2.2. **Determination of the group relations.** We would like to now determine the group relations of the cuts we computed (and by extension, the torsion line bundles they corresponds to) in the torsion of the Picard group of $\mathbb{P}^2_{\text{fake}}$.

Specifically, we want to determine when a triple of non-trivial torsion line bundles (L_1, L_2, L_3) satisfies $L_1 + L_2 + L_3 = 0$. First we note this can only happen when all three line bundles are distinct. Let $L_4 = L_1 + L_2 + L_3$, and consider sections of $12H$ which are zero on the curves in the class $3H+L_1, 3H+L_2, 3H+L_3$. This space is isomorphic to the space of sections on $3H + L_4$ and is therefore zero iff L_4 is zero. We can find these sections explicitly by computing random points on $3H + L_1, ..., 3H + L_3$ and looking for quadratic polynomials vanishing on these random points. This is realized by constructing a matrix whose rows are the evaluations the $h^0(\mathbb{P}^2_{\text{fake}}, 12H) = 55$ quadratic monomials on the aforementioned random points. Recall that for a torsion line bundle T - $h^0(\mathbb{P}^2_{\text{fake}}, 3H + T) = 0$ if and only if T is trivial, hence this matrix has full rank if and only if $L_1 + L_2 + L_3 = 0$.

We computed the rank of these linear systems for every triple in the 15 non-trivial torsion classes to determine the group laws in terms of these notations. The complete table is in the file `3H-Torsion.nb`, here we show how $D_1, ..., D_7$ can be written in terms of the basis $\{D_1, D_2, D_3\}$:

$$D_4 = D_1 + D_2, \quad D_5 = D_2 + D_3, \quad D_6 = D_1 + D_2 + D_3, \quad D_7 = D_1 + D_3 \tag{5}$$

2.3. **Quadratics vanishing on $3H + T$.** In this section, we will describe how to construct $|3H + T|$ for each non-trivial torsion T. While we realized sections of $3H + T$ as non-reduced cuts in $6H$, taking random linear cuts and computing its intersection with these non-reduced cuts would still produce points on $3H + T$. Hence, we can repeat a similar procedure as in Section 2.2.

In particular, for each of $3H + D, 3H + D_1, 3H + D + D_1$, we computed numerically points on each of the non-reduced cuts given and solved for the coefficients of quadratics in $U_0, ..., U_9$ vanishing on them. We can populate the quadratics on the other twelve $3H + D_i$'s by successively applying the C_7 action onto the quadratics computed already. For each $3H+T$, we find 28 quadratics vanishing on its curve, which is expected as $\dim_{\mathbb{C}} H^0(\mathbb{P}^2_{\text{fake}}, 12H-(3H+T)) = 28$.

3. Computing $4H$

In this section, we compute the linear system $|4H|$. This will prove a proof for the "only if" direction of Theorem 1.1(2). We follow the general method of [BL23, Section 3.2] by Borisov and Lihn, where they computed the linear system $|4H|$ on Keum's fake projective plane. Our specific constructions are realized in the file `4H.nb` in our code repository [BJLM].

By the Riemann-Roch and the Kodaira vanishing theorems, we have $\dim_{\mathbb{C}} H^0(\mathbb{P}^2_{\text{fake}}, 4H) = 3$. The C_7 action on $H^0(\mathbb{P}^2_{\text{fake}}, 4H)$ splits it into three one-dimensional C_7-eigenspaces, which, by the holomorphic Lefschetz fixed-point formula, have eigenvalues ξ^3, ξ^6, and ξ^5 respectively. Let r_3, r_6, and r_5 be the sections generating each eigenspace. There's also a C_3 action that permutes $r_3 \to r_6 \to r_5 \to r_3$.

Let's define $s_i = r_i^{\otimes 3}$ for $i = 3, 6, 5$ and $d = r_3 \otimes r_6 \otimes r_5$. Note that $s_3, s_6, s_5, d \in H^0(\mathbb{P}^2_{\text{fake}}, 12H)$ and thus can be represented as quadratics in $U_0, ..., U_9$. Since s_3 has C_7 weight $3 \times 3 \equiv 2 \pmod 7$ and d has weight $3+5+6 \equiv 0 \pmod 7$ and is C_3 invariant, we can write s_3 and d as

$$\begin{aligned} s_3 &\coloneqq b_5U_1U_3 + b_3U_4^2 + b_8U_0U_5 + b_4U_2U_6 + b_2U_4U_7 + b_1U_7^2 + b_7U_0U_8 + b_6U_2U_9 \\ d &\coloneqq e_1U_0^2 + e_2(U_1U_4 + U_2U_5 + U_3U_6) + e_3(U_1U_7 + U_2U_8 + U_3U_9) \end{aligned} \tag{6}$$

Note that s_5 and s_6 may be obtained as C_3-translates of s_3.

