Derived Partners of Enriques Surfaces

Lev Borisov, Vernon Chan, Chengxi Wang

September 23, 2021

Abstract. Let V be a 6-dimensional complex vector space with an involution σ of trace 0, and let $W \subseteq \operatorname{Sym}^2 V^\vee$ be a generic 3-dimensional subspace of σ-invariant quadratic forms. To these data we can associate an Enriques surface as the σ-quotient of the complete intersection of the quadratic forms in W. We exhibit noncommutative Deligne-Mumford stacks together with Brauer classes whose derived categories are equivalent to those of the Enriques surfaces. This provides a more accessible treatment of [16, Theorem 6.16].

1   Introduction

Derived categories are a fascinating invariant of algebraic varieties. Two algebraic varieties are called derived equivalent if their bounded derived categories of coherent sheaves are equivalent as triangulated categories [12]. Bondal and Orlov showed that a smooth projective variety X, whose K_X or $-K_X$ is ample, is determined by its bounded derived category $\mathcal{D}^b(\operatorname{Coh} -X)$ [1], so in these cases derived equivalence implies isomorphism. However in the case when $K_X = 0$ (Calabi-Yau) there are multiple constructions of non-trivial derived equivalence [2]. The notion is also related to string theory: for Calabi-Yau varieties X, Y and Z, homological mirror symmetry predicts that if X and Y are both "mirror" to Z (we say that X and Y are a double mirror pair), then X and Y are derived equivalent.

Many of the examples come from the Homological Projective Duality of Kuznetsov [15]. In these examples, the derived partners of a Calabi-Yau variety can be slightly non-commutative and involve a DM-stack structure or a Brauer class. Of particular interest to us is the example of complete intersections of quadrics [14], especially in dimension 2, which we describe briefly here: for $V = \mathbb{C}^6$ and $W \subseteq S^2 V^\vee$ a generic dimension 3 subspace homogeneous quadratic forms on V such that the intersection X of quadrics parametrized by W is a complete intersection, there exists a Brauer class α on $\mathbb{P}W$ constructed via Clifford algebras such that the derived category of sheaves on $\mathbb{P}W$ twisted by α is equivalent to the derived category of X:

$$\mathcal{D}^b(\operatorname{Coh} -\mathbb{P}W, \alpha) \quad \simeq \quad \mathcal{D}^b(\operatorname{Coh} -X).$$

Borisov and Li proposed a toric framework to generalize the above construction. In particular, they argued the existence of a derived partner of Enriques surfaces (see [3, Section 9.2]), making use of the fact that all complex Enriques surfaces can be obtained as a quotient of $(2, 2, 2)$-complete intersection in $\mathbb{P}^5$ by a fixed-point-free involution [7].

In this paper, we look into the case of Enriques surfaces in more details and aim to prove 2 main theorems:

Theorem 2.8. There exists a Brauer class α of order 2 on $\mathscr{Y}$ such that there is an equivalence:

$$\mathcal{D}^b(\mathrm{Coh} - \mathscr{Y}, \alpha) \quad \simeq \quad \mathcal{D}^b(\mathrm{Coh} - X).$$

Theorem 2.12. There exists a Brauer class $\widetilde{\alpha}$ of order 2 on $\mathscr{Y}/\mathbb{Z}_2$ such that there is an equivalence:

$$\mathcal{D}^b(\mathrm{Coh} - (\mathscr{Y}/\mathbb{Z}_2), \widetilde{\alpha}) \quad \simeq \quad \mathcal{D}^b(\mathrm{Coh} - (X/\sigma)).$$

After that, we will construct the geometric realization of the Brauer classes as Severi-Brauer varieties over the stacks $\mathscr{Y}$ and $\mathscr{Y}/\mathbb{Z}_2$.

Structure of the paper. In section 2 we define the notations and list some properties of the quotient stacks involved. In section 3 we define the Brauer classes in terms of Clifford algebras and provide the proof of the main theorems. In section 4 we will construct the associated Severi-Brauer varieties as geometric realizations of the Brauer classes. In section 5 we discuss further research directions and open problems.

Acknowledgements. We thank Alexander Kuznetsov, Daniel Krashen and Franco Rota for insightful suggestions and discussions. We note that the main results of this paper have also appeared in [16], although our setup is more explicit. We do not know whether our equivalence of categories is the same as that of [16].

2 Quotient stacks associated to the K3 and Enriques surfaces

In this section we define all of the schemes and stacks that we will be using in the paper, and fix our notations in the process.

Definition 2.1. Let V be a 6-dimensional complex vector space with coordinates $x_1^{\pm}, x_2^{\pm}, x_3^{\pm}$ equipped with an involution σ that fixes x_i^+ and sends $x_i^- \mapsto -x_i^-$. Let $W \subseteq S^2 V^*$ be a general dimension 3 subspace of homogeneous quadratic forms which are invariant under the involution. Each quadratic form q in W defines a quadric in V. Let $X \subseteq \mathbb{P}V$ be the zero locus of the quadric forms in W. Assume that X is a complete intersection of 3 quadrics q_1, q_2, q_3 in W. Then X is a K3 surface. If we take the quotient of X by the involution σ on V, assuming σ acts freely on X, we obtain an Enriques surface X/σ. (All complex Enriques surfaces can be constructed using this process [7].) On the double mirror side, let u_1, u_2, u_3 be coordinates on $W \cong \mathbb{C}^3$ so that every quadratic equation q in W can be written as $\sum_i u_i q_i$. As each $q \in W$ is invariant under the involution, q can be written as $q = q^+ + q^-$ where each of $q^{\pm}$ involves $x_i^{\pm}$ only, and the matrix representation of q is a block matrix consisting of two 3×3 matrices representing q^+ and q^-. Now $\det(q) = 0$ is a sextic equation on W in u_i and the curve E it defines in $\mathbb{P}W$ is the union of two cubic curves E_+, E_- respectively defined by

$$f_{\pm} := \det(q^{\pm}).$$

As W is assumed to be a general subspace, and the loci for which the rank of q_+ or q_- drops to 1 is cut out by the 2×2 minors of their associated matrices, we see that all quadratic forms in W have rank ≥ 4. $\qquad\square$

In the original setting, Kuznetsov considered the double cover Y of $\mathbb{P}W$ ramified over E, which is a smooth sextic curve. In our case, the double cover would be singular because the ramification locus $E_+ \cup E_-$ has nodes, so we want to look at desingularizations of it. There are at least 2 options: blowing up at the nodes which gives nine (-2)-curves; or consider stack structures on the nodes. In this paper, we go with the second route.

Now let us define the stacky resolution $\mathscr{Y}$ of the ramified double cover Y as a global quotient stack. Define the following algebras over the coordinate ring $\mathbb{C}[u] := \mathbb{C}[u_1, u_2, u_3]$ of W:

$$A := \mathbb{C}[u_1, u_2, u_3, y_+, y_-]/\langle y_+^2 - f_+, \, y_-^2 - f_- \rangle,$$
$$B := \mathbb{C}[u_1, u_2, u_3, y]/\langle y^2 - f_+ f_- \rangle.$$

They are related by the map $B \to A$ sending $y \mapsto y_+ y_-$. Next, define group actions of $\mathbb{C}_\lambda^*$, $\mathbb{C}_t^* := \mathbb{C}^*$ on A and B respectively:

$$\lambda \cdot (u_1, u_2, u_3, y_+, y_-) := (\lambda^2 u_1, \lambda^2 u_2, \lambda^2 u_3, \lambda^3 y_+, \lambda^3 y_-),$$
$$t \cdot (u_1, u_2, u_3, y) := (t u_1, t u_2, t u_3, t^3 y),$$

where we added subscripts λ and t under the groups to distinguish two different $\mathbb{C}^*$-actions. The two actions are related by the map $\mathbb{C}_\lambda^* \to \mathbb{C}_t^*$ where $\lambda \mapsto t = \lambda^2$, which is compatible with the map $B \to A$. Passing to the quotient stacks, we obtained a map

$$[(\operatorname{Spec} A \backslash 0)/\mathbb{C}_\lambda^*] \to [(\operatorname{Spec} B \backslash 0)/\mathbb{C}_t^*].$$

The $\mathbb{C}_t^*$-action on the u_i coordinates is just the scaling action on the projective space $\mathbb{P}W$. There are $\mathbb{N}$-gradings on A and B corresponding to the two $\mathbb{C}^*$-actions, explicitly:

$$A : \begin{cases} u_1, u_2, u_3 : & \text{degree } 2 \\ y_+, y_- : & \text{degree } 3 \end{cases} \quad \text{and} \quad B : \begin{cases} u_1, u_2, u_3 : & \text{degree } 1 \\ y : & \text{degree } 3 \end{cases}.$$

Then the ramified double cover $Y \to \mathbb{P}W$ can be realized as $\operatorname{Proj} B \to \mathbb{P}W$.

Remark 2.2. The algebra A cannot be made into a graded algebra over $\mathbb{C}[u]$ that respects the natural grading of u_i's in $\mathbb{P}W$ (i.e. u_i's have degree 1), because the relations in A will not be homogeneous.

Here and for the remainder of the paper, we abbreviate

$$\operatorname{Spec} A \backslash 0 := (\operatorname{Spec} A) \setminus \{(u_1, u_2, u_3) = 0\} = (\operatorname{Spec} A) \setminus \{(u_1, u_2, u_3, y_+, y_-) = 0\},$$

and do the same to $W \backslash 0$ etc. The u_i coordinates will often be collectively referred as u. We first explore the smoothness conditions on $\operatorname{Spec} A \backslash 0$.

Proposition 2.3. $\operatorname{Spec} A \backslash 0$ *is smooth if and only if* E_+, E_- *are smooth and intersect transversely.*

Proof. E_+, E_- are smooth means that their Jacobians $[\partial f_+/\partial x_i^\pm]$ and $[\partial f_-/\partial x_i^\pm]$ are non-vanishing on themselves, and E_+ intersects E_- transversely means that the 2×6 matrix

$$\begin{bmatrix} \frac{\partial f_+}{\partial x_1^+} & \frac{\partial f_+}{\partial x_2^+} & \cdots & \frac{\partial f_+}{\partial x_3^-} \\ \frac{\partial f_-}{\partial x_1^+} & \frac{\partial f_-}{\partial x_2^+} & \cdots & \frac{\partial f_-}{\partial x_3^-} \end{bmatrix}$$

3

has rank 2 at the points of $E_+ \cap E_-$. The Jacobian for $\langle y_+^2 - f_+, \, y_-^2 - f_- \rangle$ is the 2×8 matrix

$$\begin{bmatrix} -\frac{\partial f_+}{\partial x_1^+} & -\frac{\partial f_+}{\partial x_2^+} & \cdots & -\frac{\partial f_+}{\partial x_3^-} & 2y_+ & 0 \\ -\frac{\partial f_-}{\partial x_1^+} & -\frac{\partial f_-}{\partial x_2^+} & \cdots & -\frac{\partial f_-}{\partial x_3^-} & 0 & 2y_- \end{bmatrix}$$

so it has rank 2 at the points of $\mathbb{P}W$ outside of $E_+ \cup E_-$ because of the second block matrix. Looking at the first block matrix, we see that it has rank 2 at the points of $E_+ \cup E_-$ precisely when E_+, E_- are smooth and intersect transversely. □

Proposition 2.4. *The $\mathbb{C}_\lambda^*$-action on $\mathrm{Spec}\, A \backslash 0$ has stabilizer groups at a point $(u, y_\pm)$:*

$$\begin{cases} trivial & if \quad y_+ \neq 0 \ or \ y_- \neq 0, \\ \mathbb{Z}_2 & if \quad y_+, \, y_- = 0. \end{cases}$$

Proof. To compute the stabilizer, we require that

$$(\lambda^2 u_1, \lambda^2 u_2, \lambda^2 u_3, \lambda^3 y_+, \lambda^3 y_-) = (u_1, u_2, u_3, y_+, y_-),$$

so λ must be ± 1 by looking at the u_i's. Then $\lambda = -1$ fixes $(u, y_\pm)$ iff $y_+ = y_- = 0$. □

Definition 2.5. Let $\mathscr{Y}$ be the quotient stack $[(\mathrm{Spec}\, A \backslash 0) / \mathbb{C}_\lambda^*]$.

