ON EQUATIONS OF FAKE PROJECTIVE PLANES WITH AUTOMORPHISM GROUP OF ORDER 21

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ABSTRACT. We study Dolgachev elliptic surfaces with a double and a triple fiber and find explicit equations of two new pairs of fake projective plane with 21 automorphisms, thus finishing the task of finding explicit equations of fake projective planes with this automorphism group. This includes, in particular, the fake projective plane discovered by J. Keum.

1. INTRODUCTION

Theory of fake projective planes originated with the famous example of D. Mumford [Mu79] of a surface of general type with the same Hodge numbers as the usual projective plane $\mathbb{CP}^2$. By the nature of the construction, it did not yield any explicit equations of it. Over the subsequent decades, work by multiple authors (see for instance [AK17, I88, Ke06, Ke08, KK02, Kl03, PY07, PY10]) produced additional examples and general results and culminated in the classification of all fake projective planes by D. Cartwright and T. Steger [CS11, CS11+]. These surfaces are classified as free quotients of the complex two-dimensional ball $\mathbb{B}^2 = \{(z_1, z_2), |z_1|^2 + |z_2|^2 < 1\}$ by certain discrete arithmetic subgroups. There are exactly 50 conjugate pairs of such surfaces, separated into 28 classes. This extremely useful classification does not lead to any polynomial equations either, since there are no known methods for constructing explicit automorphic forms for these groups.

Over the last several years, the author of this paper has been involved in multiple collaborations with the goal of discovering explicit polynomial equations that define fake projective planes and related surfaces, [BK19, BY18, BF20, BBF20]. This paper is a continuation of such efforts, originally aimed at finding the equations of the Mumford's fake projective plane. While this goal is still elusive, we find equations of the fake projective plane constructed by Keum in [Ke06], which is commensurable to the Mumford's surface. We also find another interesting fake projective plane in the process. As always, new approaches had to be developed for the case at hand.

It is not surprising that most of the currently computed fake projective planes have nontrivial automorphism groups, as this provides some avenues for exploration, and this paper continues the trend. According to the classification of Cartwright and Steger, the maximum order of the automorphism group of a fake projective plane is 21. There are three conjugate pairs of fake projective planes with the automorphism group of this size, and in all

three cases the group is the semi-direct product of a normal subgroup C_7 and a non-normal subgroup C_3. Specifically, here are the planes, with their name in the CS classification and brief comments.

- $(a = 7, p = 2, \emptyset, D_3 2_7)$ is the first example of the fake projective plane for which explicit equations were found, see [BK19]. There are two other conjugate pairs of FPPs in its class.
- $(a = 7, p = 2, \{7\}, D_3 2_7)$ is the surface constructed by J. Keum in [Ke06]. It has three more pairs of FPPs in its class, including Mumford's fake projective plane.
- $(C20, p = 2, \emptyset, D_3 2_7)$ is the last of the three surfaces which does not seem to be implicated in any other construction. There are no other pairs in its class.

In all three cases, the quotient of $\mathbb{P}^2_{fake}$ by the subgroup C_7 of its automorphism group has a minimal resolution Y with rather peculiar geometry, see [Ke06, Ke08]. The quotient has three singular points of type $\frac{1}{7}(1,3)$, which are permuted by the residual C_3 action of the automorphism group of $\mathbb{P}^2_{fake}$. The minimum resolution Y of $\mathbb{P}^2_{fake}/C_7$ has three disjoint chains of three lines with self-intersections $-3, -2, -2$ which we denote by

$$S - B - C, \; S_1 - B_1 - C_1, \; S_2 - B_2 - C_2.$$

Here $-$ indicates a transversal intersection point. In addition, Y is fibered over $\mathbb{CP}^1$, with generic fibers of genus one, two multiple fibers, three nodal fibers and one fiber of type I_9 (a ring of nine $\mathbb{CP}^1$ with self-intersection (-2) each) with components

$$A - B - C - A_1 - B_1 - C_1 - A_2 - B_2 - C_2 - A.$$

The residual automorphism group C_3 preserves the fibration structure and acts by sending $A \to A_1 \to A_2 \to A$ and similarly for S, B and C curves.

The multiplicites of the multiple fibers in the case of $(a = 7, p = 2, \emptyset, D_3 2_7)$ are 2 and 4, and they are 2 and 3 in the other two cases, which are the focus of this paper. In particular, in the cases of interest, the curves S, S_1 and S_2 are 6-sections of the fibration. The two special fibers $3F_2$ and $2F_3$ have multiplicity 3 and 2 respectively. The reductions F_2 and F_3 are linearly equivalent to $2F$ and $3F$ with $F = K_Y$ and the generic fiber is equivalent to $6F$.

The first idea of this paper is to consider the ring

$$\bigoplus_{a,b \geq 0} H^0(Y, \mathcal{O}(aF + bS)).$$

It has a double grading and we can derive a formula for the graded dimension

$$\sum_{a,b \geq 0} \dim H^0(Y, \mathcal{O}(aF + bS)) \, t^a s^b = \frac{1 + 2st^4 + 2st^5 + s^2 t^9}{(1 - t^2)(1 - t^3)(1 - s)(1 - st^3)}$$

which is suggestive of a free module structure over the subring generated by
the variables u_0, u_1, v_1, v_2 with weights $(0, 2), (0, 3), (1, 0), (1, 3)$ respectively.
The appropriate GIT quotient is a birational model Y_0 of Y that collapses
all of the curves that intersect F and S trivially. These are the curves
C, B_1, C_1, B_2, C_2, and the image of the special fiber in Y_0 becomes

$$(1.1) \qquad A - B - * - A_1 - ** - A_2 - ** - A$$

so that the intersection point of (the images of) B and A_1 is a simple node
and intersection of A_2 with both A and A_1 are $\frac{1}{3}(1, 2)$ singularities. The
construction of the new fake projective planes then proceeds as follows.

Step 1. We construct a nine-parameter family of $(2, 3)$-Dolgachev sur-
faces with a rational six-section S. A general member of this family has
twelve distinct singular nodal fibers, in addition to the double and triple
fibers. The defining equations of the family are nine quadrics of weights
$3 \times (8, 2)$, $3 \times (9, 2)$ and $3 \times (10, 2)$ in the variables of weight

$$(2, 0), (3, 0), (0, 1), (3, 1), (4, 1), (4, 1), (5, 1), (5, 1).$$

The idea is to postulate the above free module structure and the weights
of the quadratic relations and to use the associativity conditions of the ring
to solve for the coefficients of the quadrics. It entails solving a system of
over 1600 equations with 92 unknowns, which is done by an ad hoc method
utilizing Mathematica software system.

Step 2. We construct seven-, five- and two-parameter subfamilies with
additional conditions on the special fiber. Respectively, we require the spe-
cial fiber to contain a line, two disjoint lines, two disjoint lines with two
nodes on one and one node on the other. In particular, a generic element of
the two-parameter family has the special fiber

$$A - B - * - A_1 - * - A_2 - * - A$$

in the sense that the intersection points of B with A_1 and A_2 with A and
A_1 are nodes.

Step 3. We find a finite-field reduction of the surface Y_0 by looking
through the parameter choices over a finite field and checking whether the
resulting surfaces have worse than nodal singularities at two special points
on the curve A_2. The smallest prime for which we were able to find such
surfaces was 79.

Step 4. We proceed by successively solving the (hard to write) conditions
on being more singular at the intersection points of A-curves for parame-
ters modulo powers of 79. We then recognize the parameters as algebraic
numbers and construct Y_0 over a number field of degree 12. We make a
coordinate change to realize Y_0 over the number field $\mathbb{Q}(\sqrt{-7})$.

Step 5. We study Y_0 to find its geometric features, such as the curves S_1
and S_2 and the birational action of C_3. We find the degree seven extension

of the field of rational functions of Y_0 that gives $\mathbb{P}^2_{fake}$ and calculate the bi-canonical linear system of the latter. We then realize $\mathbb{P}^2_{fake}$ as an intersection of 84 cubic equations in $\mathbb{CP}^9$, following the blueprint of [BK19].

Step 6. We identify the fake projective plane as $(C20, p = 2, \emptyset, D_3 2_7)$ by exhibiting too many torsion line bundles for it to be $(a = 7, p = 2, \{7\}, D_3 2_7)$. The method is to find non-reduced C_3-invariant elements of $|2K_{\mathbb{P}^2_{fake}}|$ modulo a prime (this time it is 29) by an exhaustive search, and then lift them to powers of the said prime and finally the algebraic numbers. We also use one of the the torsion line bundles to pick a more natural basis of $H^0(\mathbb{P}^2_{fake}, 2K)$ so that its equations have smaller coefficients. Finally, we verify that it is indeed a fake projective plane, as in [BK19].

Step 7. In Step 2, one actually finds *two different five-parameter families* of Dolgachev surfaces which contain two disjoint lines in the special fiber. Unfortunately, for the second family we were unable to reduce the number of parameters further by considering the condition of having nodes. However, we are still able to go through steps 3-5 in this case, by brute force approach to the finite field search. By the process of elimination, the new pair of fake projective planes is the one constructed in [Ke06].

The paper is organized as follows. Section 2 contains the first two steps of the construction. Section 3 contains steps 3 and 4. Section 4 describes steps 5 and 6. In Section 5 we discuss the last step. Finally, in Section 6 we talk about the open problems associated with our construction. We also have an Appendix 7 in which we put some equations that are too lengthy for the main body of the paper. However, many of the key formulas are far too large to even be included into the Appendix. They are collected in [B22+] instead.

2. Families of $(2,3)$ Dolgachev surfaces

We start by studying smooth projective surfaces Y with a genus one fibration $Y \to \mathbb{CP}^1$ with the following properties.

- The class of the general fiber is $6F$ where F is the canonical class of Y. In particular, $F^2 = 0$.
- There is a double fiber $2F_3$ and a triple fiber $3F_2$. The classes of F_2 and F_3 are $2F$ and $3F$ respectively.

- There is a rational six-section S with $SF = 1$ and $S^2 = -3$.
- $p_g(Y) = q(Y) = 0$.

Our motivation is that the minimal resolutions of the C_7 quotients of fake projective planes we are interested in satisfy the above, see [Ke08].

As implied in the Introduction, we first compute[1] the graded dimension of the ring

$$R = \bigoplus_{a,b \geq 0} H^0(Y, \mathcal{O}(aF + bS))$$

in a series of lemmas.