Our goal is then to solve for s_3 and d explicitly. Then we could solve for $\{s_3 = d = 0\}$ to obtain the section r_3 and use the C_3 action to find r_6 and r_5.

3.1. **Solving for s_3 and d.** The overall idea is that we want to solve for the sextic equation $s_3s_5s_6 - d^3 = 0$. We can do this by computing random points on $\mathbb{P}^2_{\text{fake}}$ with high accuracy and evaluating the expression $s_3s_5s_6 - d^3$ on these points. This will produce relations on the coefficients $b_1, ..., b_8$ and e_1, e_2, e_3, which we can then solve in the `Magma` file `4H-Quadratic.txt`.

However, the process above may take quite long computationally. To reduce the run-time, we also compute some additional constraints on the coefficients before passing it down to the `4H-Quadratic.txt`. The additional constraints are calculated as follows:

- We observe that there are three C_7 fixed points on $\mathbb{P}^2_{\text{fake}}$ given by

$$\begin{aligned} &p_1 \coloneqq [0:0:0:0:0:0:0:1:0:0], \quad p_2 \coloneqq [0:0:0:0:0:0:0:0:1:0], \\ &p_3 \coloneqq [0:0:0:0:0:0:0:0:0:1] \end{aligned} \tag{7}$$

  We note that p_2 must be in the curve $\{r_3 = 0\}$. This is because the only quadratic monomial that does not vanish on p_2 is U_8^2, which has C_7 weight $4 \pmod 7$. On the other hand, we observe that $s_3 = r_3^3$ has weight $2 \pmod 7$ and thus does not have a term on U_8^2, so r_3 must vanish on p_2. Similarly, we also have that $p_3 \in \{r_3 = 0\}$ as U_9^2 has weight $1 \pmod 7$.

- It follows that p_2, p_3 vanish on s_3 up to multiplicity 3. We then compute the order 3 formal neighborhoods of p_2 and p_3 respectively (in practice, we only needed them up to order 2). Then we solve for the conditions of s_3 being identically zero on the formal neighborhoods of p_2 and p_3 up to order 2. This reduces the number of independent coefficients on s_3 from 8 to 6.

After solving the relations on the remaining 9 coefficients produced by the random points, we obtain the following solutions:

$$\begin{aligned} s_3 = \frac{1}{8}\Bigg(&\frac{1}{29}(1 - 27\sqrt{\text{-}7})U_1U_3 + \frac{4}{29}(101 - \sqrt{\text{-}7})U_4^2 + \frac{8}{29}(15 + \sqrt{\text{-}7})U_0U_5 + \frac{8}{29}(1 + 2\sqrt{\text{-}7})U_2U_6 \\ &+ 8U_4U_7 - 4U_0U_8 + \frac{1}{29}(101 - \sqrt{\text{-}7})U_0U_8 + \frac{1}{58}(101 - \sqrt{\text{-}7})U_2U_9 + \frac{1}{58}\sqrt{\text{-}7}(101 - \sqrt{\text{-}7})U_2U_9\Bigg) \end{aligned} \tag{8}$$

(9) $d=\frac{1}{812}\Bigg((35-17\sqrt{-7})U_0^2+(70-34\sqrt{-7})(U_1U_4+U_2U_5+U_3U_6)+(21+13\sqrt{-7})(U_1U_7+U_2U_8+U_3U_9)\Bigg)$

Remark. While the authors of [BL23] solved the system $s_3s_5s_6-d^3=0$ using a finite field search and lifting the coefficients back to $\mathbb{Q}(\sqrt{-7})$, we instead solve for the irreducible components of the ideal formed by the equations in the file `4H-Quadratic.txt` with `Magma` directly and found the exact solutions in $\mathbb{Q}(\sqrt{-7})$ within a reasonable time.

3.2. **Solving for r_3.** We find random points on r_3 by computing random points on $\{s_3=d=0\}$ with high accuracy. We then solve for the systems of quadratics vanishing on r_3 and obtained $\dim_{\mathbb{C}} H^0(\mathbb{P}^2_{\text{fake}}, 12H-4H)=21$ equations. The equations can be found in `Equations/4H_one_section.txt`. Now we can give a proof that $4H+0$ is not basepoint free.