Proposition 2.6. *$\mathscr{Y}$ is a stacky resolution of Y, and Y is its coarse moduli space. $\mathscr{Y}$ has $\mathbb{Z}_2$-stack structure at the 9 intersection points of E_+ and E_-, and ordinary scheme points elsewhere.*

Proof. $\mathrm{Spec}\, A \backslash 0$ is a smooth scheme, so by definition $\mathscr{Y} = [(\mathrm{Spec}\, A \backslash 0)/\mathbb{C}_\lambda^*]$ is a smooth stack, and the second statement is just a restatement of the last proposition. Next, giving a point in $\mathscr{Y}$ over $\mathbb{C}$ is equivalent to giving a map $\mathbb{C}_\lambda^* \to \mathrm{Spec}\, A \backslash 0$ that is $\mathbb{C}_\lambda^*$-equivariant, which is determined by the image of $1 \in \mathbb{C}_\lambda^*$. If the image is not one of the intersection points, then it is of the form $(u_1, u_2, u_3, y_+, y_-)$ where $y_\pm^2 = f_\pm(u)$. So we have a bijection of $\mathbb{C}$-points of $\mathscr{Y}$ and Y outside of $E_+ \cup E_-$, and a map of smooth schemes bijective on points is an isomorphism, showing that $\mathscr{Y} \to Y$ is a birational map. Y being the coarse moduli space of $\mathscr{Y}$ follows from the fact that, for a graded $\mathbb{C}$-algebra R, the GIT quotient $(\mathrm{Spec}\, R \backslash 0)//\mathbb{C}^*$ is the coarse moduli space of $[(\mathrm{Spec}\, R \backslash 0)/\mathbb{C}^*]$, see [11] for example.

□

A common approach to non-commutative algebraic geometry focuses on the abelian categories of coherent sheaves and the corresponding derived categories.

Definition 2.7. (Kuznetsov) Let Y be an algebraic variety and $\mathcal{B}$ be a locally free sheaf of algebras on Y of finite rank as $\mathscr{O}_Y$-module. Then the category of coherent sheaves $\mathrm{Coh} - (Y, \mathcal{B})$ on the noncommutative algebraic variety $(Y, \mathcal{B})$ is defined to be the category of coherent sheaves of right $\mathcal{B}$-modules on Y, and $\mathcal{D}^b(\mathrm{Coh} - Y, \mathcal{B})$ is defined to be the bounded derived category of $\mathrm{Coh} - (Y, \mathcal{B})$.

Of particular interest is the case when $\mathcal{B}$ is a sheaf of Azumaya algebras on Y, which encodes a Brauer class.

There is another definition by Căldăraru based on twisted sheaves, see [2, Definition 1.2.1]. Both

4

definitions naturally extend to the case where Y is an algebraic stack by pulling back to a scheme cover.

Having stated all the relevant definitions, we can state our first main result in more details, which is the analog of [14] for $\mathscr{Y}$ and K3 surface X:

Theorem 2.8. *There exists a Brauer class α of order 2 on $\mathscr{Y}$ such that there is an equivalence:*

$$\mathcal{D}^b(\mathrm{Coh} - \mathscr{Y}, \alpha) \quad \simeq \quad \mathcal{D}^b(\mathrm{Coh} - X).$$

The Brauer class α will be constructed in terms of Clifford algebras in Section 3, and the proof of Theorem 2.8 will be provided there after all relevant definitions are given.

Our next goal is to incorporate the Enriques involution on the side of X, and we will take inspiration from Prof. Borisov's and Zhan Li's result, which suggests the presence of a $(2,2)$-root stack. We can construct another quotient stack with $\mathbb{Z}_2$-stack structure on each of the curves $E_\pm$ except the intersections and $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-stack structure at their intersections. We observe that $-1 \in \mathbb{C}_\lambda^*$ already acts as $(-1, -1)$ on the pair (y_+, y_-), so we define the group G as follows:

Definition 2.9. Let $G := \mathbb{Z}_2 \times \mathbb{C}_\lambda^*$ and its action on A to be:

$$(\pm 1, \lambda) \cdot (u_0, u_1, u_2, y_+, y_-) = (\lambda^2 u_0, \lambda^2 u_1, \lambda^2 u_2, \lambda^3 y_+, \pm 1 \cdot \lambda^3 y_-),$$

and let $\mathscr{Y}/\mathbb{Z}_2$ denote the quotient stack $[(\mathrm{Spec}\, A \backslash 0)/G]$.

We observe that the extra $\mathbb{Z}_2$-action together with $-1 \in \mathbb{C}_\lambda^*$ generate a $\mathbb{Z}_2 \times \mathbb{Z}_2$. Note that the selection of the $\mathbb{Z}_2$-action specifies a special role for y_-.

Proposition 2.10. *The G-action on $\mathrm{Spec}\, A \backslash 0$ has stabilizer group at a point $(u, y_\pm)$:*

$$\begin{cases} trivial & if \quad y_+, \, y_- \neq 0, \\ \mathbb{Z}_2 & if \quad exactly \ one \ of \ y_+, \, y_- = 0, \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & if \quad y_+, \, y_- = 0. \end{cases}$$

Proof. Assume $(u_0, u_1, u_2, y_+, y_-) \neq 0$ is a fixed point of $(s, \lambda) \in G$. Again this forces $\lambda = \pm 1$. We can then list all the cases:

$$\begin{cases} s = 1, \, \lambda = 1 & \text{if } y_+, \, y_- \neq 0, \\ (s, \lambda) = (1, -1) \text{ or } (-1, 1) & \text{if } y_+ = 0, \, y_- \neq 0, \\ s = \pm 1, \, \lambda = 1 & \text{if } y_+ \neq 0, \, y_- = 0, \\ s = \pm 1, \, \lambda = \pm 1 & \text{if } y_+, \, y_- = 0. \end{cases}$$

Hence the stabilizer groups are the subgroups $\{1\} \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \{1\}$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ of G. $\qquad \square$

As we have the map $\mathrm{Spec}\, A \backslash 0 \to W \backslash 0$ and the group actions are compatible, we obtain a map $\mathscr{Y}/\mathbb{Z}_2 \to [(W \backslash 0)/\mathbb{C}_t^*] \to \mathbb{P}W$ of stacks.

Proposition 2.11. *$\mathscr{Y}/\mathbb{Z}_2$ is the $(2,2)$-root stack over $\mathbb{P}W$ branched over (E_+, E_-) in the sense of Cadman [5]. In particular, $\mathscr{Y}/\mathbb{Z}_2 \to \mathbb{P}W$ is a birational map.*

5

Proof. We will use Cadman's definition in [5], in which the root stack is denoted by $\mathbb{P}W_{(E_+,E_-),(2,2)}$ in Cadman's notation. By definition the root stack $\mathscr{R} := \mathbb{P}W_{(E_+,E_-),(2,2)}$ is the fiber product

$$
\begin{array}{ccc}
 & \mathscr{R} & \\
 \swarrow & & \searrow \\
\mathbb{P}W = [(W\backslash 0)/\mathbb{C}_t^*] & & [\mathbb{C}^2/(\mathbb{C}_\lambda^*)^2] \\
 \searrow & & \swarrow \\
 & [\mathbb{C}^2/(\mathbb{C}^*)^2] &
\end{array}
$$

so it can be identified with $[\,(W\backslash 0)\times_{\mathbb{C}^2}\mathbb{C}^2\,/\,\mathbb{C}_t^*\times_{(\mathbb{C}^*)^2}(\mathbb{C}_\lambda^*)^2\,]$, where the fiber product of the groups is

$$
\mathbb{C}_t^*\times_{(\mathbb{C}^*)^2}(\mathbb{C}_\lambda^*)^2 = \{(t,\lambda_1,\lambda_2): t^3=\lambda_1^2=\lambda_2^2\} \subseteq \mathbb{C}_t^*\times\mathbb{C}_\lambda^*\times\mathbb{C}_\lambda^*
$$

via the maps

$$
\begin{array}{ccc}
t\in\mathbb{C}_t^* & & (\lambda_1,\lambda_2)\in(\mathbb{C}_\lambda^*)^2 \\
 \searrow & & \swarrow \\
 & (t^3,t^3),\ (\lambda_1^2,\lambda_2^2)\in(\mathbb{C}^*)^2 &
\end{array}
$$

and can be identified with

$$
\left\{\left(\frac{\lambda_1}{\lambda_2},\frac{\lambda_1}{t}\right)\right\} = \mathbb{Z}_2\times\mathbb{C}^* \simeq G.
$$

The maps of the underlying schemes are

$$
\begin{array}{ccc}
u\in W\backslash 0 & & (a_1,a_2)\in\mathbb{C}^2 \\
 \searrow & & \swarrow \\
 & (f_+(u),f_-(u)),\ (a_1^2,a_2^2)\in\mathbb{C}^2 &
\end{array}
$$

so it is clear that their fiber product is $(W\backslash 0)\times_{\mathbb{C}^2}\mathbb{C}^2 = \operatorname{Spec} A$. $\qquad\square$
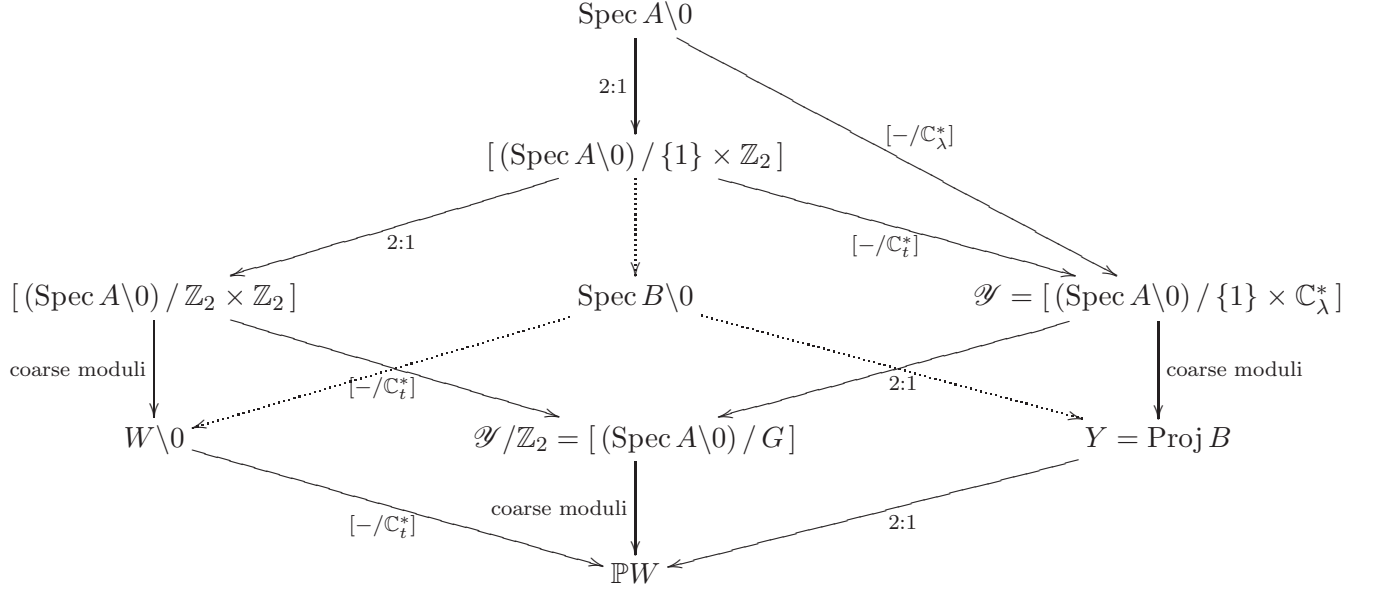
Our second main result is:

Theorem 2.12. *There exists a Brauer class $\widetilde{\alpha}$ of order 2 on $\mathscr{Y}/\mathbb{Z}_2$ such that there is an equivalence:*

$$
\mathcal{D}^b(\mathrm{Coh}-(\mathscr{Y}/\mathbb{Z}_2),\widetilde{\alpha}) \quad\simeq\quad \mathcal{D}^b(\mathrm{Coh}-(X/\sigma)).
$$

Again, elaborations and proof of Theorem 2.12 will be provided in Section 3.

Remark 2.13. The relations of the schemes and stacks above (and some addditional ones) can be summarized in the diagram:

3 Clifford Algebras

In this section, we will define the Brauer classes α and $\widetilde{\alpha}$ appearing in Theorem 2.8 and 2.12 in terms of Clifford algebras and provide the proofs to the theorems. For the sake of readability, we shall first prove the equivalence of categories, namely that

$$\mathcal{D}^b(\mathrm{Coh}-\mathscr{Y}, \alpha) \quad \simeq \quad \mathcal{D}^b(\mathrm{Coh}-X),$$

and

$$\mathcal{D}^b(\mathrm{Coh}-(\mathscr{Y}/\mathbb{Z}_2), \widetilde{\alpha}) \quad \simeq \quad \mathcal{D}^b(\mathrm{Coh}-(X/\sigma))$$

in Propositions 3.9 and 3.10. After that, we shall tackle the implicit statements that the α and $\widetilde{\alpha}$ defined are indeed sheaves of Azumaya algebras over $\mathscr{Y}$ and $\mathscr{Y}/\mathbb{Z}_2$ respectively, thus they represent Brauer classes on the corresponding spaces.

We start by defining several Clifford algebras of interest. In Kuznetsov's main theorem, the Brauer class is defined by means of a sheaf of even parts of Clifford algebras. In the presence of the involution σ in our setup, it is more natural to work with a variant of the full Clifford algebra, rather than its even part. The full Clifford algebra will be denoted by

$$Cl := Cl(V) = \mathbb{C}[u]\{v_1^+, v_2^+, v_3^+, v_1^-, v_2^-, v_3^-\} \,/\, \langle v_i^\pm v_j^\pm + v_j^\pm v_i^\pm + 2q_u(v_i^\pm, v_j^\pm),\, v_i^+ v_j^- + v_j^- v_i^+ \rangle.$$

Its odd and even parts will be denoted by Cl_{odd} and Cl_{ev} respectively, as before. The two smaller Clifford algebras on V_+ and V_- over $\mathbb{C}[u]$ will be denoted by

$$Cl_+ := Cl(V_+) = \mathbb{C}[u]\{v_1^+, v_2^+, v_3^+\} \,/\, \langle v_i^+ v_j^+ + v_j^+ v_i^+ + 2q_u(v_i^+, v_j^+) \rangle,$$

$$Cl_- := Cl(V_-) = \mathbb{C}[u]\{v_1^-, v_2^-, v_3^-\} \,/\, \langle v_i^- v_j^- + v_j^- v_i^- + 2q_u(v_i^-, v_j^-) \rangle.$$

Note that Cl is the super tensor product of Cl_+ and Cl_- over $\mathbb{C}[u]$. For reasons listed in the remark below, we will mainly work with the ordinary tensor product

$$\begin{aligned}
\widetilde{Cl} &:= Cl_+ \otimes_{\mathbb{C}[u]} Cl_- \\
&= \mathbb{C}[u]\{v_1^+, v_2^+, v_3^+, v_1^-, v_2^-, v_3^-\} \,/\, \langle v_i^\pm v_j^\pm + v_j^\pm v_i^\pm + 2q_u(v_i^\pm, v_j^\pm),\ v_i^+ v_j^- - v_j^- v_i^+ \rangle.
\end{aligned}$$

In terms of relations, the only difference between Cl and $\widetilde{Cl}$ is the anti-commutativity or commutativity of the variables v_i^+ and v_j^-.

Remark 3.1. Some reasons for working with the ordinary tensor product $\widetilde{Cl}$ are listed here, and they will be elaborated by the upcoming propositions:

(1) the original Clifford algebra Cl doesn't contain A in the center, while $\widetilde{Cl}$ does,

(2) there is a naturally defined grading on $\widetilde{Cl}$ that is compatible with the $\mathbb{C}_\lambda^*$-grading on A, which is essential in the proof of the main theorem,

(3) the ordinary tensor product $\widetilde{Cl}$ is more compatible with $A = A_+ \otimes A_-$ as shown in Proposition 3.13, and is more readily understood than a Clifford algebra of corank 2,

(4) consideration of twisted sheaves in Section 4 suggests tensor products of Brauer classes.

Remark 3.2. Ordinary tensor products of Clifford algebras were named quasi Clifford algebras or extended Clifford algebras and studied in [10, 17].

On top of the Clifford algebra structure, we will define a new $\mathbb{N}$-grading to handle the grading change between the $\mathbb{C}_\lambda^*$-action and the $\mathbb{C}_t^*$-actions. To handle the case of Enriques surfaces (Theorem 2.12), we also need to introduce another $\mathbb{Z}_2$-grading corresponding to the distinction of v_i^+ and v_i^- variables. Combined:

Proposition 3.3. *The algebras Cl, $Cl_\pm$ and $\widetilde{Cl}$ can be given a $(\mathbb{Z}_2 \times \mathbb{N})$-grading corresponding to the action of $G = \mathbb{Z}_2 \times \mathbb{C}_\lambda^*$ as follows:*

$$\begin{cases}
u_1, u_2, u_3: & \text{degree } (0, 2), \\
v_1^+, v_2^+, v_3^+: & \text{degree } (0, 1), \\
v_1^-, v_2^-, v_3^-: & \text{degree } (1, 1).
\end{cases}$$

(Here the grading group $\mathbb{Z}_2 = \{0, 1\}$ is additive.) Notice that the $\mathbb{N}$-grading extends the ordinary Clifford grading which counts the parity of the Clifford variables, in the sense that Cl_{ev} consists of elements of even degree in the $\mathbb{N}$-grading, but $\mathbb{C}[u]$ is not contained in the degree 0 part.

Proof. With the relations of the algebras listed above, one can verify that they are homogeneous w.r.t to the assigned $(\mathbb{Z}_2 \times \mathbb{N})$-degrees, keeping in mind that in the relations $q_u(v_i^\pm, v_j^\pm) = \sum_k u_k q_k(v_i^\pm, v_j^\pm)$ and $q_k(v_i^\pm, v_j^\pm) \in \mathbb{C}$. $\qquad\square$

Definition 3.4. For an $\mathbb{N}$-graded ring R, define another $\mathbb{N}$-graded ring $R^{(d)} := \bigoplus_{i \geq 0} R_{d \cdot i}$, and its i-th graded piece to be $R_{d \cdot i}$.

With this notation, the even part Cl_{ev} of the Clifford algebra Cl can be denoted by $Cl^{(2)}$ and so on. Next, we observe that, as $\mathbb{C}$-vector spaces,

$$\widetilde{Cl}_{ev} = (Cl_+ \otimes Cl_-)_{ev} = (Cl_{+ev} \otimes Cl_{-ev}) \oplus (Cl_{+odd} \otimes Cl_{-odd}) = Cl_{ev}.$$

In fact:

Proposition 3.5. $Cl^{(2)}(= Cl_{ev})$ and $\widetilde{Cl}^{(2)}$ are isomorphic as $\mathbb{C}[u]$-algebras.

Proof. We recall that the $Cl^{(2)}$ and $\widetilde{Cl}^{(2)}$ are defined using the $\mathbb{N}$-grading in Proposition 3.3. We will establish an explicit isomorphism between these two algebras. The generators of $Cl^{(2)}$ over $\mathbb{C}[u]$ are $v_i^+ v_j^+$, $v_i^- v_j^-$ and $v_i^+ v_j^-$. A product of them can be rearranged into the "standard order", i.e. so that the v_i^+'s are on the left

$$\prod_{i=1}^m v_i^+ \prod_{j=1}^n v_j^-,$$

where $m + n$ is even. We construct an isomorphism $\varphi : Cl_{ev} \to (Cl_+ \otimes Cl_-)_{ev}$ by sending the elements

$$\varphi: \quad \prod_{i=1}^m v_i^+ \prod_{j=1}^n v_j^- \quad \mapsto \quad (-1)^{\frac{m(m-1)}{2}} \prod_{i=1}^m (v_i^+ \otimes 1) \prod_{j=1}^n (1 \otimes v_j^-).$$

To verify that it is an algebra homomorphism, we need to show that the image of a product before rearranging

$$\left(\prod_{i=1}^m v_i^+ \prod_{j=1}^n v_j^- \right) \left(\prod_{i=m+1}^{m+k} v_i^+ \prod_{j=n+1}^{n+\ell} v_j^- \right) \quad \mapsto \quad (-1)^{\frac{m(m-1)+k(k-1)}{2}} \prod_{i=1}^{m+k} (v_i^+ \otimes 1) \prod_{j=1}^{n+\ell} (1 \otimes v_j^-),$$

is the same as the image we get if we arrange the product into the standard order beforehand,

$$(-1)^{nk} \prod_{i=1}^{m+k} v_i^+ \prod_{j=1}^{n+\ell} v_j^- \quad \mapsto \quad (-1)^{nk + \frac{(m+k)(m+k-1)}{2}} \prod_{i=1}^{m+k} (v_i^+ \otimes 1) \prod_{j=1}^{n+\ell} (1 \otimes v_j^-).$$

We see that the only difference is the exponent of the scalar (-1), so we check that

$$\left[nk + \frac{(m+k)(m+k-1)}{2} \right] - \left[\frac{m(m-1)+k(k-1)}{2} \right] \equiv mk + \frac{2mk}{2} \equiv 0 \mod 2,$$

which shows that the 2 exponents have the same parity. $\qquad\square$

Remark 3.6. Note that the isomorphism works in the more general context of $\mathbb{Z}/2$-graded algebras. See [9] and [4, Chapter III, Section 4.6-4.7] for similar constructions.