Lemma 2.1. *The graded dimension of $R_0 = \bigoplus_{a \geq 0} H^0(Y, \mathcal{O}(aF))$ is given by*

$$\sum_{a \geq 0} t^a \dim H^0(Y, \mathcal{O}(aF)) = \frac{1}{(1 - t^2)(1 - t^3)}.$$

Proof. The ring R_0 is freely generated by the elements $u_0 \in H^0(Y, \mathcal{O}(2F))$ and $u_1 \in H^0(Y, \mathcal{O}(3F))$ whose divisors are F_2 and F_3 respectively. Indeed, these generate a subring of R_0 and we will show that there are no other forms. The subring for $a = 0 \mod 6$ is isomorphic to the homogeneous ring of the $\mathbb{CP}^1$ of the base of the fibration, so it is a polynomial ring in u_0^3 and u_1^2. If a is odd, then $\mathcal{O}(aF)$ restricts to a nontrivial bundle to F_3 (because the normal bundle of the double fiber is nontrivial), so all global sections of it vanish on F_3 and the corresponding elements in R_0 are divisible by u_1. Similarly, if a is not divisible by 3 then all global sections of $\mathcal{O}(aF)$ vanish on F_2 and the elements are divisible by u_0. Together, these observations imply the result. $\square$

Lemma 2.2. *The dimension of $H^0(Y, \mathcal{O}(aF + S))$ is 1 for $a = 0$ and is $(a - 1)$ for $a > 0$.*

Proof. For $0 \leq a \leq 2$ we have $(aF + S)S < 0$, so any section of $\mathcal{O}(aF + S)$ must vanish on S, and the statement follows from Lemma 2.1. For $a \geq 3$ we have $\chi(aF + S) = \frac{1}{2}(aF + S)((a - 1)F + S) + 1 = a - 1$, so it suffices to show that the invertible sheaf $\mathcal{O}(aF + S)$ has no higher cohomology. The vanishing of $H^2(Y, \mathcal{O}(aF + S))$ for all integer a is clear from Serre duality and the fact that every effective divisor on Y must have a nonnegative intersection with F. To see the vanishing of $H^1(Y, \mathcal{O}(aF + S))$, we run induction on a. Specifically, consider the long exact sequence below.

$$0 \to H^0(Y, \mathcal{O}((a - 2)F + S)) \to H^0(Y, \mathcal{O}(aF + S)) \to H^0(Y, i_* \mathcal{O}_{F_2}(aF + S))$$
$$\to H^1(Y, \mathcal{O}((a - 2)F + S)) \to H^1(Y, \mathcal{O}(aF + S)) \to H^1(Y, i_* \mathcal{O}_{F_2}(aF + S))$$

Since $(aF + S)F_2 = 2$, the last term is 0, so it suffices to prove that $H^1(Y, \mathcal{O}(aF + S)) = 0$ for $a \in 1, 2$, which follows from our computation of the global sections of these divisors. $\square$

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[1]Technically, we could just conjecture everything, with the justification for it being the final outcome, but it is worth proving what we can.

Lemma 2.3. *For $b \geq 1$ and $a \geq 3b$ the dimension of $H^0(Y, \mathcal{O}(aF + bS))$ for $a \geq 0$ and $b \geq 1$ is equal to $\chi(\mathcal{O}(aF + bS)) = \frac{1}{2}(2ab - b - 3b^2 + 2)$.*

Proof. We will prove it by induction on b with the base case provided by Lemma 2.2. As before, $\dim H^2(Y, \mathcal{O}(aF + bS)) = \dim H^0(Y, \mathcal{O}((1-a)F - bS)) = 0$, so the statement amounts to $\dim H^1(Y, \mathcal{O}(aF + bS)) = 0$. For the induction step, the short exact sequence

$$0 \to \mathcal{O}(aF + (b-1)S) \to \mathcal{O}(aF + bS)) \to i_{S*}\mathcal{O}(aF + bS) \to 0$$

leads to

$$\to H^1(Y, \mathcal{O}(aF + (b-1)S) \to H^1(Y, \mathcal{O}(aF + bS)) \to H^1(S, i_S^*\mathcal{O}(aF + bS)).$$

The terms on the left and on the right are zero by the induction hypothesis and $(aF + bS)S = a - 3b \geq 0$. $\qquad\square$

The above lemmas allow us to compute the graded dimension of R.

Proposition 2.4.

$$\sum_{a,b \geq 0} \dim H^0(Y, \mathcal{O}(aF + bS))\, s^b t^a = \frac{1 + 2st^4 + 2st^5 + s^2 t^9}{(1 - t^2)(1 - t^3)(1 - s)(1 - st^3)}.$$

Proof. We denote $\dim H^0(Y, \mathcal{O}(aF + bS)) = c_{a,b}$ to simplify notation. Since $(aF + bS)S < 0$ implies that S is a fixed component of $|aF + bS|$, we see that for $a < 3b$ there holds $c_{a,b} = c_{a,b-1}$. Therefore, $\sum_{a,b \geq 0} c_{a,b} t^a s^b$ equals

$$\tfrac{s}{1-s} \sum_{b \geq 0} (c_{3b,b} t^{3b} + c_{3b+1,b} t^{3b+1} + c_{3b+2,b} t^{3b+2}) s^b + \sum_{b \geq 0} \sum_{a \geq 3b} c_{a,b} t^a s^b.$$

We then use Lemmas 2.1 and 2.3 to write the above as

$$\tfrac{s}{1-s}(1 + t^2) + \tfrac{1}{(1-t^2)(1-t^3)} + \sum_{b \geq 1} s^b \Big(\tfrac{s}{1-s}(\tfrac{1}{2}(2(3b)b - b - 3b^2 + 2)t^{3b}$$
$$+ \tfrac{1}{2}(2(3b+1)b - b - 3b^2 + 2)t^{3b+1} + \tfrac{1}{2}(2(3b+2)b - b - 3b^2 + 2)t^{3b+2})$$
$$+ \sum_{a \geq 3b} \tfrac{1}{2}(2ab - b - 3b^2 + 2)t^a \Big)$$

which is then easily computed by Mathematica.[2] $\qquad\square$

As a consequence of Proposition 2.4 we conjecture that R has a structure of a rank six graded free module over the ring

$$\mathbb{C}[u_0, u_1, v_1, v_2]$$

generated by the sections u_0, u_1, v_1 and v_2 of $\mathcal{O}(2F)$, $\mathcal{O}(3F)$, $\mathcal{O}(S)$ and $\mathcal{O}(3F + S)$ respectively. Note that while u_0, u_1, v_1 are defined uniquely up to scaling, the section v_2 can also be changed by adding a scalar multiple of $u_1 v_1$. We denote the generators of the module at weights $(4, 1)$ and $(5, 1)$ by v_3, v_4, v_5, v_6. We conjecture that the generator at degree $(9, 2)$ is equal to $v_3 v_5$. It can likely be proved that the ring R is Gorenstein, which would then imply that such choice is possible, but we just take it as a sensible guess.

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[2] It is, of course, computable by hand, but this seems to be a fool's errand given subsequent use of various software.

We will consider polynomial relations on $u_0, \ldots, v_6$. By looking at the graded dimension of R we observe that these variables must satisfy three linearly independent relations of degree $(8, 2)$, three linearly independent relations of degree $(9, 2)$ and three relations of degree $(10, 2)$, which are linearly independent together with u_0 multiplies of the $(8, 2)$ relations. We will refer to these relations as quadrics (in v_i). In general, genus one curves of degree 6 in $\mathbb{P}^5$ can be written as an intersection of nine degree two polynomials and we expect that for most values of u_0 and u_1 these quadrics describe the corresponding fiber of $Y \to \mathbb{CP}^1$.

We then set up the possible quadrics with undetermined coefficients, taking care to undo multiple symmetries of the construction. For example, we make sure that the coefficients of the $(8, 2)$ quadrics in $v_3^2, v_3 v_4, v_4^2$ are the standard basis vectors, and similarly for $(9, 2)$ and $(10, 2)$ quadrics. We assume that (v_3, v_4) and (v_5, v_6) are dual to each other in the socle pairing of $R/\langle u_0, u_1, v_1, v_2 \rangle$. We can also make additional assumptions on the coefficients in view of the possible changes of v_i such as $v_2 \to \alpha v_2 + \beta u_1 v_1$. The one such assumption which appears crucial to the success of the method is to require that the coefficients of the first $(8, 2)$ quadric at $v_3 v_1 u_0^2$ an $v_4 v_1 u_0^2$ are 1 and 0, the coefficients of the second quadric at these monomials are both 0 and the coefficients of the third quadric are 0 and 1 respectively. It can be shown that a generic collection of the quadrics can be manipulated into this form by appropriate linear changes of v_3, v_4 together with adding multiples of $u_0^2 v_1$ to them. However, there are six ways of doing so, which means that while we expect Y_0 to be defined over a quadratic imaginary field, we can not expect the coefficients of these quadrics to be this simple.

Once the quadrics are written down, one can compute the multiplication table for the generators of R as a free module over $\mathbb{C}[u_0, u_1, v_1, v_2]$ and then set up the associativity relations as equations on the coefficients of the quadrics, see [B22+, DolgachevSurfaces.nb]. The associativity relations led to over 1600 equations in 92 variables. Fortunately, some of these equations were quite simple, but still the task appeared daunting. We were able to use the Mathematica "Solve" command to eventually reduce to nine free parameters. The basic idea was to try to solve the easier equations first, with either byte count or the number of terms used as measure of complexity. The drawback of this technique is that one can at best hope to recover rational parameterizations by a subset of the set of variables, but we were fortunate in this case. The resulting equations are presented in (7.1) in the Appendix. The variables d_i are the parameters and the variables $u_0, u_1, v_1, \ldots, v_6$ are the coordinates.

Having found the equations of a nine-parameter family, we then tried to impose additional geometric conditions on it. We postulated without loss of generality that the I_9 fiber of Y occurs at $u_0^3 = u_1^2$ (or simply $u_0 = u_1 = 1$ if one wants to dehomogenize). We know from the work of Keum [Ke08] that possible intersections of S with (A, A_1, A_2) fall into cases $(2, 2, 1)$ and

$(1, 1, 3)$, so in either setting the image of the I_9 fiber in Y_0 should have two disjoint lines. We first solved for the condition that equations (7.1) vanish on one line, by parameterizing the said line. This gave a seven-parameter family and we used its explicit description to put in a condition of having two such lines, taking care not to get the degenerate cases where the lines intersect. We used Mathematica to solve for one variable at a time, and then plugged the results into the remaining equations. Our choices were guided by the desire to keep the size of the equations manageable. There was a lot of trial and error involved and even in the best case the computations took a long time. Due to an unfortunate mistake on the author's part (accidental deletion of a key file), some of the intermediate steps were lost, however, the final five-parameter answer survived and is explicitly written in [B22+, DolgachevSurfaces.nb].

The general member of this five parameter family has a double and a triple fiber at $u_1 = 0$ and $u_0 = 0$ respectively and a special fiber at $u_0^3 = u_1^2$ which has the following configuration of curves A, B, A_1, A_2 and their intersection points $p_1, \ldots, p_4$.
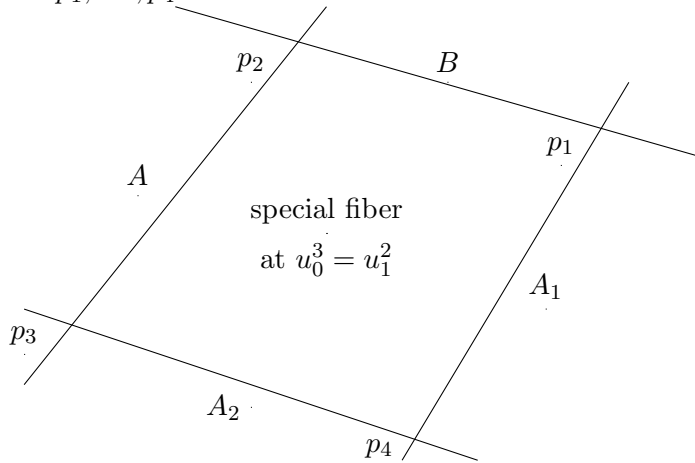
Here, A_2 and B are the two disjoint lines in the special fiber and the degrees of A and A_1 are 2.