Proof of "only if" direction of Theorem 1.1(3). Observe that the equations in `Equations/4H_one_section.txt` have no monomials of the form U_7^2, U_8^2, or U_9^2. This means that $p_1,p_2,p_3\in\{r_3=0\}$. The C_3 action on $\mathbb{P}^2_{\text{fake}}$ permutes $p_1\to p_2\to p_3\to p_1$, so it follows that r_5 and r_6 also vanish on these 3 points. □

4. Computing $5H+D$ and $5H$

In this section, we prove Theorem 1.1(3). For $5H+D$, we will construct an embedding of $\mathbb{P}^2_{\text{fake}}$ in $\mathbb{C}P^5$ given by its sections to show it is very ample in Section 4.1. This simplifies the embedding of $\mathbb{P}^2_{\text{fake}}$ in [BK20] dimension wise. For $5H$, we will construct the image of $\mathbb{P}^2_{\text{fake}}$ under its map and show the image is singular in Section 4.2. We will also verify that $5H$ is basepoint free after we compute the quadratics vanishing on the global sections of $5H$ in Section 4.3. Most of the relevant computations are laid out in the `Mathematica` file `5H-Torsion.nb` in our code repository [BJLM] except for the check that the sections of $5H$ do not have any common zeros, which is done in the `Magma` file `5H-intersection.txt`.

4.1. **For $5H+D$.** With the computations of $3H+D$ in Section 2 and $4H$ in Section 3, we can now find six linearly independent sections on $5H+D$. Observe that $12H=(5H+D)+(3H+D)+4H$, so we can compute for linear combination of quadratic monomials vanishing on random points from both $4H$ and $3H+D$. This produced 6 quadratics in variables $U_0,...,U_9$ that represent the 6 global sections of $5H+D$.

These 6 quadratics produce a map $\mathbb{P}^2_{\text{fake}}\to\mathbb{C}P^5$, so we can enumerate random points of the image of $\mathbb{P}^2_{\text{fake}}$ in $\mathbb{C}P^5$. We then solved for sextic equations in terms of the 6 quadratic polynomials and found 56 sextics with coefficients in $\mathbb{Q}(\sqrt{-7})$. To check that this is indeed an embedding, we follow the verification process carried out in [BL23]. The relevant verification files are in the folder `Verification/5H+D`.

4.2. **For $5H$.** Because $h^0(\mathbb{P}^2_{\text{fake}},3H)=0$, we can't use the method in Section 4.1 to find the sections of $5H$. Instead, we observe that $30H=5H+4H+\sum_{i=1}^{7}(3H+D_i)$, so we can look for linear combinations of quintic monomials vanishing on random points enumerated on $4H, 3H+D_1,...,3H+D_7$. This produced 6 quadratics in variables $U_0,...,U_9$ which we will name $Z_1,...,Z_6$.

Let X denote the image of $\mathbb{P}^2_{\text{fake}}$ in $\mathbb{C}P^5$ given by $Z_1,...,Z_6$. We can enumerate random points on X and solve for sextics in variables $Z_1,...,Z_6$ that vanish on the image. This produced 59 (as opposed to 56) sextic equations with coefficients in $\mathbb{Q}(\sqrt{-7})$. Moreover, the Hilbert polynomial of the quotient of the polynomial ring in six variables by the ideal generated by the above 59 sextics is $p(n)=\frac{1}{2}(5n-1)(5n-2)-3$, which is less than the dimension $\frac{1}{2}(5n-1)(5n-2)$ of $H^0(\mathbb{P}^2_{\text{fake}},5nH)$. By [Har77, Exercise II.5.9(b)], this implies that $5H$ is not very ample.

We investigated the image X of $\mathbb{P}^2_{\text{fake}}$ under this map further. We checked that there were no higher degree equations besides the sextic equations calculated for X, because these 59 equations generate a prime ideal, see `Magma` file `5H-is-prime.txt`. Furthermore, we checked that X is singular at the following 3 points:

$$q_1:=[1:0:0:0:0:0], q_2:=[0:1:0:0:0:0], q_3:=[0:0:0:1:0:0] \tag{10}$$

by computing the rank of the Jacobian matrix at these points.

Remark. We suspect that q_1, q_2, and q_3 are the only singular points of X and that X is not normal at these points.