The existence of an $\mathbb{N}$-grading on $\widetilde{Cl}$ allows us to utilize the following "Veronese" theorem from [19]:

Lemma 3.7. *Let R be a $\mathbb{N}$-graded left Noetherian ring which is generated by R_1 as a R_0-algebra. Then the category of coherent (or quasicoherent) sheaves, obtained by localizing the categories of left $R^{(d)}$-modules and R-modules, are equivalent for any natural number d.*

An immediate consequence of Lemma 3.7 is that it connects our result to Kuznetsov's main theorem, which is included here for readers' benefit.

Proposition 3.8. *(Kuznetsov) Let $V = \mathbb{C}^6$ with coordinates $x_1, \cdots, x_6$ and $W \subseteq S^2V^*$ with coordinates u_1, u_2, u_3 a dimension 3 subspace of rank ≥ 4 (except the origin) homogeneous quadratic forms on V such that the intersection X of quadrics parametrized by W is a complete intersection. Let $q \in \mathbb{C}[u_i, x_j]$ be the quadratic form corresponding to the choice of W. Let Y be the double cover of $\mathbb{P}W$ ramified over the sextic curve of degenerate quadrics, and $Cl(V)$ be the sheaf of Clifford algebras on Y associated to q. Then we have derived equivalences:*

$$\mathcal{D}^b(\mathrm{Coh} - X) \quad \simeq \quad \mathcal{D}^b(\mathrm{Coh} - \mathbb{P}W, Cl(V)_{ev}) \quad \simeq \quad \mathcal{D}^b(\mathrm{Coh} - Y, Cl(V)_{ev}).$$

Here $Cl(V)_{ev}$ is equipped with the $\mathbb{C}^*_t$-grading and $Cl(V)$ with the $\mathbb{C}^*_\lambda$-grading.

Now we have all the tools to carry out the main step of the proofs of Theorems 2.8 and 2.12:

Proposition 3.9. *There exists an equivalence of the categories (provided that they are well-defined):*

$$\mathcal{D}^b(\mathrm{Coh} - \mathscr{Y}, \alpha) \quad \simeq \quad \mathcal{D}^b(\mathrm{Coh} - X).$$

Proof. Let α be the sheaf of algebras associated to $\widetilde{Cl}$ on $\mathscr{Y}$ with the $\mathbb{N}$-grading corresponding to the $\mathbb{C}^*_\lambda$-action. Now Lemma 3.7 tells us that the categories

$$\mathrm{Cohproj}(\widetilde{Cl}_{ev}) \quad \text{and} \quad \mathrm{Cohproj}(\widetilde{Cl})$$

are equivalent, which translates to the equivalence of $\mathrm{Coh}(\mathrm{Proj}\,\widetilde{Cl}_{ev})$ and $\mathrm{Coh}(\mathrm{Proj}\,\widetilde{Cl})$ when both Proj's are considered as non-commutative varieties. This allows us to apply Kuznetsov's theorem (Proposition 3.8) to conclude that

$$D^b(\mathrm{Coh} - X) \quad \simeq \quad D^b(\mathrm{Coh} - \mathbb{P}W, Cl_{ev}) \quad \simeq \quad D^b(\mathrm{Coh} - \mathbb{P}W, \widetilde{Cl}_{ev}) \quad \simeq \quad D^b(\mathrm{Coh} - \mathscr{Y}, \widetilde{Cl}).$$

$\square$

Proposition 3.10. *There exists an equivalence of the categories (provided that they are well-defined):*

$$\mathcal{D}^b(\mathrm{Coh} - (\mathscr{Y}/\mathbb{Z}_2), \widetilde{\alpha}) \quad \simeq \quad \mathcal{D}^b(\mathrm{Coh} - (X/\sigma)).$$

Proof. Let $\widetilde{\alpha}$ be the sheaf of algebras associated to $\widetilde{Cl}$ on $\mathscr{Y}/\mathbb{Z}_2$ with the $(\mathbb{Z}_2 \times \mathbb{N})$-grading corresponding to the G-action. Considering the induced $\mathbb{N}$-grading on $\widetilde{Cl}$, we can again apply Lemma 3.7 to conclude that the equivalence of categories $\mathrm{Coh}(\mathrm{Proj}\,\widetilde{Cl}_{ev})$ and $\mathrm{Coh}(\mathrm{Proj}\,\widetilde{Cl})$ when both Proj's are considered as non-commutative varieties. We can then use [3, Theorem 6.2] by Borisov and Li to get

$$D^b(\mathrm{Coh} - (X/\sigma)) \simeq D^b(\mathrm{Coh} - [\mathbb{P}W/\mathbb{Z}_2], Cl_{ev}) \simeq D^b(\mathrm{Coh} - [\mathbb{P}W/\mathbb{Z}_2], \widetilde{Cl}_{ev}) \simeq D^b(\mathrm{Coh} - (\mathscr{Y}/\mathbb{Z}_2), \widetilde{Cl}).$$

$\square$

Remark 3.11. In [3, Section 9.2], a semidirect product of a Clifford algebra and $\mathbb{C}[h]/\langle h^2 - 1 \rangle$ is considered. However, the extra element h should be thought of as an extra $\mathbb{Z}_2$-grading on the sheaf of algebras, since adjoining the element h would change the rank of the algebras and they will not be matrix algebras anymore.

Now we will show that $\widetilde{Cl}$ represents a Brauer class on $\mathscr{Y}$ and $\mathscr{Y}/\mathbb{Z}_2$. The $\mathbb{C}[u]$-algebra $\widetilde{Cl}$ can be made into an algebra over A by $y_+ \mapsto d_+ \otimes 1$ and $y_- \mapsto 1 \otimes d_-$, where $d_\pm$ is the central element in $Cl_\pm$, which can be computed explicitly:

Proposition 3.12. *The center of Cl_+ is isomorphic to $\mathbb{C}[u][d_+]/\langle d_+^2 - f_+ \rangle$ where*

$$d_+ = v_1^+ v_2^+ v_3^+ - q_{32} v_1^+ + q_{31} v_2^+ - q_{21} v_3^+.$$

Similar statement holds for Cl_-.

Proof. Since we know the central element is the product $v_1 v_2 v_3 \ldots v_n$ when q is in the standard form (dropping the $+$ signs), in any coordinates the central element in Cl_+ is odd degree of the form

$$d_+ = v_1 v_2 v_3 + r_1 v_1 + r_2 v_2 + r_3 v_3$$

where $r_i \in \mathbb{C}[u]$. Then we compute

$$v_1 v_2 v_3 v_1 = v_1 v_2 (2q_{31} - v_1 v_3) = 2v_1 v_2 q_{31} - v_1 (2q_{21} - v_1 v_2) v_3 = 2v_1 v_2 q_{31} - 2v_1 q_{21} v_3 + q_{11} v_2 v_3,$$

$$d_+ v_1 = v_1 v_2 v_3 v_1 + r_1 q_{11} + r_2 v_2 v_1 + r_3 v_3 v_1 = (2v_1 v_2 q_{31} - 2v_1 q_{21} v_3 + q_{11} v_2 v_3) + r_1 q_{11} + 2r_2 q_{21} - r_2 v_1 v_2 + 2r_3 q_{31} - r_3 v_1 v_3,$$

$$v_1 d_+ = q_{11} v_2 v_3 + r_1 q_{11} + r_2 v_1 v_2 + r_3 v_1 v_3,$$

$$[d_+, v_1] = d_+ v_1 - v_1 d_+ = 2v_1 v_2 q_{31} - 2v_1 q_{21} v_3 + 2r_2 q_{21} - 2r_2 v_1 v_2 + 2r_3 q_{31} - 2r_3 v_1 v_3.$$

Setting this to 0, we get $r_2 = q_{31}$ and $r_3 = -q_{21}$. Consideration of the commutators $[d_+, v_2]$ and $[d_+, v_3]$ gives $r_1 = -q_{32}$ and consistent values of r_1, r_2, r_3. Furthermore, it can be verified that

$$\begin{aligned}
d_+^2 &= v_1 v_2 v_3 v_1 v_2 v_3 - q_{32} v_1 v_2 v_3 v_1 + q_{31} v_1 v_2 v_3 v_2 - q_{21} v_1 v_2 q_{33} \\
&\quad - q_{32} q_{11} v_2 v_3 + q_{32}^2 q_{11} - q_{32} q_{31} v_1 v_2 + q_{32} q_{21} v_1 v_3 \\
&\quad + q_{31} v_2 v_1 v_2 v_3 - q_{31} q_{32} v_2 v_1 + q_{31}^2 q_{22} - q_{31} q_{21} v_2 v_3 \\
&\quad - q_{21} v_3 v_1 v_2 v_3 - q_{21} q_{32} v_3 v_1 - q_{21} q_{31} v_3 v_2 + q_{21}^2 q_{33} \\
&= q_{11} q_{22} q_{33} + q_{12} q_{23} q_{31} + q_{13} q_{21} q_{32} - q_{11} q_{23} q_{32} - q_{12} q_{21} q_{33} - q_{13} q_{22} q_{31} \\
&= det(q^+) \\
&= f_+(u).
\end{aligned}$$

Now we check that the center contains nothing in the even part. First we assume there is a central element $z \in (Cl_+)_{ev}$ of the form $z = r_1 v_2 v_3 + r_2 v_3 v_1 + r_3 v_1 v_2$ (which is symmetric in v_i's). Setting $[z, v_1] = 0$ gives

$$\begin{cases} -r_1 q_{12} + r_2 q_{11} = 0 \\ r_1 q_{13} - r_3 q_{11} = 0 \\ r_2 q_{13} + a_3 q_{12} = 0 \end{cases}$$

We can simplify the equations by localizing at u_i's and use the fact that the centers satisfy

$$Z(Cl_+) = Z(Cl_+ \otimes_{\mathbb{C}[u]} \mathbb{C}(u_1, u_2, u_3)) \cap \mathbb{C}[u].$$

Over $\mathbb{C}(u) := \mathbb{C}(u_1, u_2, u_3)$, we can diagonalize the quadratic form q^+ and assume that

$$[q_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \det(q^+) \end{bmatrix},$$

so the equations above reduces to $r_2 = r_3 = 0$. By symmetry in v_i's, we also get that $r_1 = 0$.

Finally, a central element of the form $z = r_0 + r_1 v_2 v_3 + r_2 v_3 v_1 + r_3 v_1 v_2 \in (Cl_+)_{ev}$ cannot exist because $r_0 \in \mathbb{C}[u]$ is central and that would imply $z - r_0$ is central as well, which contradicts the conclusion above. □

Note that $d_+ \otimes 1$ and $1 \otimes d_-$ are themselves central elements in $\widetilde{Cl}$, so A is contained in the center of $\widetilde{Cl}$. We can think of $\operatorname{Spec} A \backslash 0$ as the base (commutative) space, and with the Clifford algebra it turns to a non-commutative space. When restricted to A, the elements $y_\pm$ $(= d_+ \otimes 1$ and $1 \otimes d_-)$ have degree 3, and the $\mathbb{N}$-grading on $\widetilde{Cl}$ corresponds to the $\mathbb{C}_\lambda^*$-action on $\operatorname{Spec} A$.