Our next step was to impose the conditions that Y_0 is singular at p_1, p_3 and p_4, as would be expected by the geometry of the surface. One immediate technical difficulty was that the points p_1 and p_2 were not defined over the parameter space. However, we were able to make a change of variables, similar to a rational parametrization of a plane conic, to resolve this difficulty. We also made a simple coordinate change to simplify the formulas somewhat, see [B22+, DolgachevSurfaces.nb] for details.

The conditions of being singular at each p_i were computed as follows. We looked at the Jacobian matrix of the quadrics at p_i and computed its minors. We then computed the greatest common factor of these minors, which left us with large, but manageable expressions in parameters $t_1, \ldots, t_5$. An ad hoc manipulation of the equations and the parameters allowed us to replace

t_i by $s_1, \ldots, s_5$ and then solve for s_1 and s_5. The polynomial equation on the remaining parameters s_2, s_3 and s_4 has several factors, with all but one of them leading to undesirable degeneracies. We were left with a polynomial with integer coefficients of degree 3, 13 and 12 in s_2, s_3 and s_4 respectively. It is about $130 Kb$ long in its expanded form and thus is not worth trying to write down explicitly in the paper, see [B22+, DolgachevSurfaces.nb]. It appears that the resulting two-dimensional parameter space is not rational, but it was simple enough for our purposes.

3. FINDING THE SPECIAL DOLGACHEV SURFACE

This section describes how we found the surface Y_0. We expected that Y_0 has $\frac{1}{3}(1, 2)$ singularities at p_3 and p_4, and it would be reasonable to try to encode these in terms of s_2, s_3 and s_4 subject to the large polynomial equation. Unfortunately, this direct approach was not computationally feasible, as equations ballooned to hundreds of Mbytes in length. So we used an alternative method of finite field reduction. Specifically, we wanted to find a reduction of Y_0 modulo a prime p and then lift it first to p-adics and then to a number field.

We looked at various primes p and parameters $(s_2, s_3, s_4) \in \mathbb{F}_p^3$, with p chosen so that $\sqrt{-7}$ exists in $\mathbb{F}_p$, as it appeared likely that we would need this in the field of definition. For each triple of parameters we first checked whether it fits the polynomial relation that defines the parameter space. Then we used Magma to compute the structure of singularities at p_3 and p_4 (we have lengthy but explicit formulas for p_i in terms of the parameters). If both singularities looked right, then we took note of the values of the parameters. For one reason or another, we had to go to $p = 79$ before any suitable examples were found! However, at $p = 79$ there were six possible solutions, see [B22+, Search79].

$$(s_2, s_3, s_4) \in \quad \{(14, 47, 52), (15, 65, 27), (19, 32, 14),$$
$$(44, 14, 32), (58, 27, 65), (72, 52, 47)\}$$

It was then an interesting challenge to lift these solutions to powers of 79. By solving for v_5 and v_6 one can reduce the problem to codimension two. Then it is possible to encode the condition of having worse than nodal singularity as a certain polynomial in terms of first and second derivatives of the defining equations. Then we endeavored to have these polynomials produce values that are zero modulo higher and higher powers of 79, as we are adjusting our parameters s_i to a more accurate p-adic approximation. Of course, we also have to keep the defining polynomial of the family zero to the appropriate power of 79. The resulting code is messy, but not particularly slow, and we were able to compute the values of s_i up to 79^{101} in reasonable amount of time.

Once a p-adic approximation was found, it was then routine to find "simple" algebraic numbers that give these parameters s_i. As in [BF20, BBF20]

we used a lattice reduction algorithm to find a small linear combination of powers of s_i modulo 79^{101} and 79^{101} itself. This suggested that these are algebraic numbers of degree 12. By looking at the standard polynomial, we indeed found 12 possible triples (s_2, s_3, s_4). Perhaps not surprisingly, our finite field search only picked up the six cases that correspond to the simple roots of the reduction of the defining polynomial of s_2

$$1048576 + 9633792s_2 + 47179776s_2^2 + 156022272s_2^3 + 376708864s_2^4$$
$$+693988960s_2^5 + 1003433368s_2^6 + 1148276192s_2^7 + 1023247890s_2^8$$
$$+681835980s_2^9 + 317640295s_2^{10} + 91989513s_2^{11} + 12492403s_2^{12}$$

modulo 79.

Since we expected that Y_0 can be defined over a quadratic field, as is the FPP it came from, we wanted to find a coordinate change in v_i to see it. Specifically, we aimed to get the points p_i to have simple coordinates. As it is a lot faster in Mathematica, this was done numerically and then the coefficients were approximated by algebraic numbers. The resulting equations (7.2) are listed in the Appendix where we use w_i to denote new variables, with u_0 and u_1 not affected by the coordinate change.

4. CONSTRUCTION AND IDENTIFICATION OF THE FAKE PROJECTIVE PLANE

This section describes how we found the fake projective plane (up to conjugation) and identified it as $(C20, p = 2, \emptyset, D_3 2_7)$. The relevant calculations are contained in the Mathematica file SevenfoldCover.nb and Magma files Torsion and CheckSmoothness, as well as Macaulay2 file CheckFPP in [B22+].

We know that S is cut out by $w_1 = 0$, but it takes a bit of an effort to find equations of S_1 and S_2. The first idea is that since $(4F + S)S = 4 - 3 = 1$, the sections w_3 and w_4 are linear on S. Thus, points on S can be parameterized by $t = \frac{w_4}{w_3}$. This is true for any member of the nine-dimensional family of Dolgachev surfaces (7.1). So we know both the equation and the parameterization of S and we would like to do the same for S_1 and S_2.

It follows from the intersection form considerations [B22+, Intersection-Forms.nb] that S_2 is cut out by an equation of bidegree $(8, 1)$ which also passes through A and A_2 with multiplicities one and three respectively. We pick multiple points on A and A_2 and encode the relevant conditions on the $(8, 1)$ polynomial to find it to be

(4.1)
$$u_0^4 w_1 + \tfrac{1}{149}(-124 - 9\,\mathrm{i}\sqrt{7})u_0 u_1^2 w_1 + \tfrac{1}{8344}(-553 + 509\,\mathrm{i}\sqrt{7})u_0 u_1 w_2$$
$$+ \tfrac{1}{149}(20 - 37\,\mathrm{i}\sqrt{7})u_0^2 w_3 + \tfrac{1}{596}(-155 + 175\,\mathrm{i}\sqrt{7})u_0^2 w_4$$
$$+ \tfrac{1}{8344}(3255 + 1093\,\mathrm{i}\sqrt{7})u_1 w_5 + \tfrac{1}{2086}(-763 - 369\,\mathrm{i}\sqrt{7})u_1 w_6.$$

We also needed to compute the parametric equation of S_2. After some of the more straightforward approaches failed to finish, we did the following. For a given value of the parameter t on S we can compute the corresponding point $p(t)$ on S and six points of S_2 in the same fiber of genus one fibration.

One of these points is the image $p_1(t)$ of $p(t)$ under the predicted order three birational automorphism of Y_0. If t is an integer, then this point on S_2 has to be defined over $\mathbb{Q}(i\sqrt{7})$ which allows us to distinguish $p_1(t)$ from five others. After normalization $w_1 = 1$, the coordinates of $p_1(t)$ are polynomial in t of degree at most 5, and we use several values of t to find their coefficients. The resulting parameterization is given in (7.3) in the Appendix.

One could follow a similar approach to find S_1 which has a degree $(10, 1)$ equation vanish on it, but we chose to use the group law of the elliptic fibers instead. Indeed, we know that for a given value of t the points $p(t)$, $p_1(t)$ and $p_2(t)$ satisfy $p(t) + p_1(t) = 2p_2(t)$ under the group law of the fiber with any choice of origin. This allows us to find $p_1(t)$ and then compute the parameterization (7.4). We then find the degree $(10, 1)$ polynomial

(4.2)
$$u_0^5 w_1 - u_0^2 u_1^2 w_1 + \tfrac{1}{7112}(-1897 - 3\,i\sqrt{7})u_0^2 u_1 w_2) + \tfrac{1}{254}(-3 + 17\,i\sqrt{7})u_0^3 w_3$$
$$+ \tfrac{1}{254}(39 + 33\,i\sqrt{7})u_1^2 w_3 + \tfrac{1}{508}(-71 - 21\,i\sqrt{7})u_0^3 w_4 + \tfrac{1}{254}(-71 - 21\,i\sqrt{7})u_1^2 w_4$$
$$+ \tfrac{1}{7112}(6559 + 509\,i\sqrt{7})u_0 u_1 w_5 + \tfrac{1}{1778}(-1435 - 81\,i\sqrt{7})u_0 u_1 w_6$$

that vanishes on S_1.

We then computed the order three birational automorphism by postulating its general form and using the parametrization of S, S_1 and S_2 to find the coefficients. It can be viewed as an order three automorphism of a genus one curve defined over $\mathbb{C}(u_0, u_1)$. The formulas for the automorphism are in (7.5).

We were then ready to compute the equations of the sevenfold cover of Y. The method was essentially the same as that of [BK19] but we briefly describe it here for the benefit of the reader. We know that the seven-fold Galois cover $\mathbb{P}^2_{fake} \to Y$ is ramified at $S, B, C, S_1, B_1, C_1, S_2, B_2, C_2$. We consider the rational function

(4.3)
$$\frac{f_{1,10} f_{1,8}^2}{u_0^7 (u_0^3 - u_1^2)^2 w_1^3}$$

where $f_{1,8}$ and $f_{1,10}$ are given in (4.1) and (4.2) respectively. Its divisor is supported on the special curves, with all but the nine $S, \ldots, C_2$ curves above having multiplicity divisible by 7. So the function field of $\mathbb{P}^2_{fake}$ is obtained from that of Y by adding the seventh root of the function (4.3). We then compute ten sections of the bicanonical linear system on $\mathbb{P}^2_{fake}$ by projecting them to Y and realizing them as sections of various line bundles on Y_0. Care is taken to have the action of the automorphism group look nice in these coordinates. Specifically, we have ten variables $P_0, \ldots, P_9$ on which the order seven element acts by

$$g_7(P_0, \ldots, P_9) = (P_0, \zeta_7 P_1, \zeta_7^2 P_2, \zeta_7^4 P_3, \zeta_7^3 P_4, \zeta_7^6 P_5, \zeta_7^5 P_6, \zeta_7^3 P_7, \zeta_7^6 P_8, \zeta_7^5 P_9)$$

and order three element acts by

$$g_3(P_0, \ldots, P_9) = (P_0, P_2, P_3, P_1, P_5, P_6, P_4, P_8, P_9, P_7).$$

See [B22+, SevenfoldCover.nb] for more details.