4.3. **Quadratics vanishing on $5H$.** For each quintic equation corresponding to each Z_i, we can find random roots of the quintic equation by computing the intersection of the quintic equation with random linear cuts. Out of the roots we computed, we only keep the points that are non-zero on the equations of $4H$ and $3H + D_i$ for $i = 1, ..., 7$. The points left will then be on the section Z_i.

We then solve for quadratics in $U_0, .., U_9$ vanishing on the remaining points and finds $h^0(\mathbb{P}^2_{\text{fake}}, 12H - 5H) = 15$ quadratics equations defining Z_i. We also check in the file `5H-intersection.txt` that all 90 quadratics do not have any common zeros, so $5H$ is indeed basepoint free.

5. Computing $4H + T$

In this section, we use the results of Section 4.3 to compute quadratics vanishing on sections of $4H+T$ for all non-trivial torsion classes $T \in \text{Pic}(\mathbb{P}^2_{\text{fake}})$. These computations will also be used to prove Theorem 1.1(1) and the "if" direction of Theorem 1.1(2). The relevant computations for the quadratics vanishing on $4H+T$ and the proof of Theorem 1.1(1) can be found in the `Mathematica` file `4H-Torsion.nb` from our code repository [BJLM]. The checks that $4H+T$ is basepoint free are in the `Magma` file `4H-Torsion-intersection.txt`.

It suffices for us to compute the quadratics on $4H + D$, $4H + D_1$, and $4H + D_8$, as the rest can be generated by the C_7 action. For $L \in \{D, D_1, D_8\}$, we observe that $12H = (4H + L) + (3H + L) + 5H$. We can compute the quadratics vanishing on $4H + L$ in two steps:

(1) First, we can find the 3 sections of $4H+L$ as linear combinations of quadratics in $U_0, ..., U_9$ vanishing on random points of $3H + L$ and one section of $5H$ (in our case, we chose Z_2).

(2) These sections are technically quadratic polynomials. Since we have computed explicit equations on sections of $5H$ in Section 4.3 and of $3H+L$ in Section 2.3, we can take random cuts on each quadratic polynomials, only keeping the points that are non-zero on these equations. The remaining points will be random points on the sections of $4H+L$ as divisors. We can then find $h^0(\mathbb{P}^2_{\text{fake}}, 12H-(4H+L)) = 21$ quadratics for each section.

Note that we also only need to do this on one section of $4H + L$, as the other two can be populated by the C_3 action. Now we will finish the proofs of Theorem 1.1.

Proof of "if" direction of Theorem 1.1(2). It suffices for us to check this for $4H+D, 4H+D_1$, and $4H+D_8$. Since we have computed the quadratics vanishing on each already, we can simply check that they don't have common zeros in the `Magma` file `4H-Torsion-intersection.txt`. □

Proof of Theorem 1.1(1). The case for $2H$ and $2H+D$ is implied by Theorem 5.5 of [GKS23]. For the other 14 torison classes, it suffices for us to check this on $2H + D_1$ and $2H + D_8$ because of the C_7 action.

If $h^0(\mathbb{P}^2_{\text{fake}}, 2H + D_1) \neq 0$, then let $\alpha \in H^0(\mathbb{P}^2_{\text{fake}}, 2H + D_1)$ be some non-trivial section. Consider the section $\alpha \otimes r \in H^0(\mathbb{P}^2_{\text{fake}}, 6H)$ where r is any non-trivial section of $H^0(\mathbb{P}^2_{\text{fake}}, 4H + D_1)$. Since the basis of $H^0(\mathbb{P}^2_{\text{fake}}, 6H)$ is given by $U_0, ..., U_9$, this means that $\alpha \otimes r$ lies in some hyperplane in $\mathbb{C}P^9$. However, since we have found random points on $4H + D_1$ previously in this section, a direct check in `4H-Torsion.nb` shows that there exists 10 points of one section of $H^0(\mathbb{P}^2_{\text{fake}}, 4H + D_1)$ whose determinant is non-zero, hence we have a contradiction. A similar check is done for $4H + D_8$ to show that $h^0(\mathbb{P}^2_{\text{fake}}, 4H + D_8) = 0$. □

Remark. Since $h^0(\mathbb{P}^2_{\text{fake}}, 2H) = 0$, we know that $h^0(\mathbb{P}^2_{\text{fake}}, H + T) = 0$. This is because the existence of any non-trivial section in $h^0(\mathbb{P}^2_{\text{fake}}, H + T)$ would imply $h^0(\mathbb{P}^2_{\text{fake}}, 2H) \neq 0$ by tensoring with itself.

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