Proposition 3.13. *The center of $\widetilde{Cl}$ is A.*

Proof. Define

$$A_+ := \mathbb{C}[u_1, u_2, u_3, y_+]/\langle y_+^2 - f_+ \rangle,$$
$$A_- := \mathbb{C}[u_1, u_2, u_3, y_-]/\langle y_-^2 - f_- \rangle,$$

then $A = A_+ \otimes_{\mathbb{C}[u]} A_-$ and $A_\pm = Z(Cl_\pm)$ by Proposition 3.12. For algebras over a field, the center of the tensor product is the tensor product of the centers, so we can localize at u_i's and pass to that case:

$$
\begin{aligned}
Z(\widetilde{Cl}) &= Z(\widetilde{Cl} \otimes_{\mathbb{C}[u]} \mathbb{C}(u)) \cap \mathbb{C}[u] \\
&= [Z(Cl_+ \otimes_{\mathbb{C}[u]} \mathbb{C}(u)) \otimes_{\mathbb{C}(u)} Z(Cl_- \otimes_{\mathbb{C}[u]} \mathbb{C}(u))] \cap \mathbb{C}[u] \\
&= Z(Cl_+) \otimes_{\mathbb{C}[u]} Z(Cl_-) \\
&= A_+ \otimes_{\mathbb{C}[u]} A_- \\
&= A.
\end{aligned}
$$

□

This proposition shows that $\operatorname{Spec} A$ is the natural base commutative variety to use when we consider $\widetilde{Cl}$, among all other possibilities.

Remark 3.14. Now we see that $y_\pm \in A$ should naturally correspond to $d_\pm \in Cl_\pm$. We cannot embed A in the full Clifford algebra Cl because y_+ and y_- commute in A, while d_+ and d_- do not commute in Cl. This is why we must use the ordinary tensor product $\widetilde{Cl}$ of Cl_+ and Cl_-.

One definition of a Brauer class is as a sheaf of Azumaya algebras modulo Morita equivalence. Hence we want to show that $\widetilde{Cl}$ gives rise to a sheaf of Azumaya algebras on $\mathscr{Y}$ and $\mathscr{Y}/\mathbb{Z}_2$. The definition of Azumaya algebra over a stack is a natural extension from that of a scheme, which we state here.

Definition 3.15. An Azumaya algebra $\mathcal{A}$ over a stack $\mathscr{X}$ is an $\mathscr{O}_{\mathscr{X}}$-algebra (of finite presentation as an $\mathscr{O}_{\mathscr{X}}$-module) that is étale locally isomorphic to a matrix algebra sheaf.

Being an Azumaya algebra is a local property, so to check that for an algebraic stack, one just needs to pull it back to a scheme covering and check it there.

To reduce the number of symbols, we shall use Cl, $\widetilde{Cl}$, etc. to denote both the algebras themselves and the sheaves associated to them on the suitable space.

Proposition 3.16. $\widetilde{Cl}$ *is an Azumaya algebra over A.*

Proof. First, we look at the diagram of algebra inclusions, in which each small diamond is a tensor product:
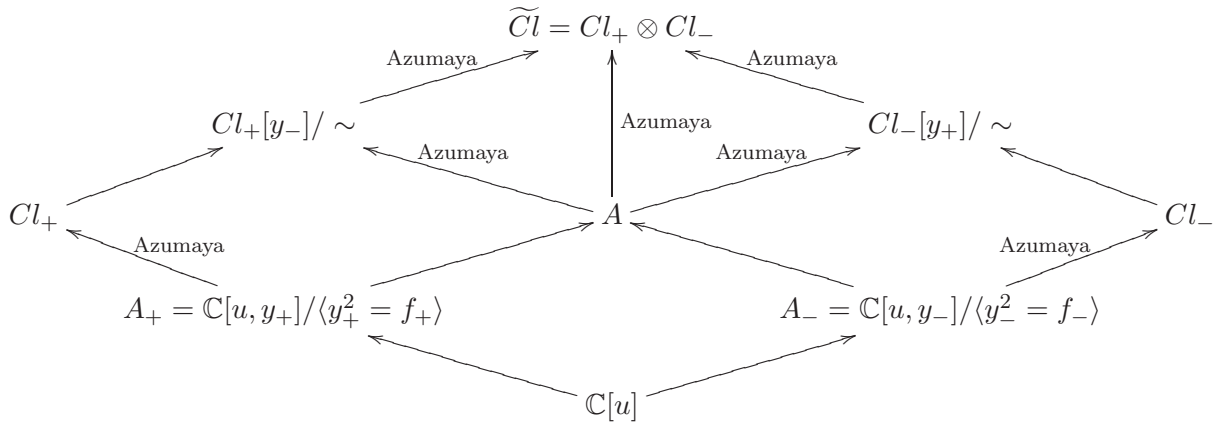
We will prove that the maps labelled by "Azumaya" give Azumaya algebras below. We use the fact that:

(i) the pullback of an Azumaya algebra is Azumaya, and

(ii) the tensor product of two Azumaya algebras is also Azumaya.

So in order to show that $A \to \widetilde{Cl}$ is Azumaya, we just need to show it for the maps $A_+ \to Cl_+$ and $A_- \to Cl_-$ at the lower left and right. It will be proved in a similar fashion as proposition 3.13 in Kuznetsov's paper [14].

As being an Azumaya algebra is a local property, it suffices to check that the fiber of Cl_+ at each point of $\operatorname{Spec} A_+ \backslash 0$ is a matrix algebra. On the algebra level, y_+ is sent to d_+. For a point $(u, y_+) \in \operatorname{Spec} A_+ \backslash 0$ such that $y_+ \neq 0$, taking the fiber of Cl_+ means fixing u and y_+. For fixed u, the quadratic form q_u^+ is of full rank and it is well-known that fiber $Cl(q_u^+)$ is a product of two rank 2 matrix algebras, which comes from its two irreducible Clifford modules classified by the action of d_+, so fixing $y_+ = d_+$ means choosing one of the two components. For points of the form $(u, 0)$, the quadratic form q_u^+ is of corank 1 and the maximal ideal at $(u, 0)$ is generated by y_+, so the fiber of Cl_+ is $Cl(q_u^+)/\langle d_+ \rangle$ which is isomorphic to a matrix algebra of rank 2. $\square$

Proposition 3.17. *The sheaf associated to $\widetilde{Cl}$ on $\operatorname{Spec} A \backslash 0$ can be given a $\mathbb{C}_\lambda^*$-equivariant structure and a G-equivariant structure, so there are sheaves associated to $\widetilde{Cl}$ on $\mathscr{Y}$ and $\mathscr{Y}/\mathbb{Z}_2$. The two sheaves on $\mathscr{Y}$ and $\mathscr{Y}/\mathbb{Z}_2$ are also Azumaya algebras.*

13

Proof. Note that $\mathscr{Y} = [(\operatorname{Spec} A \backslash 0) / \mathbb{C}_\lambda^*]$ and $\mathscr{Y} / \mathbb{Z}_2 = [(\operatorname{Spec} A \backslash 0) / G]$, and for $\lambda \in \mathbb{C}_\lambda^*$ and $(\pm 1, \lambda) \in G = \mathbb{Z}_2 \times \mathbb{C}_\lambda^*$ we can define the actions

$$\lambda \cdot v_i^\pm = \lambda v_i^\pm \quad \text{and} \quad (-1, \lambda) \cdot v_i^\pm = \pm \lambda v_i^\pm,$$

which are compatible with the actions on A and give the associated sheaf the equivariant structures. By definition, $\widetilde{Cl}$ gives rise to sheaves on $\mathscr{Y}$ and $\mathscr{Y} / \mathbb{Z}_2$, and they are Azumaya algebras because the condition can be pulled back to $\operatorname{Spec} A$ and we know $\widetilde{Cl}$ is Azumaya over $\operatorname{Spec} A$ by Proposition 3.16. $\qquad\square$

The proofs to Theorem 2.8 and 2.12 are completed when we combine Proposition 3.9, 3.10 and 3.17.

4 Brauer-Severi Varieties

There are several formuations of the Brauer group and its elements Brauer classes on a scheme. One way is to define them as Azumaya algebras under Morita equivalence, which we had done in the last section; another way is to represent them by twisted sheaves [6], whose projectivizations are projective bundles that are locally trivial in the étale topology, namely the Brauer-Severi varieties. We will now construct a projective bundle on $\mathscr{Y}$ and $\mathscr{Y} / \mathbb{Z}_2$ and show that it represents the Azumaya algebra $\widetilde{Cl}$. This gives us the connection between the Clifford algebra and the base space. (In contrast, the derived equivalence itself has little relevance to the base space.)

To construct the Brauer class on $\operatorname{Spec} A \backslash 0$ as a projective bundle involves several steps:

(1) Construct two $\mathbb{P}^1$-bundles on $\operatorname{Spec} A_\pm \backslash 0$ from the data of $q^\pm$.

(2) Construct a $\mathbb{P}^3$-bundle on $\operatorname{Spec} A \backslash 0$ from the two $\mathbb{P}^1$-bundles.

(3) Exhibit an isomorphism from the projectivization of the Clifford module to the $\mathbb{P}^1$-bundle.

In the second half of this section, we will also relate the $\mathbb{P}^3$-bundle in our case to the $\mathbb{P}^3$-bundle that comes from Kuznetsov's construction which considers the quadratic form q as a whole.
Recall from section 2 that W parametrizes the quadratic forms q and each q can be decomposed into $q = q^+ + q^-$. Consider the conic fibration C over $W \backslash 0$ whose fiber over a point $u \in W \backslash 0$ is the conic $\{q_u^+ = 0\} \subseteq \mathbb{P} V_+$. Because we restrict to a general family of such quadratic forms q, we can assume C is smooth by Bertini's Theorem, because C is a hyperplane of degree $(1, 2)$ in $(W \backslash 0) \times \mathbb{P} V_+$. These are nonsingular conics for $u \in W \backslash \operatorname{cone}(E_+)$ so we have a generic $\mathbb{P}^1$-fibration, while the fibers over $u \in \operatorname{cone}(E_+)$ are pairs of lines. We will present 2 ways to construct a $\mathbb{P}^1$-bundle from it.

First method is the following: take the conic fibration, denoted $\operatorname{adj}(C)$, over $W \backslash 0$ whose fiber over a point $u \in W \backslash 0$ is the conic $\{\operatorname{adj}(q_u^+) = 0\} \subseteq \mathbb{P} V_+^\vee$ in the dual projective space, where $\operatorname{adj}(q_u^+)$ denotes the quadratic form whose associated matrix is the adjugate matrix of q_u^+. The general fibers are then dual conics and the fibers over $u \in E_+$ become double lines. Over $W \backslash \operatorname{cone}(E_+)$, we have a birational map $C \to \operatorname{adj}(C)$ arising from the dual curve map. For convenience, denote the pullback of $\operatorname{adj}(C)$ to the ramified double cover $\operatorname{Spec} A_+ \backslash 0$ by

$$\operatorname{adj}(C_+) := \operatorname{adj}(C) \times_{(W \backslash 0)} (\operatorname{Spec} A_+ \backslash 0).$$

Take the normalization P_+ of $\mathrm{adj}(C_+)$, then the composition $P_+ \to \operatorname{Spec} A_+ \backslash 0$ will be a $\mathbb{P}^1$-bundle.