Having constructed a fake projective plane with 21 automorphisms, we want to try to identify it in accordance with the Cartwright-Steger classification. Since the Dolgachev surface it is built from has a double and a triple fiber, there are two possibilities: the Keum's surface $(a = 7, p = 2, \{7\}, D_3 2_7)$ and $(C20, p = 2, \emptyset, D_3 2_7)$. In the former case, the torsion in the Picard group is C_2^3, with the automorphism group G acting transitively on seven nontrivial elements. Therefore, for every cyclic subgroup of order three in G there should be a unique G-invariant torsion element on $\mathbb{P}^2_{fake}$. This naturally lead to us trying to construct such torsion classes.

If L is a torsion class in the Picard group of a fake projective plane, then $H^0(\mathbb{P}^2_{fake}, K + L)$ is one-dimensional, see for example [GKS16]. If L is furthermore a two-torsion element, then the square of the corresponding section is in $H^0(\mathbb{P}^2_{fake}, 2K)$. In the other direction, two-torsion elements of the Picard group can be constructed from sections of $2K$ which give non-reduced curves. Additional condition of C_3 invariance means that we should consider sections

$$(4.4) \quad a_0 P_0 + a_1(P_1 + P_2 + P_3) + a_2(P_4 + P_5 + P_6) + a_3(P_7 + P_8 + P_9)$$

up to scaling.

We looked at the finite field reduction of $\mathbb{P}^2_{fake}$ to look for such a_i. Specifically, $p = 29$ was the smallest prime that both had $\sqrt{-7}$ in it and had the expected Hilbert polynomial of the reduction of $\mathbb{P}^2_{fake}$. We then ran through all possible a_i with $a_0 = 1$ and used Magma to check if the resulting scheme is non-reduced, see [B22+, Torsion]. We got three solutions, which was already a likely indicator that the $\mathbb{P}^2_{fake}$ is not $(a = 7, p = 2, \{7\}, D_3 2_7)$ but rather is $(C20, p = 2, \emptyset, D_3 2_7)$ which has a larger torsion subgroup C_2^6 in its Picard group. However, we needed to ascertain it by lifting to the characteristic zero.

The lifting procedure was pretty typical of such approach: we lifted the a_i in (4.4) to   mod p^k for large enough k and then guessed the algebraic numbers that could give these reductions. It is worth mentioning how exactly we did the lifting. We found via Magma several points on the non-reduced linear cut of the reduction of $\mathbb{P}^2_{fake}$ modulo p, then at each point we found two linearly independent tangent vectors in $\mathbb{P}^9(\mathbb{F}_{29})$ which are orthogonal to the gradients of all 84 cubic polynomials and the linear polynomial (4.4). We lifted these vectors to small integers and then were successively adjusting them so that the aforementioned orthogonality held modulo higher powers of p. At each stage we modified the points, the tangent vectors and the polynomial (4.4) to the next power of p; this amounted to solving a system of linear equations modulo p which was not time-consuming. Afterwards, we identified the corresponding algebraic numbers.

One of the solutions was the linear polynomial

$$(4.5) \quad \begin{array}{l} P_0 + \tfrac{1}{2}(1 + \mathrm{i}\sqrt{7})(\mathrm{P}_1 + \mathrm{P}_2 + \mathrm{P}_3) + (-122 + 2\,\mathrm{i}\sqrt{7})(\mathrm{P}_4 + \mathrm{P}_5 + \mathrm{P}_6) \\ + \tfrac{1}{7}(84 - 4\,\mathrm{i}\sqrt{7})(\mathrm{P}_7 + \mathrm{P}_8 + \mathrm{P}_9) \end{array}$$

and the other two were defined over a degree four number field. For each of these equations we then verified that the resulting cuts are non-reduced.[3] We note that the resulting two-torsion line bundles are not C_7 invariant, so we have established at least 21 nontrivial two-torsion elements of the Picard group which indicates that our surface is (up to complex conjugation) $(C20, p = 2, \emptyset, D_3 2_7)$.

Last but not least, we used the description of the above non-reduced linear cut to find a more pleasant basis for $H^0(\mathbb{P}^2_{fake}, 2K)$, see [B22+, Sevenfold-Cover.nb]. Namely we used a coordinate change from P_i to Q_i in which the aforementioned non-reduced cut (4.5) becomes

$$Q_0 + Q_1 + Q_2 + Q_3 + Q_7 + Q_8 + Q_9.$$

This allowed us to find a model of it with smaller coefficients of the 84 equations, which we recorded in [B22+, EqsFPPwithrrQ]. We then went through the verification procedure developed in [BK19] to make sure that the scheme cut out by the 84 cubic relations in 10 variables is indeed a fake projective plane in its bicanonical embedding, see [B22+, CheckFPP, CheckSmoothness]. The smoothness calculation was performed by looking at three specific minors which are nonzero at the C_7 invariant points of $\mathbb{P}^2_{fake}$. Each minor took a few hours to compute on our hardware.

5. Constructing the fake projective plane of Keum

While looking for Dolgachev surfaces with two disjoint lines, we have found a 5-parameter subfamily of Dolgachev surfaces with two disjoint lines in the special fiber with the intersections of S with A, A_1 and A_2 given by $(1, 1, 3)$. This makes it is provably different from the family in Step 2. This family was noticeably harder to work with. It was not even entirely straightforward to find the intersection points of the components of the special fiber. In particular, conditions of being singular at the three of these points were far too complicated to simplify. The details are in [B22+, SecondFamily.nb]

To find a member of this family with additional singularities (1.1), we used a brute force approach. Namely, we ran a search over the parameter space $\mathbb{F}_p^5$ for small p, computed the intersection points of the components of the special fiber and checked to see if the tangent space at the three points of interest had the correct dimension. This was too time-consuming in Mathematica so we used lower level languages. We first tried PARI/GP

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[3]This verification was only done numerically rather than symbolically, but it is sufficiently convincing for our purposes. An interested reader is welcome to try their hand at verifying it symbolically.

and then eventually C [B22+, finitefieldnodes.c]. Clearly, this is very easy
to parallelize, and we ran multiple computations at a time on the Amarel
cluster [Ama]. This allowed us to reduce the set of possible parameters to
roughly p^2. For each of these we used Mathematica to check if there are
worse-than-nodal singularities at the two expected points.

The first successful prime was $p = 53$. We proceeded to lift the solu-
tion to the p-adics, as described in Section 3. We computed the seven-fold
cover along the lines of Section 4, see [B22+, FindingKeumFPP.nb]. We
do not write down the parametric equations or the C_3 symmetry, but do
list the equations in (7.6) for the record, also available at [B22+, Quadric-
sWforKeum]. The field extension needed to construct the cyclic cover was
obtained by adding the seventh root of

$$\begin{aligned}
\big(4u_0^4 w_1 &- 4u_0 u_1^2 w_1 + (18 - 6i\sqrt{7})u_0 u_1 w_2 - (3 - i\sqrt{7})u_0^2 w_3 + (5 + 3i\sqrt{7})u_0^2 w_4 \\
&+ (-1 - 3i\sqrt{7})u_1 w_5 + (-19 - 7i\sqrt{7})u_1 w_6\big)\big(-66u_0^2 u_1 w_2 + (8 + 4i\sqrt{7})u_0^3 w_3 \\
&+ (-2 + 10i\sqrt{7})u_1^2 w_3 + (-31 + i\sqrt{7})u_0^3 w_4 + (-34 - 6i\sqrt{7})u_1^2 w_4 \\
&+ (-17 - 3i\sqrt{7})u_0 u_1 w_5 + (142 - 6i\sqrt{7})u_0 u_1 w_6\big)^2 / (u_0^{14} w_1^3)
\end{aligned}$$

We verified that the resulting equations give a fake projective plane by
the same method. By elimination, it must be the one constructed by Keum
in [Ke06]. We did not try to simplify the equations of the Keum's FPP
using the torsion elements in its Picard group, even though it must be pos-
sible to do. The plan is to have an undergraduate researcher to try to use
our approach to simplify equations of this and other known fake projective
planes.

6. OPEN PROBLEMS

The construction of the this paper raises several interesting questions.

One can try to apply our approach to computing order two elements of the
Picard group of the fake projective plane $(C20, p = 2, \emptyset, D_3 2_7)$ to other fake
projective planes. It is important to have at least a C_3 symmetry, otherwise
the finite field search may take too long. One can even try to do it with
C_3-invariant elements of higher order by looking for reducible linear cuts in
the bicanonical embedding, although it is less clear how to do the lifting in
this case.

It is hoped that the knowledge of Keum's FPP will allow one to find ex-
plicit equations of Mumford's FPP. This appears to boil down to finding
a degree 7 non-Galois cover of the quotient of $\mathbb{P}^2_{\text{Keum}}$ by its C_3 automor-
phism group, with the Galois group of the corresponding splitting field be-
ing $PSL(2, 7)$. Perhaps the key to this is understanding such covers for the
fibers of the map $Y \to \mathbb{P}^1$. In fact, the triple fiber leads one to look for such
degree 7 covers $C \to E$ of a genus 7 curve C which are ramified over two
explicitly known points on an explicitly known genus one curve E. This is
currently work in progress.

7. Appendix

Equations of the nine-parameter family of Dolgachev surfaces.

(7.1)