Proposition 4.1. P_+ *is a* $\mathbb{P}^1$-*bundle over* $\operatorname{Spec} A_+ \backslash 0$.

Proof. To see this, restrict to a Zariski neighborhood U of a point u_0 lying over $\mathrm{cone}(E_+)$ in $\operatorname{Spec} A_+ \backslash 0$. The quadratic forms can be simultaneously diagonalized to have the form (abusing notations):

$$
q_u^+ = \begin{bmatrix} a(u) & 0 & 0 \\ 0 & b(u) & 0 \\ 0 & 0 & \epsilon(u) \end{bmatrix}
$$

where $\epsilon(u_0) = 0$ and $a(u), b(u) \neq 0$ for $u \in U$, i.e. a and b are units in $\mathbb{C}[U]$. Its adjugate matrix is

$$
\mathrm{adj}(q_u^+) = \begin{bmatrix} b(u)\epsilon(u) & 0 & 0 \\ 0 & a(u)\epsilon(u) & 0 \\ 0 & 0 & a(u)b(u) \end{bmatrix}
$$

so at u_0 it is

$$
\mathrm{adj}(q_{u_0}^+) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a(u_0)b(u_0) \end{bmatrix}
$$

and in the coordinates x_1^+, x_2^+, x_3^+ of V_+ the fiber is the double line $(x_3^+)^2 = 0$.

Now we verify that the normalization is smooth. By picking complementary coordinates u_1, u_2 to the local coordinate ϵ, we identify the coordinate ring $\mathbb{C}[U]$ with $\mathbb{C}[u_1, u_2, \epsilon]$. The coordinate ring of $\mathrm{adj}(C)$ over U is thus

$$
\mathbb{C}[U][y_+, x_1^+, x_2^+, x_3^+]/\langle y_+^2 - ab\epsilon, \, b\epsilon(x_1^+)^2 + a\epsilon(x_2^+)^2 + ab(x_3^+)^3 \rangle.
$$

In which we see that

$$
\left(\frac{x_3^+}{y_+}\right)^2 = \frac{(x_3^+)^2}{(ab\epsilon)^2} = -\frac{(x_1^+)^2}{a^2 b} - \frac{(x_2^+)^2}{ab^2}
$$

belongs to the coordinate ring but x_3^+/y_+ does not. We adjoin $\widetilde{x_3}^+ = x_3^+/y_+$ and show that the resultant ring

$$
\mathbb{C}[U][y_+, x_1^+, x_2^+, \widetilde{x_3}^+] \Big/ \left\langle y_+^2 - ab\epsilon, \, \frac{(x_1^+)^2}{a^2 b} + \frac{(x_2^+)^2}{ab^2} + (\widetilde{x_3}^+)^2 \right\rangle
$$

is regular. Taking partial derivatives of the relations w.r.t. u_1, u_2, ϵ, y_+, x_1^+, x_2^+, $\widetilde{x_3}^+$ to zero we get

$$
\begin{bmatrix} -\epsilon\frac{\partial(ab)}{\partial u_1} & -\epsilon\frac{\partial(ab)}{\partial u_2} & -ab - \epsilon\frac{\partial(ab)}{\partial \epsilon} & 2y_+ & 0 & 0 & 0 \\ * & * & * & 0 & \frac{2x_1^+}{a^2 b} & \frac{2x_2^+}{ab^2} & 2\widetilde{x_3}^+ \end{bmatrix}
$$

Since $x_1^+, x_2^+, \widetilde{x_3}^+$ are coordinates on $\mathbb{P}V_+$ and cannot be all zero, the matrix has rank 1 precisely when the first row is a multiple of the second row. This forces it to be the zero row, so $y_+ = 0$ and thus $\epsilon = 0$ by the relation, but then $ab \neq 0$ in the third entry. So there is no point at which the matrix has rank 1.

Since all the fibers of P_+ are smooth conics given by $\frac{(x_1^+)^2}{a^2 b} + \frac{(x_2^+)^2}{ab^2} + (\widetilde{x_3}^+)^2 = 0$, we obtain a $\mathbb{P}^1$-bundle. $\qquad\square$

In the second method, we start from the pullback

$$C_+ := C \times_{(W \backslash 0)} (\operatorname{Spec} A_+ \backslash 0)$$

of C to $\operatorname{Spec} A_+ \backslash 0$, and consider the blowup $P_{1,+}$ of C_+ along its singular locus:

Proposition 4.2. *The singular locus of C_+ consists of the singular points in the singular fibers over $\operatorname{cone}(E_+)$, and the blowup $P_{1,+}$ of C_+ along the singular locus is smooth. Similar statement holds for C_-.*

Proof. Observe that C_+ is cut out by $y_+^2 - f_+$ and q^+ inside $(\operatorname{Spec} A_+ \backslash 0) \times \mathbb{P}V_+$. Taking partial derivatives w.r.t. to the variables u_1, u_2, u_3, y_+, x_1^+, x_2^+, x_3^+, we have

$$\begin{bmatrix} -\frac{\partial f_+}{\partial u_1} & -\frac{\partial f_+}{\partial u_2} & -\frac{\partial f_+}{\partial u_3} & 2y_+ & 0 & 0 & 0 \\ \frac{\partial q^+}{\partial u_1} & \frac{\partial q^+}{\partial u_2} & \frac{\partial q^+}{\partial u_3} & 0 & \frac{\partial q^+}{\partial x_1^+} & \frac{\partial q^+}{\partial x_2^+} & \frac{\partial q^+}{\partial x_3^+} \end{bmatrix}$$

Note that one of $\frac{\partial f_+}{\partial u_i}$ must be nonzero because $E_+ = \{f_+ = 0\}$ is smooth, and also it is assumed that $C_+ = \{q^+ = 0\}$ is smooth. If $y_+ \neq 0$, then the Jacobian must have rank 2. If $y_+ = 0$, then setting rank ≤ 1 implies that second row is a multiple of the first row, so $\frac{\partial q^+}{\partial x_1^+} = \frac{\partial q^+}{\partial x_2^+} = \frac{\partial q^+}{\partial x_3^+} = 0$. So the singular locus of C_+ is contained in the singular loci of the singular fibers of C_+.

In the other direction, we likewise locally diagonalize q^+ as $q^+ = r_1(u)(x_1^+)^2 + r_2(u)(x_2^+)^2 + r_3(u)(x_3^+)^2$ where r_1, r_2 are inveritble and r_3 is a local coordinate. Pulling back to $\operatorname{Spec} A_+ \backslash 0$, we have $r_3 = y_+^2 / r_1 r_2$, so we see that C_+ is singular when $y_+ = x_1^+ = x_2^+ = 0$. (Let u_1, u_2 be complementary coordinates to r_3, and note that q^+ lies in the square of the maximal ideal.) Hence we have the reverse inclusion.

Since $(x_1^+, x_2^+, x_3^+) \neq (0,0,0)$, we see that $x_3^+ \neq 0$ at the singular points. The form of q^+ shows that we have an A_1-singularity along the singular locus. Therefore the blowup $P_{1,+}$ of the singular locus is resolution of C_+. □

Note that for smooth conics in $\mathbb{P}V_+$ over $W \backslash \operatorname{cone}(E_+)$, there is a natural map of dual conics mapping a point to the tangent line at that point and it gives a rational map $C_+ \dashrightarrow \operatorname{adj}(C_+)$. fiber-wise it maps to $\left(\frac{\partial q^+}{\partial x_1^+} : \frac{\partial q^+}{\partial x_2^+} : \frac{\partial q^+}{\partial x_3^+} \right)$. We claim that it extends to a morphism $P_{1,+} \to C_+$, hence factor through the normalization $P_{1,+} \to P_+ \to \operatorname{adj}(C_+)$. Moreover, let $H_0 \subseteq C_+$ denote the pullback of $\operatorname{cone}(E_+) \subseteq \operatorname{Spec} A_+ \backslash 0$, which is fiber-wise the singular quadric $\{q^+ = 0\}$, and $H \subseteq P_{1,+}$ be the strict transform of H_0 in $P_{1,+}$, then:

Proposition 4.3. *There exists a morphism $P_{1,+} \to \operatorname{adj}(C_+)$, and fiber-wise it contracts the two lines in $H \subseteq P_{1,+}$ into two points.*

Proof. Restricting to a Zariski neighborhood U of a point u_0 lying over $\operatorname{cone}(E_+)$ in $\operatorname{Spec} A_+ \backslash 0$, the quadratic forms can be simultaneously diagonalized to be $q^+ = r_1(u)(x_1^+)^2 + r_2(u)(x_2^+)^2 + r_3(u)(x_3^+)^2$ where r_1, r_2 are inveritble and r_3 is a local coordinate. In this case the equations of the singular locus in C_+ reduce to

$$y_+ = 0, \quad x_1^+ = 0, \quad x_2^+ = 0.$$

16

Since $(x_1^+, x_2^+, x_3^+) \neq (0,0,0)$, we see that $x_3^+ \neq 0$ at u_0 and we can assume it is invertible in U. The blowup is locally $\mathrm{Proj}(\mathscr{O}_{C_+} + \mathcal{I} + \mathcal{I}^2 + \dots)$ where $\mathcal{I} = \langle y_+, x_1^+, x_2^+ \rangle$ is the ideal sheaf. Let $\widetilde{y_+}, \widetilde{x_1}^+$, $\widetilde{x_2}^+$ denote the corresponding sections in the degree 1 piece of the Rees algebra $\mathscr{O}_{C_+} + \mathcal{I} + \mathcal{I}^2 + \dots$, then the exceptional divisor has equation

$$(\widetilde{x_1}^+)^2 + (\widetilde{x_2}^+)^2 + (x_3^+)^2(\widetilde{y_+})^2 = 0 \quad \text{(see proof of Proposition 4.2).}$$

Then we see that the map $\left(\frac{\partial q^+}{\partial x_1^+} : \frac{\partial q^+}{\partial x_2^+} : \frac{\partial q^+}{\partial x_3^+} \right)$ extends to the formula

$$\left(2\widetilde{x_1}^+ : 2\widetilde{x_2}^+ : 2x_3^+ y_+ \widetilde{y_+} \right)$$

to the exceptional divisor in the new coordinates. which is well-defined because the first two components cannot be simultaneously 0: if $\widetilde{x_1}^+ = \widetilde{x_2}^+ = 0$ then $\widetilde{y_+} = 0$ from the equation of the exceptional divisor, but $(\widetilde{y_+}, \widetilde{x_1}^+, \widetilde{x_2}^+) \neq (0,0,0)$.

Next, we see that over a point $u \in \mathrm{cone}(E_+)$ the quadric q^+ degenerates to 2 lines passing through the origin, so the points on q^+ has two fixed ratios $\widetilde{x_1}^+ : \widetilde{x_2}^+$ for a given u. The map sends these points to $(2\widetilde{x_1}^+ : 2\widetilde{x_2}^+ : 0)$, i.e. fiber-wise H is contracted to 2 points. $\qquad\square$

Remark 4.4. The two constructions can be summarized in the diagram:
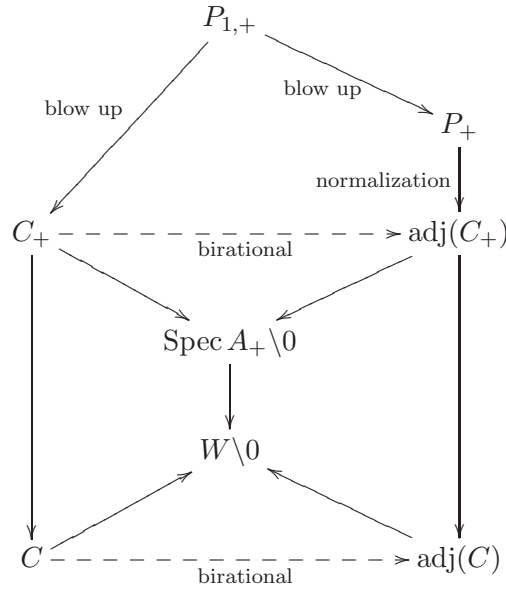
We have described the $\mathbb{P}^1$-bundle P_+ associated to q^+ and E_+ on $\mathrm{Spec}\, A_+ \backslash 0$. Likewise there is a $\mathbb{P}^1$-bundle P_- associated to q^- and E_- on $\mathrm{Spec}\, A_- \backslash 0$. We can now construct a $\mathbb{P}^3$-bundle on $\mathrm{Spec}\, A \backslash 0$ from P_+ and P_-. We denote the pullbacks of $P_\pm$ from $\mathrm{Spec}\, A_\pm \backslash 0$ to $\mathrm{Spec}\, A \backslash 0$ by $P_\pm^*$.

Proposition 4.5. *There exists a Severi-Brauer variety P over $\mathrm{Spec}\, A \backslash 0$ such that P_+^* and P_-^* are fiber-wise embedded in P via the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, and P corresponds to the Brauer class represented by $\widetilde{Cl}$.*

Proof. We will exhibit 2 canonical ways to construct the $\mathbb{P}^3$-bundle. First we can take the twisted sheaves $\mathscr{P}_\pm$ associated to $P_\pm$ [6] (unique up to twisting of a line bundle). We can then pull back $\mathscr{P}_\pm$ from $\operatorname{Spec} A_\pm \backslash 0$ to $\operatorname{Spec} A \backslash 0$ to get $\mathscr{P}^*_\pm$, and their tensor product $\mathscr{P}^*_+ \otimes \mathscr{P}^*_-$ is then a twisted sheaf on $\operatorname{Spec} A \backslash 0$ whose fiber is $\mathbb{C}^4$, hence its projectivization P' becomes a $\mathbb{P}^3$-bundle. By [6, 13], P' corresponds to the Brauer class represented by $\widetilde{Cl} = Cl^*_+ \otimes Cl^*_-$. From this description it is not obvious that we get an algebraic variety because we went through the étale or analytic topology having twisted sheaves involved, but we observe that fiber-wise P_+ and P_- are embedded in P' via the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, which allows us to show that P' is algebraic.

To do that, we make use of the fact that both $P_\pm \to \operatorname{Spec} A_\pm \backslash 0$ are projective by construction, so are their pullbacks $P^*_\pm$ to $\operatorname{Spec} A \backslash 0$, and their fiber product $P^*_+ \times P^*_-$ there. By definition we can embed it in a trivial $\mathbb{P}^N$-bundle over $\operatorname{Spec} A \backslash 0$. Now to every $(1,1)$-divisor in the fiber we can associate its image in $\mathbb{P}^N$, which is a closed subscheme in $\mathbb{P}^N$. This assignment gives a map from $P^*_+ \times P^*_-$ to the Hilbert scheme $\operatorname{Hilb}(\mathbb{P}^N)$. Since there is a $\mathbb{P}^3$-family of $(1,1)$-divisor for each fiber, the closure of the image of $P^*_+ \times P^*_-$ in $\operatorname{Hilb}(\mathbb{P}^N)$ will form an algebraic $\mathbb{P}^3$-bundle P''. We see that $P^*_+ \times P^*_- \to P''$ is fiber-wise a Segre embedding so it must agree with P', as both constructions are canonical. Hence we obtain the desired algebraic $\mathbb{P}^3$-bundle P, which is a Severi-Brauer variety. $\quad\square$

Next we will exhibit an explicit relation of the Brauer class $\widetilde{Cl}$ and the $\mathbb{P}^3$-bundle P when formulated via the quadric fibrations. We state here [2, Theorem 1.3.5] that serves as an equivalent definition of an Azumaya algebra:

Theorem 4.6. *Let $\mathscr{A}$ be an Azumaya algebra over X, and let $\alpha \in \operatorname{Br}'(X)$ be the element that $\mathscr{A}$ represents. Then there exists a locally free α-twisted sheaf $\mathscr{E}$ of finite rank (not necessarily unique) such that $\mathscr{A}$ is isomorphic to the sheaf of endormorphism algebra of $\mathscr{E}$. Conversely, for any $\alpha \in \operatorname{Br}'(X)$ such that there exists a locally free α-twisted sheaf of finite rank, the sheaf of endormorphism algebra of $\mathscr{E}$ is an Azumaya algebra whose class in $\operatorname{Br}'(X)$ is α.*

So it suffices to show that

Proposition 4.7. *P_+ is locally isomorphic to the projectivization of an irreducible Clifford module on $\operatorname{Spec} A_+ \backslash \operatorname{cone}(E_+)$. Similar statement holds for P_-.*

Proof. Let M be an irreducible Cl_+-module of rank 2, when Cl_+ is considered a sheaf of algebras on $\operatorname{Spec} A_+ \backslash \operatorname{cone}(E_+)$. We want to get an isomorphism $\mathbb{P}M \to P_+$. Fix $m \neq 0$ of M and consider $\mathbb{C}m \in \mathbb{P}M$, the map $V_+ \to M$ sending $v \mapsto v \cdot m$ has a kernel by counting dimension. The annihilator of m consists of isotropic vectors because $0 = vvm = q^+(v,v)m$ and $m \neq 0$. So for q^+ of full rank, the annihilator must be 1-dimensional because $\{q^+ = 0\}$ doesn't contain any line in $\mathbb{P}V_+$. This gives a natural map $\mathbb{P}M \to \{q^+ = 0\} \subseteq \mathbb{P}V_+$ sending $\mathbb{C}m \mapsto \operatorname{Ann}(m)$. For the inverse, send $\mathbb{C}v \in \{q^+ = 0\} \subseteq \mathbb{P}V_+$ to

$$\ker(v) = \{m \in M : v \cdot m = 0\}.$$

We need to make sure $\ker(v)$ is always 1-dimensional to get an element in $\mathbb{P}M$. Since q^+ is assumed of full rank, $q^+(v,v) \neq 0$. If $\ker(v) = M$, take $v' \neq v$ in $\{q^+ = 0\} \subseteq \mathbb{P}V_+$ and $m' \in M$ linearly independent with m such that $v'm' = 0$, then $\operatorname{Ann}(m')$ contains both v and v', a contradiction to $\dim \operatorname{Ann}(m') = 1$. Hence we have a map $\{q^+ = 0\} \to \mathbb{P}M$ sending $\mathbb{C}v \mapsto \ker(v)$. It is then evident that the two maps are inverses to each others. $\quad\square$

Over the points $u \in W \backslash \mathrm{cone}(E_+ \cup E_-)$, we are in a similar situation as in Kuznetsov's case for which the Clifford algebras in the fiber are of full rank, so we can make comparison of the $\mathbb{P}^3$-bundles. First we give a description to the $\mathbb{P}^3$-bundle in Kuznetsov's case: when q is a quadratic form in $\mathbb{P}V$ of full rank, in the quadric $\{q = 0\} \subseteq (W \backslash 0) \times \mathbb{P}V$ there are two families of rulings, parametrized by $\mathbb{P}^3$, of isotropic subspaces isomorphic to $\mathbb{P}^2$'s. These two $\mathbb{P}^3$ families naturally form two $\mathbb{P}^3$-bundles P' and P'' over $W \backslash \mathrm{cone}(E_+ \cup E_-)$. Note that P' is isomorphic to P''.

Proposition 4.8. *P' and P'' are isomorphic to the projectivizations of the two irreducible modules of Cl_{ev} over $W \backslash \mathrm{cone}(E_+ \cup E_-)$.*

Proof. When restricted to $u \in W \backslash \mathrm{cone}(E_+ \cup E_-)$, the rank of the quadratic form is 6, the Clifford algebra Cl_{ev} has two irreducible modules M_0 and M_1 of rank 4, and $M = M_0 \oplus M_1$ is the irreducible module for Cl. The embedding $V \subseteq Cl$ gives rise to two maps $V \otimes M_0 \to M_1$ and $V \otimes M_1 \to M_0$ via the Clifford action of Cl on M.

Fix $m \neq 0$ of M_0 and consider $\mathbb{C}m \in \mathbb{P}M_0$. The map $V \to M_1$ sending $v \mapsto v \cdot m$ has a matrix of rank 3 and hence a kernel $\mathrm{Ann}(m)$ of dimension 3. To see this, we may identify the quadratic form with a natural pairing on $U \oplus U^\vee$ where $U \simeq \mathbb{C}^3$, then the Clifford algebra and modules are built from wedge products and contractions [8]. Fix bases $\{v_1, v_2, v_3\}$ for U, $\{v_4, v_5, v_6\}$ for $U^\vee$ and $\{v_1, v_2, v_3, v_1 \wedge v_2 \wedge v_3\}$ for M_0. Let $m = a_0 + a_1 v_2 \wedge v_3 + a_2 v_3 \wedge v_1 + a_3 v_1 \wedge v_2$. Then the matrix of m is

$$\begin{bmatrix} a_0 & & & & & a_1 \\ & a_0 & & & & a_2 \\ & & a_0 & & & a_3 \\ & & a_3 & -a_2 & & \\ -a_3 & & & a_1 & & \\ a_2 & -a_1 & & & & \end{bmatrix}$$

which contains 3×3 minors of the form a_i^3 so the rank is at least 3, and the columns C_i's satisfy the relation $a_1 C_1 + a_2 C_2 + a_3 C_3 - a_0 C_4 = 0$.

The kernel $\mathrm{Ann}(m)$ is an isotropic subspace w.r.t. q because $0 = vvm = q(v, v)m$ implies $q(v, v) = 0$. From here we obtained a map from $\mathbb{P}M_0 \to P' \cup P''$ sending $\mathbb{C}m \mapsto \mathrm{Ann}(m)$. By symmetry, there is also a map from $\mathbb{P}M_1 \to P' \cup P''$ sending $\mathbb{C}m \mapsto \mathrm{Ann}(m)$.

It follows from the continuity of the map that two m, m' from the same M_i will be sent to the same ruling, i.e. to the same P' or P''.