$$d_2d_5d_9u_0u_1^2v_1^2 + d_2d_3d_9^2u_0u_1^2v_1^2 - d_2d_6d_9^2u_0u_1^2v_1^2 + d_3d_9u_0u_1v_1v_2 + d_2d_4d_9u_0u_1v_1v_2$$
$$+d_2d_5d_9u_0u_1v_1v_2 + 2d_2d_3d_9^2u_0u_1v_1v_2 - 2d_2d_6d_9^2u_0u_1v_1v_2 + d_3d_9u_0v_2^2 + d_2d_4d_9u_0v_2^2$$
$$+d_2d_3d_9^2u_0v_2^2 - d_2d_6d_9^2u_0v_2^2 + d_2u_0^2v_1v_3 + d_2v_3^2 - d_2d_9u_1v_1v_5 - v_2v_5 - d_2d_9v_2v_5$$
$$+d_2d_9u_1v_1v_6 + d_2d_9v_2v_6, -d_2d_5d_9u_0u_1^2v_1^2 - d_2d_3d_8d_9u_0u_1^2v_1^2 + d_2d_6d_8d_9u_0u_1^2v_1^2$$
$$-d_3d_8u_0u_1v_1v_2 - d_2d_4d_9u_0u_1v_1v_2 - d_2d_5d_9u_0u_1v_1v_2 - d_2d_3d_7d_9u_0u_1v_1v_2$$
$$+d_2d_6d_7d_9u_0u_1v_1v_2 - d_2d_3d_8d_9u_0u_1v_1v_2 + d_2d_6d_8d_9u_0u_1v_1v_2 - d_3d_7u_0v_2^2$$
$$-d_2d_4d_9u_0v_2^2 - d_2d_3d_7d_9u_0v_2^2 + d_2d_6d_7d_9u_0v_2^2 + d_2v_3v_4 + d_2d_9u_1v_1v_5 + d_2d_9v_2v_5$$
$$-d_2d_9u_1v_1v_6 - d_2d_9v_2v_6, d_2d_5d_9u_0u_1^2v_1^2 + d_2d_3d_8d_9u_0u_1^2v_1^2 - d_2d_6d_8d_9u_0u_1^2v_1^2$$
$$+d_1d_9u_0u_1v_1v_2 + d_2d_4d_9u_0u_1v_1v_2 + d_2d_5d_9u_0u_1v_1v_2 + d_2d_3d_7d_9u_0u_1v_1v_2$$
$$-d_2d_6d_7d_9u_0u_1v_1v_2 + d_2d_3d_8d_9u_0u_1v_1v_2 - d_2d_6d_8d_9u_0u_1v_1v_2 + d_1d_9u_0v_2^2$$
$$+d_2d_4d_9u_0v_2^2 + d_2d_3d_7d_9u_0v_2^2 - d_2d_6d_7d_9u_0v_2^2 + d_2u_0^2v_1v_4 + d_2v_4^2 - d_2d_9u_1v_1v_5$$
$$-d_2d_9v_2v_5 + d_2d_8u_1v_1v_6 + d_2d_7v_2v_6, d_3u_0^3v_1v_2 + d_2u_1^2v_1v_2 + d_2u_1v_2^2$$
$$+d_2d_3d_9u_0u_1v_1v_4 - d_2d_6d_9u_0u_1v_1v_4 + d_3u_0v_2v_4 + d_2d_3d_9u_0v_2v_4 - d_2d_6d_9u_0v_2v_4$$
$$+d_2v_3v_6, -d_2d_5d_9u_0^3u_1v_1^2 + d_2d_6d_8d_9u_0^3u_1v_1^2 - d_2d_3d_9^2u_0^3u_1v_1^2 + d_2^2d_8d_9u_1^3v_1^2$$
$$-d_2^2d_9^2u_1^3v_1^2 - d_2d_4d_9u_0^3v_1v_2 + d_2d_6d_7d_9u_0^3v_1v_2 - d_2d_3d_9^2u_0^3v_1v_2 + d_2d_8u_1^2v_1v_2$$
$$+d_2^2d_7d_9u_1^2v_1v_2 + d_2^2d_8d_9u_1^2v_1v_2 - 2d_2^2d_9^2u_1^2v_1v_2 + d_2d_7u_1v_2^2 + d_2^2d_7d_9u_1v_2^2$$
$$-d_2^2d_9^2u_1v_2^2 - d_2d_5d_9u_0u_1v_1v_3 + d_2d_6d_8d_9u_0u_1v_1v_3 - d_1d_2d_9^2u_0u_1v_1v_3 - d_1d_9u_0v_2v_3$$
$$-d_2d_4d_9u_0v_2v_3 + d_2d_6d_7d_9u_0v_2v_3 - d_1d_2d_9^2u_0v_2v_3 - d_2d_5d_9u_0u_1v_1v_4$$
$$-d_2d_4d_9u_0v_2v_4 + d_2d_9u_0^2v_1v_5 + d_2d_9v_3v_5 - d_2d_9u_0^2v_1v_6 - d_2d_9v_4v_6, -d_3d_8u_0^3u_1v_1^2$$
$$+d_3d_9u_0^3u_1v_1^2 - d_2d_8u_1^3v_1^2 + d_2d_9u_1^3v_1^2 - d_3d_7u_0^3v_1v_2 + d_3d_9u_0^3v_1v_2 - d_2d_7u_1^2v_1v_2$$
$$-d_2d_8u_1^2v_1v_2 + 2d_2d_9u_1^2v_1v_2 - d_2d_7u_1v_2^2 + d_2d_9u_1v_2^2 - d_3d_8u_0u_1v_1v_3 + d_1d_9u_0u_1v_1v_3$$
$$-d_3d_7u_0v_2v_3 + d_1d_9u_0v_2v_3 + v_4v_5, d_2d_3d_9u_0^5v_1^2 + d_2^2d_9u_0^2u_1^2v_1^2 - d_2d_3d_5d_8d_9u_0^2u_1^2v_1^2$$
$$+d_1d_2d_5d_9^2u_0^2u_1^2v_1^2 + d_2^2d_9u_0^2u_1v_1v_2 - d_2d_3d_5d_7d_9u_0^2u_1v_1v_2 - d_2d_3d_4d_8d_9u_0^2u_1v_1v_2$$
$$+d_1d_2d_4d_9^2u_0^2u_1v_1v_2 + d_1d_2d_5d_9^2u_0^2u_1v_1v_2 - d_2d_3d_4d_7d_9u_0^2v_2^2 + d_1d_2d_4d_9^2u_0^2v_2^2$$
$$+d_1d_2d_9u_0^3v_1v_3 + d_2^2d_8u_1^2v_1v_3 + d_2^2d_7u_1v_2v_3 + d_2d_3d_9u_0^3v_1v_4 + d_2^2d_9u_1^2v_1v_4$$
$$+d_2^2d_9u_1v_2v_4 - d_2d_5d_9u_0u_1v_1v_5 + d_2d_6d_8d_9u_0u_1v_1v_5 - d_1d_2d_9^2u_0u_1v_1v_5$$
$$-d_1d_9u_0v_2v_5 - d_2d_4d_9u_0v_2v_5 + d_2d_6d_7d_9u_0v_2v_5 - d_1d_2d_9^2u_0v_2v_5 + d_2d_9v_5^2$$
$$-d_2d_3d_8d_9u_0u_1v_1v_6 + d_1d_2d_9^2u_0u_1v_1v_6 - d_2d_3d_7d_9u_0v_2v_6 + d_1d_2d_9^2u_0v_2v_6,$$
$$d_2d_3u_0^5v_1^2 + d_2^2u_0^2u_1^2v_1^2 + d_2d_3^2d_8d_9u_0^2u_1^2v_1^2 - d_2d_3d_6d_8d_9u_0^2u_1^2v_1^2 - d_1d_2d_3d_9^2u_0^2u_1^2v_1^2$$
$$+d_1d_2d_6d_9^2u_0^2u_1^2v_1^2 + d_2^2u_0^2u_1v_1v_2 + d_3^2d_8u_0^2u_1v_1v_2 - d_1d_3d_9u_0^2u_1v_1v_2$$
$$+d_2d_3^2d_7d_9u_0^2u_1v_1v_2 - d_2d_3d_6d_7d_9u_0^2u_1v_1v_2 + d_2d_3^2d_8d_9u_0^2u_1v_1v_2$$
$$-d_2d_3d_6d_8d_9u_0^2u_1v_1v_2 - 2d_1d_2d_3d_9^2u_0^2u_1v_1v_2 + 2d_1d_2d_6d_9^2u_0^2u_1v_1v_2 + d_3^2d_7u_0^2v_2^2$$
$$-d_1d_3d_9u_0^2v_2^2 + d_2d_3^2d_7d_9u_0^2v_2^2 - d_2d_3d_6d_7d_9u_0^2v_2^2 - d_1d_2d_3d_9^2u_0^2v_2^2$$
$$+d_1d_2d_6d_9^2u_0^2v_2^2 + d_2d_3u_0^3v_1v_3 + d_2^2u_1^2v_1v_3 + d_2^2u_1v_2v_3 + d_2d_3u_0^3v_1v_4 + d_2^2u_1^2v_1v_4$$
$$+d_2^2u_1v_2v_4 + d_2v_5v_6, d_2d_3d_9u_0^5v_1^2 + d_3^2d_9u_0^2u_1^2v_1^2 - d_2^2d_3d_5d_9u_0^2u_1^2v_1^2$$
$$+d_2^2d_5d_6d_9u_0^2u_1^2v_1^2 + d_2^2d_3d_6d_8d_9u_0^2u_1^2v_1^2 - d_2^2d_6^2d_8d_9u_0^2u_1^2v_1^2 - d_1d_2^2d_3d_9^3u_0^2u_1^2v_1^2$$
$$+d_1d_2^2d_6d_9^3u_0^2u_1^2v_1^2 + d_3^3d_9u_0^2u_1v_1v_2 - d_2d_3d_5d_9u_0^2u_1v_1v_2 + d_2d_3d_6d_8d_9u_0^2u_1v_1v_2$$
$$-2d_1d_2d_3d_9^2u_0^2u_1v_1v_2 - d_2^2d_3d_4d_9^2u_0^2u_1v_1v_2 - d_2^2d_3d_5d_9^2u_0^2u_1v_1v_2$$
$$+d_1d_2d_6d_9^2u_0^2u_1v_1v_2 + d_2^2d_4d_6d_9^2u_0^2u_1v_1v_2 + d_2^2d_5d_6d_9^2u_0^2u_1v_1v_2$$
$$+d_2^2d_3d_6d_7d_9^2u_0^2u_1v_1v_2 - d_2^2d_6^2d_7d_9^2u_0^2u_1v_1v_2 + d_2^2d_3d_6d_8d_9^2u_0^2u_1v_1v_2$$
$$-d_2^2d_6^2d_8d_9^2u_0^2u_1v_1v_2 - 2d_1d_2^2d_3d_9^3u_0^2u_1v_1v_2 + 2d_1d_2^2d_6d_9^3u_0^2u_1v_1v_2 - d_1d_3d_9u_0^2v_2^2$$
$$-d_2d_3d_4d_9u_0^2v_2^2 + d_2d_3d_6d_7d_9u_0^2v_2^2 - 2d_1d_2d_3d_9^2u_0^2v_2^2 - d_2^2d_3d_4d_9^2u_0^2v_2^2 + d_1d_2d_6d_9^2u_0^2v_2^2$$
$$+d_2^2d_4d_6d_9^2u_0^2v_2^2 + d_2^2d_3d_6d_7d_9^2u_0^2v_2^2 - d_2^2d_6^2d_7d_9^2u_0^2v_2^2 - d_1d_2^2d_3d_9^3u_0^2v_2^2 + d_1d_2^2d_6d_9^3u_0^2v_2^2$$
$$+d_2^2d_3d_9u_0^3v_1v_3 + d_3^3d_9u_1^2v_1v_3 + d_3^3d_9u_1v_2v_3 + d_2^2d_6d_9u_0^3v_1v_4 + d_3^3d_9u_1^2v_1v_4 + d_2^2u_1v_2v_4$$
$$+d_3^3d_9u_1v_2v_4 + d_2^2d_3d_9^2u_0u_1v_1v_5 - d_2^2d_6d_9^2u_0u_1v_1v_5 + d_2d_3d_9u_0v_2v_5 + d_2^2d_3d_9^2u_0v_2v_5$$
$$-d_2^2d_6d_9^2u_0v_2v_5 + d_2^2d_5d_9u_0u_1v_1v_6 + d_2^2d_4d_9u_0v_2v_6 + d_2^2d_9v_6^2$$

Equations of Y_0 in the case of $(C20, p = 2, \emptyset, D_3 2_7)$.

(7.2)

$\frac{1}{32}(-63 + 259\,\mathrm{i}\sqrt{7})u_0^4 w_1^2 + \frac{1}{32}(63 - 259\,\mathrm{i}\sqrt{7})u_0 u_1^2 w_1^2 + \frac{1}{64}(21 + 31\,\mathrm{i}\sqrt{7})u_0 u_1 w_1 w_2$

$+\frac{1}{32}(-1 - 3\,\mathrm{i}\sqrt{7})u_0 w_2^2 + \frac{1}{16}(133 - 17\,\mathrm{i}\sqrt{7})u_0^2 w_1 w_3 + w_3^2 + \frac{1}{32}(-217 + 21\,\mathrm{i}\sqrt{7})u_0^2 w_1 w_4$

$+\frac{1}{64}(77 - 57\,\mathrm{i}\sqrt{7})u_1 w_1 w_5 + \frac{1}{32}(-67 + 23\,\mathrm{i}\sqrt{7})w_2 w_5 + \frac{1}{16}(-49 + 13\,\mathrm{i}\sqrt{7})u_1 w_1 w_6$

$+\frac{1}{8}(9 - 5\,\mathrm{i}\sqrt{7})w_2 w_6,\ \frac{1}{4}(7 + 30\,\mathrm{i}\sqrt{7})u_0^4 w_1^2 + \frac{1}{4}(-7 - 30\,\mathrm{i}\sqrt{7})u_0 u_1^2 w_1^2$

$+\frac{1}{32}(11 + 9\,\mathrm{i}\sqrt{7})u_0 u_1 w_1 w_2 + \frac{1}{56}(-7 - 5\,\mathrm{i}\sqrt{7})u_0 w_2^2 + \frac{1}{4}(23 - 4\,\mathrm{i}\sqrt{7})u_0^2 w_1 w_3$

$+\frac{1}{16}(-77 + 9\,\mathrm{i}\sqrt{7})u_0^2 w_1 w_4 + w_3 w_4 + \frac{1}{32}(51 - 7\,\mathrm{i}\sqrt{7})u_1 w_1 w_5 + \frac{1}{56}(-77 + 41\,\mathrm{i}\sqrt{7})w_2 w_5$

$+\frac{1}{8}(-23 + 3\,\mathrm{i}\sqrt{7})u_1 w_1 w_6 + \frac{1}{14}(7 - 9\,\mathrm{i}\sqrt{7})w_2 w_6,\ \frac{1}{64}(275 + 471\,\mathrm{i}\sqrt{7})u_0^4 w_1^2$

$+\frac{1}{64}(-275 - 471\,\mathrm{i}\sqrt{7})u_0 u_1^2 w_1^2 + \frac{1}{448}(161 - 11\,\mathrm{i}\sqrt{7})u_0 u_1 w_1 w_2 + \frac{1}{28}(-7 - 2\,\mathrm{i}\sqrt{7})u_0 w_2^2$

$+\frac{1}{64}(229 + 17\,\mathrm{i}\sqrt{7})u_0^2 w_1 w_3 + \frac{1}{4}(-10 - 3\,\mathrm{i}\sqrt{7})u_0^2 w_1 w_4 + w_4^2 + \frac{1}{112}(175 - 31\,\mathrm{i}\sqrt{7})u_1 w_1 w_5$

$+\frac{1}{28}(-21 + 22\,\mathrm{i}\sqrt{7})w_2 w_5 + \frac{1}{14}(-42 + 11\,\mathrm{i}\sqrt{7})u_1 w_1 w_6 - \frac{5}{7}\mathrm{i}\sqrt{7}w_2 w_6,$

$\frac{1}{1472}(2107 - 567\,\mathrm{i}\sqrt{7})u_0^3 u_1 w_1^2 + \frac{1}{1472}(-2107 + 567\,\mathrm{i}\sqrt{7})u_1^3 w_1^2 + \frac{1}{1472}(-973 + 617\,\mathrm{i}\sqrt{7})u_0^3 w_1 w_2 +$

$\frac{1}{736}(427 - 607\,\mathrm{i}\sqrt{7})u_1^2 w_1 w_2 + \frac{1}{736}(-177 + 73\,\mathrm{i}\sqrt{7})u_1 w_2^2 + \frac{1}{16}(-63 + 47\,\mathrm{i}\sqrt{7})u_0 u_1 w_1 w_3 +$

$\frac{1}{184}(254 - 29\,\mathrm{i}\sqrt{7})u_0 w_2 w_3 + \frac{1}{184}(861 - 497\,\mathrm{i}\sqrt{7})u_0 u_1 w_1 w_4 + \frac{1}{736}(-1071 + 147\,\mathrm{i}\sqrt{7})u_0 w_2 w_4 +$

$\frac{1}{64}(161 - 61\,\mathrm{i}\sqrt{7})u_0^2 w_1 w_5 + \frac{1}{368}(-1169 + 413\,\mathrm{i}\sqrt{7})u_0^2 w_1 w_6 + w_3 w_6 + \frac{1}{92}(-63 - 13\,\mathrm{i}\sqrt{7})w_4 w_6,$

$\frac{1}{736}(2457 - 735\,\mathrm{i}\sqrt{7})u_0^3 u_1 w_1^2 + \frac{1}{736}(-2457 + 735\,\mathrm{i}\sqrt{7})u_1^3 w_1^2 + \frac{1}{368}(-413 + 201\,\mathrm{i}\sqrt{7})u_0^3 w_1 w_2 +$

$\frac{1}{92}(119 - 70\,\mathrm{i}\sqrt{7})u_1^2 w_1 w_2 + \frac{1}{736}(-257 + 139\,\mathrm{i}\sqrt{7})u_1 w_2^2 + \frac{1}{8}(-49 + 23\,\mathrm{i}\sqrt{7})u_0 u_1 w_1 w_3 +$

$\frac{1}{736}(1565 - 167\,\mathrm{i}\sqrt{7})u_0 w_2 w_3 + \frac{1}{92}(651 - 203\,\mathrm{i}\sqrt{7})u_0 u_1 w_1 w_4 + \frac{1}{184}(-399 + 71\,\mathrm{i}\sqrt{7})u_0 w_2 w_4 +$

$\frac{1}{16}(105 - 37\,\mathrm{i}\sqrt{7})u_0^2 w_1 w_5 + w_3 w_5 + \frac{1}{92}(-707 + 171\,\mathrm{i}\sqrt{7})u_0^2 w_1 w_6 + \frac{1}{23}(-14 - 8\,\mathrm{i}\sqrt{7})w_4 w_6,$

$\frac{1}{5888}(4249 - 171\,\mathrm{i}\sqrt{7})u_0^3 u_1 w_1^2 + \frac{1}{5888}(-4249 + 171\,\mathrm{i}\sqrt{7})u_1^3 w_1^2 + \frac{1}{2944}(-1551 + 589\,\mathrm{i}\sqrt{7})u_0^3 w_1 w_2 +$

$\frac{1}{368}(371 - 41\,\mathrm{i}\sqrt{7})u_1^2 w_1 w_2 + \frac{1}{5888}(-903 + 565\,\mathrm{i}\sqrt{7})u_1 w_2^2 + \frac{1}{128}(-399 + 77\,\mathrm{i}\sqrt{7})u_0 u_1 w_1 w_3$

$+\frac{1}{5888}(6223 - 653\,\mathrm{i}\sqrt{7})u_0 w_2 w_3 + \frac{1}{184}(707 - 33\,\mathrm{i}\sqrt{7})u_0 u_1 w_1 w_4 + \frac{1}{736}(-765 + 151\,\mathrm{i}\sqrt{7})u_0 w_2 w_4 +$

$\frac{1}{16}(57 - 27\,\mathrm{i}\sqrt{7})u_0^2 w_1 w_5 + w_4 w_5 + \frac{1}{368}(-1755 + 433\,\mathrm{i}\sqrt{7})u_0^2 w_1 w_6 + \frac{1}{184}(-159 - 35\,\mathrm{i}\sqrt{7})w_4 w_6,$

$\frac{1}{32}(245 + 217\,\mathrm{i}\sqrt{7})u_0^5 w_1^2 + \frac{1}{32}(-245 - 217\,\mathrm{i}\sqrt{7})u_0^2 u_1^2 w_1^2 + \frac{1}{256}(287 - 157\,\mathrm{i}\sqrt{7})u_0^2 u_1 w_1 w_2$

$+\frac{1}{256}(-71 - 11\,\mathrm{i}\sqrt{7})u_0^2 w_2^2 + \frac{1}{32}(119 - 21\,\mathrm{i}\sqrt{7})u_0^3 w_1 w_3 + \frac{1}{256}(-7 + 181\,\mathrm{i}\sqrt{7})u_1^2 w_1 w_3$

$+\frac{1}{256}(-121 - 53\,\mathrm{i}\sqrt{7})u_1 w_2 w_3 + \frac{1}{8}(-49 - 7\,\mathrm{i}\sqrt{7})u_0^3 w_1 w_4 + \frac{1}{32}(119 - 21\,\mathrm{i}\sqrt{7})u_1^2 w_1 w_4$

$+\frac{1}{32}(-7 + 5\,\mathrm{i}\sqrt{7})u_1 w_2 w_4 + \frac{1}{32}(-343 + 17\,\mathrm{i}\sqrt{7})u_0 u_1 w_1 w_5 + \frac{1}{32}(33 + 29\,\mathrm{i}\sqrt{7})u_0 w_2 w_5 + w_5^2$

$+\frac{1}{16}(133 + 25\,\mathrm{i}\sqrt{7})u_0 u_1 w_1 w_6 + \frac{1}{16}(-17 - 13\,\mathrm{i}\sqrt{7})u_0 w_2 w_6,\ \frac{1}{16}(98 + 119\,\mathrm{i}\sqrt{7})u_0^5 w_1^2$

$+\frac{1}{16}(-98 - 119\,\mathrm{i}\sqrt{7})u_0^2 u_1^2 w_1^2 + \frac{1}{32}(28 - 29\,\mathrm{i}\sqrt{7})u_0^2 u_1 w_1 w_2 + \frac{1}{64}(-27 + \mathrm{i}\sqrt{7})u_0^2 w_2^2$

$+\frac{1}{32}(133 + 77\,\mathrm{i}\sqrt{7})u_0^3 w_1 w_3 + \frac{1}{64}(-49 + 19\,\mathrm{i}\sqrt{7})u_1^2 w_1 w_3 + \frac{1}{64}(11 - 17\,\mathrm{i}\sqrt{7})u_1 w_2 w_3$

$+\frac{1}{32}(-147 - 105\,\mathrm{i}\sqrt{7})u_0^3 w_1 w_4 + \frac{1}{8}(28 - 7\,\mathrm{i}\sqrt{7})u_1^2 w_1 w_4 + \frac{1}{8}(-7 + 2\,\mathrm{i}\sqrt{7})u_1 w_2 w_4$

$+\frac{1}{64}(-651 - 17\,\mathrm{i}\sqrt{7})u_0 u_1 w_1 w_5 + \frac{1}{16}(19 + 13\,\mathrm{i}\sqrt{7})u_0 w_2 w_5 + \frac{1}{8}(56 + 21\,\mathrm{i}\sqrt{7})u_0 u_1 w_1 w_6$

$+\frac{1}{16}(-17 - 13\,\mathrm{i}\sqrt{7})u_0 w_2 w_6 + w_5 w_6,\ \frac{1}{128}(735 + 1057\,\mathrm{i}\sqrt{7})u_0^5 w_1^2$

$+\frac{1}{128}(-735 - 1057\,\mathrm{i}\sqrt{7})u_0^2 u_1^2 w_1^2 + \frac{1}{256}(273 - 333\,\mathrm{i}\sqrt{7})u_0^2 u_1 w_1 w_2 + \frac{1}{32}(-19 + 2\,\mathrm{i}\sqrt{7})u_0^2 w_2^2$

$+\frac{1}{32}(56 + 175\,\mathrm{i}\sqrt{7})u_0^3 w_1 w_3 + \frac{1}{64}(-63 + 13\,\mathrm{i}\sqrt{7})u_1^2 w_1 w_3 + \frac{1}{64}(59 - 21\,\mathrm{i}\sqrt{7})u_1 w_2 w_3$

$+\frac{1}{128}(49 - 749\,\mathrm{i}\sqrt{7})u_0^3 w_1 w_4 + \frac{1}{64}(161 - 77\,\mathrm{i}\sqrt{7})u_1^2 w_1 w_4 + \frac{1}{64}(-105 + 21\,\mathrm{i}\sqrt{7})u_1 w_2 w_4$

$+\frac{1}{256}(-2471 - 405\,\mathrm{i}\sqrt{7})u_0 u_1 w_1 w_5 + \frac{1}{8}(9 + 7\,\mathrm{i}\sqrt{7})u_0 w_2 w_5 + \frac{1}{64}(315 + 273\,\mathrm{i}\sqrt{7})u_0 u_1 w_1 w_6$

$+\frac{1}{16}(-13 - 15\,\mathrm{i}\sqrt{7})u_0 w_2 w_6 + w_6^2$

Parametric equation of S_2 in Y_0 in the case of $(C20, p = 2, \emptyset, D_3 2_7)$.