For the inverse, and send an isotropic subspace $\mathbb{P}V' \simeq \mathbb{P}^2$ in $\mathbb{P}V$ to

$$\ker(V') = \{m \in M : V' \cdot m = 0 \text{ in } M\}.$$

Pick a complementary isotropic subspace $\mathbb{P}V'' \subseteq \mathbb{P}V$ from the other ruling of the quadric, then $V = V' \oplus V''$. We can then identify $M_0 = (\wedge V'')_{ev} = (\wedge V')_{odd}$, $M_1 = (\wedge V'')_{odd} = (\wedge V')_{ev}$ (note that we cannot distinguish M_0 and M_1 and the codomain of this map depends on the choice of the roles of V' and V''), and from this description we see that $\ker(V') = \wedge^3 V' \subseteq M_0$ has dimension 1, hence we get an element in $\mathbb{P}M_0$. This gives a map $P' \cup P'' \to \mathbb{P}M_0$. It is then evident that the two maps are inverses to each others.

19

Again from continuity, two V', V'' from the same ruling will both be sent to the same one of $\mathbb{P}M_0$ or $\mathbb{P}M_1$. □

Let $Y' \to W \setminus \operatorname{cone}(E_+ \cup E_-)$ be the restriction of the ramified double cover $\operatorname{Spec} B \to W$. Then Y' is unramified and we have the following:

Proposition 4.9. *P' and P'' can be combined into one $\mathbb{P}^3$-bundle P_d over Y'.*

Proof. The two irreducible modules of Cl_{ev} are classified by the action of the central element, which can be one of the two roots y_1, y_2 of $y^2 - \det(q) = y^2 - f_+ f_-$. Once we made a choice of correspondence between P', P'' and the two modules as in Proposition 4.8, say $P' \leftrightarrow y_1$ and $P'' \leftrightarrow y_2$, we can define the $\mathbb{P}^3$-bundle P_d for which the fiber over $(u, y_1) \in \operatorname{Spec} B$ is the fiber of P', and the fiber over (u, y_2) is the fiber of P''. It is well-defined as we can choose a fixed square root over an étale neighborhood, showing that it is a $\mathbb{P}^3$-bundle in the étale topology. □

Let $C'_\pm$ denote the quadric fibration $\{q_u^\pm = 0\} \subseteq \mathbb{P}V_\pm$ over the unramified double cover Y'.

Proposition 4.10. *There exist embedding of $C'_+ \times_{Y'} C'_-$ into P_d such that fiber-wise it is the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$.*

Proof. For $(u, y) \in Y'$, $p_+ \in \{q_u^+ = 0\} \subseteq \mathbb{P}V_+$ and $p_+ \in \{q_u^+ = 0\} \subseteq \mathbb{P}V_-$, the line ℓ joining p_+ and p_- in $\mathbb{P}V$ is isotropic w.r.t. q_u:

$$q_u(ap_+ + bp_-, ap_+ + bp_-) = q_u(ap_+, ap_+) + 2q_u(ap_+, bp_-) + q_u(bp_-, bp_-) = 0.$$

The line ℓ is contained in the two isotropic subspaces of dimension 2 w.r.t. q_u. From here we obtain a globally defined map $C'_+ \times_{Y'} C'_- \to P_d$ by sending ℓ to the point in the fiber of P_d over (u, y) which corresponds to the isotropic subspace that contains ℓ. Now we show that étale locally they are Segre embeddings.

Fix an u. We can model the full rank quadric as $q_u = x_1 x_2 + x_3^2 - x_4^2 + x_5 x_6$ where V_+ has coordinates x_1, x_2, x_3 and V_- has coordinates x_4, x_5, x_6. By the change of variables

$$x_1 = z_{12}, \quad x_2 = z_{34}, \quad x_3 = \frac{z_{13} - z_{24}}{2}, \quad x_4 = \frac{z_{13} + z_{24}}{2}, \quad x_5 = z_{14}, \quad x_6 = z_{23},$$

we can identify the quadric as the image $z_{12} z_{34} - z_{13} z_{24} + z_{14} z_{23} = 0$ of the Plücker embedding $G(2, 4) \hookrightarrow \wedge^2 \mathbb{C}^4 \simeq \mathbb{C}^6$ where we use the basis $\{e_{ij} = e_i \wedge e_j : i < j\}$. For the Plücker embedding we can parametrize one $\mathbb{P}^3$-family of isotropic subspaces of dimension 2 by

$$(y_1 : y_2 : y_3 : y_4) \in \mathbb{P}^3 \to \mathbb{P}\operatorname{Span}\left\{\sum_{i=1}^4 y_i e_{ij} : j = 1, \ldots, 4\right\}.$$

Points $p_+ \in \{q_u^+ = x_1 x_2 + x_3^2 = 0\}$ can be parametrized by $(a_0 : a_1) \in \mathbb{P}^1$ as $p_+ = (a_0^2 : -a_1^2 : a_0 a_1 : 0 : 0 : 0)$. Similarly we let $p_- = (0 : 0 : 0 : a_2 a_3 : a_2^2 : a_3^2)$ for $(a_2 : a_3) \in \mathbb{P}^1$. Switching back to the coordinates z_{ij}'s, we have $p_+ = a_0^2 e_{12} - a_1^2 e_{34} + a_0 a_1 e_{13} - a_0 a_1 e_{24}$ and $p_- = a_2 a_3 e_{13} + a_2 a_3 e_{24} + a_2^2 e_{14} + a_3^2 e_{23}$. They come from the following matrices in $G(2, 4)$:

$$\begin{pmatrix} a_0 & 0 & 0 & a_1 \\ 0 & a_0 & a_1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_2 & a_3 & 0 & 0 \\ 0 & 0 & a_3 & a_2 \end{pmatrix}.$$

We are then looking for $(y_1 : y_2 : y_3 : y_4)$ whose associated isotropic subspace contains both p_+ and p_-. In $\mathbb{C}^4$, p_+ and p_- intersect at the line spanned by $(a_0a_2 : a_0a_3 : a_1a_3 : a_1a_2)$, so they are contained in a 3-dimensional subspace, which corresponds to an isotropic 2-dimensional subspace in $\{q_u = 0\} \subseteq \mathbb{P}V$. It can be checked that the subspace corresponding to $(y_1 : y_2 : y_3 : y_4) = (a_0a_2 : a_0a_3 : a_1a_3 : a_1a_2)$ contains p_+ and p_-. This formula clearly is the Segre embedding. $\qquad\square$

The same is true after pulling $C'_+ \times_{Y'} C'_-$ and P_d back to $\operatorname{Spec} A \backslash \operatorname{cone}(E_+ \cup E_-)$. Since the constructions in Proposition 4.5 and 4.10 are both canonical, we see that the restrictions of P in Proposition 4.5 to $\operatorname{Spec} A \backslash \operatorname{cone}(E_+ \cup E_-)$ and the pullback P_d^* can be identified. This established the relation between Kuznetsov's construction and our construction outside of the ramification locus $\operatorname{cone}(E_+ \cup E_-)$.

5 Generalizations and Future Work

We can ask whether some conditions on the root stacks and the Brauer class ensure certain properties on the Enriques surfaces, and vice versa:

(i) When is the Enriques surface nodal, i.e. contains a nonsingular rational curve?

(ii) How do the elements of the Picard group of the Enriques surface corresponds to (complexes of) sheaves on the root stack? What are the corresponding auto-equivalences?

(iii) In our setup, the 2 cubic curves come from determinants of 3×3 matrices. If two quadratic forms give rise to the same cubic curves, i.e. if the root stacks are identical, is there any relation to the corresponding Enriques surfaces?

(iv) What can be said about the Enriques surface if the Brauer class is trivial?

(v) Is the orbifold cohomology of the two sides related? There are no twisted sectors in the Enriques surface, but the root stack should have the cubic curves and their intersection points as twisted sectors.

In another direction, direct generalizations of the tensor product construction of the Clifford algebras of invariant subspaces to other dimensions can be studied. For example, the double mirror of a dimension 3 intersection of 4 quadrics in $\mathbb{P}^7$ quotient by a larger group of actions.

References

[1] Bondal, A., Orlov, D. *Reconstruction of a variety from the derived category and groups of autoequivalences.* Compositio Mathematica **125**, p. 327-344 (2001).

[2] Borisov, L., Căldăraru, A. *The Pfaffian-Grassmannian derived equivalence.* Journal of Algebraic Geometry **18**, p. 201-222 (2006).

[3] Borisov, L., Li, Z. *On Clifford Double Mirrors of Toric Complete Intersections.* Advances in Mathematics **328**, p. 300-355 (2018).

[4] Bourbaki, N. *Algebra I, Chapters 1-3.* Springer-Verlag Berlin Heidelberg (1989).

[5] Cadman, C. *Using Stacks to Impose Tangency Conditions on Curves.* American Journal of Mathematics **129**, no. 2, p. 405-427 (2007).

[6] Căldăraru, A. *Derived Categories of Twisted Sheaves on Calabi-Yau Manifolds.* Ph.D. thesis, Cornell University (2000), also available at https://people.math.wisc.edu/~andreic/publications/ThesisSingleSpaced.pdf (last accessed July 2021).

[7] Cossec, F. *Projective models of Enriques surfaces.* Math. Ann. **265**, p. 283-334 (1983).

[8] Deligne, P. *Notes on spinors*, in Quantum Fields and Strings: A Course for Mathematicians. Available at http://publications.ias.edu/sites/default/files/79_NotesOnSpinors.pdf.

[9] Deligne, P., Morgan J. *Notes on supersymmetry*, in Quantum Fields and Strings: A Course for Mathematicians.

[10] Gastineau-Hills, H. *Quasi Clifford algebras and systems of orthogonal designs.* Journal of the Australian Mathematical Society. Series A. Pure Mathematics and Statistics, **32**(1), p. 1-23 (1982).

[11] Gulbrandsen, M. *Stack structures on GIT quotients parametrizing hypersurfaces.* Math. Nachr. 284, no. 7, p. 885–898 (2011).

[12] Huybrechts, D. *Lectures on K3 surfaces*, 1st ed. Cambridge University Press, 2016.

[13] Kollár, J. *Severi–Brauer varieties a geometric treatment.* Available at https://arxiv.org/abs/1606.04368 (last update on 14 Jun 2016).

[14] Kuznetsov, A. *Derived Categories of Quadraic Fibrations and Intersections of Quadrics.* Advances in Mathematics **218**, p. 1340-1369 (2008).

[15] Kuznetsov, A. *Homological projective duality.* Publ. Math. IHES **105**, p. 157-220 (2007).

[16] Kuznetsov, A., Perry A., *Categorical joins.* J. Amer. Math. Soc. **34**, no. 2, 505-564 (2021).

[17] Marchuk, N.G. *Classification of Extended Clifford Algebras.* Izv. Vyssh. Uchebn. Zaved. Mat. 2018, no. 11, 27–32; translation in Russian Math. (Iz. VUZ) **62**, no. 11, p. 23-27 (2018).

[18] Polishchuk, A. *Noncommutative Proj and Coherent Algebras.* Mathematical Research Letters **12**(1), p. 63-74 (2003).

[19] Verevkin, A. B. *On a Non-commutative Analogue of the Category of Coherent Sheaves on a Projective Scheme.* AMS Transl. **151**, p. 41-53 (1992).