(7.3)

$(u_0, u_1, w_1, w_2, w_3, w_4, w_5, w_6) =$

$\Big(\frac{1}{2}(5 + 9\,\mathrm{i}\sqrt{7} - 16(1 + \mathrm{i}\sqrt{7})\mathrm{t} + (13 + 7\,\mathrm{i}\sqrt{7})\mathrm{t}^2), \frac{1}{2}(-47 + 29\,\mathrm{i}\sqrt{7} + 84(1 - \mathrm{i}\sqrt{7})\mathrm{t}$

$+(-14 + 86\,\mathrm{i}\sqrt{7})\mathrm{t}^2 + (-21 - 31\,\mathrm{i}\sqrt{7})\mathrm{t}^3), 1, \frac{1}{2}(3121 - 403\,\mathrm{i}\sqrt{7} + (-9820 + 1052\,\mathrm{i}\sqrt{7})\mathrm{t}$

$+4(2545 - 221\,\mathrm{i}\sqrt{7})\mathrm{t}^2 + (-3473 + 237\,\mathrm{i}\sqrt{7})\mathrm{t}^3), \frac{1}{2}(3505 - 2963\,\mathrm{i}\sqrt{7} + 8(-2259 + 1391\,\mathrm{i}\sqrt{7})\mathrm{t}$

$+16(2039 - 955\,\mathrm{i}\sqrt{7})\mathrm{t}^2 + (-24950 + 9054\,\mathrm{i}\sqrt{7})\mathrm{t}^3 + 7(985 - 277\,\mathrm{i}\sqrt{7})\mathrm{t}^4),$

$4(-1 + \mathrm{t})(-108 + 316\,\mathrm{i}\sqrt{7} + 32(25 - 27\,\mathrm{i}\sqrt{7})\mathrm{t} + (-1239 + 731\,\mathrm{i}\sqrt{7})\mathrm{t}^2 + 8(67 - 23\,\mathrm{i}\sqrt{7})\mathrm{t}^3),$

$\frac{1}{2}(2265 - 2539\,\mathrm{i}\sqrt{7} + 8(-2941 + 1523\,\mathrm{i}\sqrt{7})\mathrm{t} + 736(99 - 29\,\mathrm{i}\sqrt{7})\mathrm{t}^2 + (-99624 + 16264\,\mathrm{i}\sqrt{7})\mathrm{t}^3$

$+48(1323 - 95\,\mathrm{i}\sqrt{7})\mathrm{t}^4 + (-15479 - 5\,\mathrm{i}\sqrt{7})\mathrm{t}^5), \frac{1}{2}(-1 + \mathrm{t})(4720 + 336\,\mathrm{i}\sqrt{7} - 32(313 + 43\,\mathrm{i}\sqrt{7})\mathrm{t}$

$+(-952 - 40\,\mathrm{i}\sqrt{7})\mathrm{t}^2 + 8(1571 + 393\,\mathrm{i}\sqrt{7})\mathrm{t}^3 + (-6325 - 2047\,\mathrm{i}\sqrt{7})\mathrm{t}^4)\Big)$

Parametric equation of S_1 in Y_0 in the case of $(C20, p = 2, \emptyset, D_3 2_7)$.

(7.4)

$(u_0, u_1, w_1, w_2, w_3, w_4, w_5, w_6) =$

$\Big(\frac{1}{2}(5 + 9\,\mathrm{i}\sqrt{7} - 16(1 + \mathrm{i}\sqrt{7})\mathrm{t} + (13 + 7\,\mathrm{i}\sqrt{7})\mathrm{t}^2), \frac{1}{2}(-47 + 29\,\mathrm{i}\sqrt{7} + 84(1 - \mathrm{i}\sqrt{7})\mathrm{t}$

$+(-14 + 86\,\mathrm{i}\sqrt{7})\mathrm{t}^2 + (-21 - 31\,\mathrm{i}\sqrt{7})\mathrm{t}^3), 1, \frac{1}{2}(1049 - 379\,\mathrm{i}\sqrt{7}) + (-908 + 564\,\mathrm{i}\sqrt{7})\mathrm{t}$

$+(357 - 529\,\mathrm{i}\sqrt{7})\mathrm{t}^2 + 2(15 + 77\,\mathrm{i}\sqrt{7})\mathrm{t}^3, \frac{1}{2}(-(1271 - 363\,\mathrm{i}\sqrt{7}) + 16(193 + 37\,\mathrm{i}\sqrt{7})\mathrm{t}$

$+(-5446 - 530\,\mathrm{i}\sqrt{7})\mathrm{t}^2 + 4096\mathrm{t}^3 + \frac{1}{4}(-4417 + 477\,\mathrm{i}\sqrt{7})\mathrm{t}^4), -1105 - 77\,\mathrm{i}\sqrt{7} + 16(261 + 7\,\mathrm{i}\sqrt{7})\mathrm{t}$

$+(-6096 + 208\,\mathrm{i}\sqrt{7})\mathrm{t}^2 + 4(997 - 113\,\mathrm{i}\sqrt{7})\mathrm{t}^3 + (-965 + 209\,\mathrm{i}\sqrt{7})\mathrm{t}^4, 2(-1 + \mathrm{t})(-112 + 48\,\mathrm{i}\sqrt{7}$

$+8(71 + 11\,\mathrm{i}\sqrt{7})\mathrm{t} + (-944 - 400\,\mathrm{i}\sqrt{7})\mathrm{t}^2 + 64(13 + 7\,\mathrm{i}\sqrt{7})\mathrm{t}^3 + (-369 - 179\,\mathrm{i}\sqrt{7})\mathrm{t}^4),$

$\frac{1}{4}(-1 + \mathrm{t})(16(75 + \mathrm{i}\sqrt{7}) + 64(21 + 47\,\mathrm{i}\sqrt{7})\mathrm{t} + (-10696 - 6232\,\mathrm{i}\sqrt{7})\mathrm{t}^2 + 48(275 + 89\,\mathrm{i}\sqrt{7})\mathrm{t}^3$

$+(-5197 - 967\,\mathrm{i}\sqrt{7})\mathrm{t}^4)\Big)$

(Birational) automorphism of order 3 of Y_0 for $(C20, p = 2, \emptyset, D_3 2_7)$.

(7.5)

$(u_0, u_1, w_1, w_2, w_3, w_4, w_5, w_6) \rightarrow \Big(u_0, u_1, \frac{1}{1792(u_0^3 - u_1^2)}(7(383 - 29\,\mathrm{i}\sqrt{7})u_0^4 w_1$

$+u_0 u_1(7(-331 + \mathrm{i}\sqrt{7})u_1 w_1 + (-91 + 177\,\mathrm{i}\sqrt{7})w_2) + 7u_0^2(w_3 - 99\,\mathrm{i}\sqrt{7}w_3 + 40(-1 + 3\,\mathrm{i}\sqrt{7})w_4)$

$+16u_1((77 + 17\,\mathrm{i}\sqrt{7})w_5 + (-77 - 25\,\mathrm{i}\sqrt{7})w_6), \frac{1}{256(u_0^3 - u_1^2)}(u_0^4((2471 - 405\,\mathrm{i}\sqrt{7})u_1 w_1$

$+16(3 + 7\,\mathrm{i}\sqrt{7})w_2) + u_0 u_1^2((-2387 + 281\,\mathrm{i}\sqrt{7})u_1 w_1 + (-325 + 47\,\mathrm{i}\sqrt{7})w_2)$

$+u_0^2 u_1((1961 - 603\,\mathrm{i}\sqrt{7})w_3 + 8(-301 + 71\,\mathrm{i}\sqrt{7})w_4) + 128u_0^3((-1 + \mathrm{i}\sqrt{7})w_5$

$+(6 - 2\,\mathrm{i}\sqrt{7})w_6) + 16u_1^2((19 + 15\,\mathrm{i}\sqrt{7})w_5 + (-19 - 7\,\mathrm{i}\sqrt{7})w_6)), \frac{1}{256(u_0^3 - u_1^2)}((287 + 131\,\mathrm{i}\sqrt{7})u_0^6 w_1$

$+32u_1^3((35 + 11\,\mathrm{i}\sqrt{7})u_1 w_1 + (-5 - \mathrm{i}\sqrt{7})w_2) + u_0^3 u_1((-1547 - 447\,\mathrm{i}\sqrt{7})u_1 w_1$

$+(259 + 87\,\mathrm{i}\sqrt{7})w_2) + 16u_0 u_1^2((-45 + 7\,\mathrm{i}\sqrt{7})w_3 + 8(7 - \mathrm{i}\sqrt{7})w_4) + u_0^4((673 - 579\,\mathrm{i}\sqrt{7})w_3$

$+8(-133 + 79\,\mathrm{i}\sqrt{7})w_4) + 16u_0^2 u_1((11 + 7\,\mathrm{i}\sqrt{7})w_5 + 5(w_6 - 3\,\mathrm{i}\sqrt{7}w_6))),$

$\frac{1}{896(u_0^3 - u_1^2)}(14(39 + 43\,\mathrm{i}\sqrt{7})u_0^6 w_1 + 14u_1^3((291 + 55\,\mathrm{i}\sqrt{7})u_1 w_1 + 3(-9 - 5\,\mathrm{i}\sqrt{7})w_2)$

$+u_0^3 u_1(56(-89 - 21\,\mathrm{i}\sqrt{7})u_1 w_1 + (693 + 257\,\mathrm{i}\sqrt{7})w_2) + 14u_0^4((185 - 139\,\mathrm{i}\sqrt{7})w_3$

$+8(-33 + 19\,\mathrm{i}\sqrt{7})w_4) + 7u_0 u_1^2((-467 + 153\,\mathrm{i}\sqrt{7})w_3 + 8(71 - 21\,\mathrm{i}\sqrt{7})w_4)$

$+16u_0^2 u_1((7 + 11\,\mathrm{i}\sqrt{7})w_5 + (21 - 31\,\mathrm{i}\sqrt{7})w_6)), \frac{1}{256}((77 + 57\,\mathrm{i}\sqrt{7})u_0^2 u_1 w_1$

$+16u_1((-17 - 13\,\mathrm{i}\sqrt{7})w_3 + 16\,\mathrm{i}\sqrt{7}w_4) + \frac{32}{u_0^3 - u_1^2}u_0^3(u_0^2(w_2 + \mathrm{i}\sqrt{7}w_2)$

$+(-3 - 7\,\mathrm{i}\sqrt{7})u_1 w_3 + 8\,\mathrm{i}\sqrt{7}u_1 w_4 + (6 + 2\,\mathrm{i}\sqrt{7})u_0 w_5 + (-4 - 4\,\mathrm{i}\sqrt{7})u_0 w_6)$

$+u_0((11 - 65\,\mathrm{i}\sqrt{7})u_0 w_2 + 16((-3 + 9\,\mathrm{i}\sqrt{7})w_5 + (15 - 13\,\mathrm{i}\sqrt{7})w_6))),$

$\frac{1}{32}(u_0^2((7 + 11\,\mathrm{i}\sqrt{7})u_1 w_1 + (-5 - 5\,\mathrm{i}\sqrt{7})w_2) + 4u_1(-4(4 + \mathrm{i}\sqrt{7})w_3 + (7 + 5\,\mathrm{i}\sqrt{7})w_4)$

$+6(7 + 5\,\mathrm{i}\sqrt{7})u_0 w_5 + 8(-3 - 5\,\mathrm{i}\sqrt{7})u_0 w_6)\Big)$

Equations of Y_0 in the case of Keum's FPP.

(7.6)

$\frac{1}{4096}(4439 - 677i\sqrt{7})(u_0^3 - u_1^2)u_0w_1^2 + \frac{1}{256}(-117 - 417i\sqrt{7})u_0u_1w_1w_2$
$+ \frac{1}{16}(-27 - 9i\sqrt{7})u_0w_2^2 + \frac{1}{128}(-193 + 3i\sqrt{7})u_0^2w_1w_3 + w_3^2 + \frac{1}{256}(119 + 315i\sqrt{7})u_0^2w_1w_4$
$+ \frac{1}{32}(-15 - 7i\sqrt{7})u_1w_1w_5 + \frac{1}{16}(-3 + 15i\sqrt{7})w_2w_5 + \frac{1}{32}(7 + 43i\sqrt{7})u_1w_1w_6$
$+ \frac{1}{8}(21 - 9i\sqrt{7})w_2w_6, \ \frac{1}{2048}(169 + 165i\sqrt{7})(u_0^3 - u_1^2)u_0w_1^2 + \frac{1}{128}(9 + 21i\sqrt{7})u_0u_1w_1w_2$
$+ \frac{1}{16}(-9 - 27i\sqrt{7})u_0w_2^2 + \frac{1}{64}(5 + i\sqrt{7})u_0^2w_1w_3 + \frac{1}{128}(-331 + 33i\sqrt{7})u_0^2w_1w_4 + w_3w_4$
$+ \frac{1}{8}u_1w_1w_5 + \frac{1}{16}(-33 - 3i\sqrt{7})w_2w_5 + \frac{1}{16}(-27 - 7i\sqrt{7})u_1w_1w_6 + \frac{1}{8}(45 + 15i\sqrt{7})w_2w_6,$
$\frac{1}{1024}(-57 + 11i\sqrt{7})(u_0^3 - u_1^2)u_0w_1^2 + \frac{1}{64}(-39 - 3i\sqrt{7})u_0u_1w_1w_2 + \frac{1}{16}(45 - 9i\sqrt{7})u_0w_2^2$
$+ \frac{1}{32}(-3 + i\sqrt{7})u_0^2w_1w_3 + \frac{1}{64}(-39 - 27i\sqrt{7})u_0^2w_1w_4 + w_4^2 + \frac{1}{16}(-1 + i\sqrt{7})u_1w_1w_5$
$+ \frac{1}{16}(-3 - 9i\sqrt{7})w_2w_5 + \frac{1}{8}(9 - 3i\sqrt{7})u_1w_1w_6 + \frac{1}{8}(-27 + 15i\sqrt{7})w_2w_6,$
$\frac{1}{4096}(-161 + 67i\sqrt{7})u_0^3u_1w_1^2 + \frac{1}{4096}(161 - 67i\sqrt{7})u_1^3w_1^2 + \frac{1}{256}(93 + 9i\sqrt{7})u_0^3w_1w_2$
$+ \frac{1}{512}(-63 + 93i\sqrt{7})u_1^2w_1w_2 + \frac{1}{128}(-81 - 45i\sqrt{7})u_1w_2^2 + \frac{1}{256}((31 + 3i\sqrt{7})u_0u_1w_1w_3$
$+ \frac{1}{64}(-33 + 3i\sqrt{7})u_0w_2w_3 + \frac{1}{512}(-233 + 27i\sqrt{7})u_0u_1w_1w_4 + \frac{1}{128}(51 - 57i\sqrt{7})u_0w_2w_4$
$+ \frac{1}{64}(11 - i\sqrt{7})u_0^2w_1w_5 + \frac{1}{64}(-69 - 17i\sqrt{7})u_0^2w_1w_6 + w_3w_6 + \frac{1}{4}(3 + 3i\sqrt{7})w_4w_6,$
$\frac{1}{4096}(-1663 + 285i\sqrt{7})(u_0^3 - u_1^2)u_1w_1^2 + \frac{1}{256}(-477 - 9i\sqrt{7})u_0^3w_1w_2$
$+ \frac{1}{512}(-561 + 243i\sqrt{7})u_1^2w_1w_2 + \frac{1}{32}(9 - 27i\sqrt{7})u_1w_2^2 + \frac{1}{256}(121 + 53i\sqrt{7})u_0u_1w_1w_3$
$+ \frac{1}{16}(-3 + 9i\sqrt{7})u_0w_2w_3 + \frac{1}{512}(-1911 - 251i\sqrt{7})u_0u_1w_1w_4 + \frac{1}{32}(21 - 63i\sqrt{7})u_0w_2w_4$
$+ \frac{1}{64}(-29 + 23i\sqrt{7})u_0^2w_1w_5 + w_3w_5 + \frac{1}{64}(203 - 65i\sqrt{7})u_0^2w_1w_6 + \frac{1}{4}(7 + 11i\sqrt{7})w_4w_6,$
$\frac{1}{256}(-13 + 7i\sqrt{7})(u_0^3 - u_1^2)u_1w_1^2 + \frac{1}{8}((-3 - 3i\sqrt{7})u_0^3w_1w_2 + \frac{1}{256}(147 + 39i\sqrt{7})u_1^2w_1w_2$
$+ \frac{1}{64}(-99 + 9i\sqrt{7})u_1w_2^2 + \frac{1}{128}(-9 - 5i\sqrt{7})u_0u_1w_1w_3 + \frac{1}{32}(-3 + 9i\sqrt{7})u_0w_2w_3$
$+ \frac{1}{256}(-17 - 109i\sqrt{7})u_0u_1w_1w_4 + \frac{1}{64}(105 - 27i\sqrt{7})u_0w_2w_4 + \frac{1}{32}(-5 - i\sqrt{7})u_0^2w_1w_5$
$+ w_4w_5 + \frac{1}{32}(19 + 7i\sqrt{7})u_0^2w_1w_6 + \frac{1}{2}(-3 + i\sqrt{7})w_4w_6,$
$\frac{1}{32768}(5579 + 3199i\sqrt{7})(u_0^3 - u_1^2)u_0^2w_1^2 + \frac{1}{4096}(12009 - 1563i\sqrt{7})u_0^2u_1w_1w_2$
$+ \frac{1}{2048}(-3897 + 3339i\sqrt{7})u_0^2w_2^2 + \frac{1}{4096}(-1267 - 7i\sqrt{7})u_0^3w_1w_3$
$+ \frac{1}{4096}(317 - 1175i\sqrt{7})u_1^2w_1w_3 + \frac{1}{1024}(843 + 1023i\sqrt{7})u_1w_2w_3$
$+ \frac{1}{4096}(49 - 1267i\sqrt{7})u_0^3w_1w_4 + \frac{1}{1024}(1267 + 7i\sqrt{7})u_1^2w_1w_4 + \frac{1}{2048}(-8589 + 903i\sqrt{7})u_1w_2w_4$
$+ \frac{1}{512}(141 - 135i\sqrt{7})u_0u_1w_1w_5 + \frac{1}{3}87 + 87i\sqrt{7})u_0w_2w_5 + w_5^2$
$+ \frac{1}{512}(-1043 + 345i\sqrt{7})u_0u_1w_1w_6 + \frac{1}{256}(147 - 729i\sqrt{7})u_0w_2w_6,$
$\frac{1}{32768}(623 + 275i\sqrt{7})(u_0^3 - u_1^2)u_0^2w_1^2 + \frac{1}{4096}(1227 - 129i\sqrt{7})u_0^2u_1w_1w_2$
$+ \frac{1}{2048}(-1359 + 909i\sqrt{7})u_0^2w_2^2 + \frac{1}{4096}(315 + 47i\sqrt{7})u_0^3w_1w_3 + \frac{1}{4096}(-221 - 137i\sqrt{7})u_1^2w_1w_3$
$+ \frac{1}{1024}(573 + 105i\sqrt{7})u_1w_2w_3 + \frac{1}{4096}(-329 + 315i\sqrt{7})u_0^3w_1w_4 + \frac{1}{256}(31 + 3i\sqrt{7})u_1^2w_1w_4$
$+ \frac{1}{2048}(-1851 + 465i\sqrt{7})u_1w_2w_4 + \frac{1}{512}(-5 - 17i\sqrt{7})u_0u_1w_1w_5 + \frac{1}{256}(-39 + 21i\sqrt{7})u_0w_2w_5$
$+ \frac{1}{512}((-213 + 95i\sqrt{7})u_0u_1w_1w_6 + \frac{1}{256}(57 - 267i\sqrt{7})u_0w_2w_6 + w_5w_6,$
$\frac{1}{32768}(623 + 275i\sqrt{7})(u_0^3 - u_1^2)u_0^2w_1^2 + \frac{1}{4096}(1227 - 129i\sqrt{7})u_0^2u_1w_1w_2$
$+ \frac{1}{2048}(135 + 459i\sqrt{7})u_0^2w_2^2 + \frac{1}{4096}(161 - 67i\sqrt{7})u_0^3w_1w_3 + \frac{1}{4096}(-67 - 23i\sqrt{7})u_1^2w_1w_3$
$+ \frac{1}{!024}(171 - 33i\sqrt{7})u_1w_2w_3 + \frac{1}{4096}(147 + 295i\sqrt{7})u_0^3w_1w_4 + \frac{1}{1024}(5 + 17i\sqrt{7})u_1^2w_1w_4$
$+ \frac{1}{2048}(-141 + 135i\sqrt{7})u_1w_2w_4 + \frac{1}{512}(-5 - 17i\sqrt{7})u_0u_1w_1w_5 + \frac{1}{256}(27 + 15i\sqrt{7})u_0w_2w_5$
$+ \frac{1}{512}(-213 + 95i\sqrt{7})u_0u_1w_1w_6 + \frac{1}{256}(-309 - 129i\sqrt{7})u_0w_2w_6 + w_6^2$

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