The Pfaffian-Grassmannian derived equivalence

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Abstract

We argue that there exists a derived equivalence between Calabi-Yau threefolds obtained by taking dual hyperplane sections (of the appropriate codimension) of the Grassmannian $\mathbf{G}(2,7)$ and the Pfaffian $\mathbf{Pf}(7)$. The existence of such an equivalence has been conjectured by physicists for almost ten years, as the two families of Calabi-Yau threefolds are believed to have the same mirror. It is the first example of a derived equivalence between Calabi-Yau threefolds which are provably non-birational.

Introduction

0.1. Let V be a vector space of dimension seven over $\mathbf{C}$ (or any algebraically closed field of characteristic zero), and let

$$\mathbf{G} = \mathbf{G}(2, V)$$

be the Grassmannian of planes in V. The Plücker map embeds $\mathbf{G}$ as a smooth subvariety of dimension 10 of

$$\mathbf{P} = \mathbf{P}^{20} = \mathbf{P}(\wedge^2 V).$$

Regard the dual projective space

$$\mathbf{P}^* = \mathbf{P}(\wedge^2 V^*)$$

as the projectivization of the space of two-forms on V. The Pfaffian locus

$$\mathbf{Pf} \subset \mathbf{P}^*$$

is defined to be the projectivization of the locus of degenerate two-forms on V (forms of rank ≤ 4). Equations for $\mathbf{Pf}$ can be obtained by taking the Pfaffians of the diagonal minors of a skew-symmetric 7×7 matrix of linear forms on V.

While the Grassmannian $\mathbf{G}$ is smooth, the Pfaffian $\mathbf{Pf}$ is a singular subvariety of $\mathbf{P}^*$ of dimension 17. Indeed, a point $\omega \in \mathbf{Pf}$ will be singular precisely when the rank of ω is two. (Recall that the rank of a two-form ω – or, equivalently, of a skew symmetric matrix – is always even. In our case, since we eliminate $\omega = 0$ by projectivizing, this rank could be two, four, or six. The general two-form on V has rank six; it has rank four at the smooth points of $\mathbf{Pf}$, and it has rank two at its singular points, which coincide with $\mathbf{G}(2, V^*)$ in its Plücker embedding.)

The Pfaffian is the classical projective dual of the Grassmannian:

$$\mathbf{Pf} = \{y \in \mathbf{P}^* \ : \ \mathbf{G} \cap H_y \text{ is singular}\},$$

where H_y is the hyperplane in $\mathbf{P}$ corresponding to y.

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0.2. Consider a seven-dimensional linear subspace

$$W \subset \wedge^2 V^*,$$

and denote by $\boldsymbol{W}$ its image in $\mathbf{P}^*$. Let Y be the intersection of $\boldsymbol{W}$ with $\mathbf{Pf}$.

On the dual side, let

$$M = \mathsf{Ann}(W) \subset \wedge^2 V$$

be the 14-dimensional annihilator of W; again, we will use bold-face $\boldsymbol{M}$ to denote its projectivization in $\mathbf{P}$, which has codimension seven. Let X be the intersection of $\boldsymbol{M}$ and $\mathbf{G}$.

The main result of this paper is the following theorem.

0.3. Theorem. *For a given choice of W, if either X or Y has dimension three, then X is smooth if and only if Y is. When this happens, X and Y are Calabi-Yau threefolds with*

$$h^{1,1} = 1, \quad h^{1,2} = 50,$$

and there exists an equivalence of derived categories

$$\Phi : \mathbf{D}^b_{\mathsf{coh}}(Y) \xrightarrow{\sim} \mathbf{D}^b_{\mathsf{coh}}(X).$$

0.4. Such a result has been conjectured for a while. Indeed, Rødland [11] argued, by comparing solutions to the Picard-Fuchs equation, that the families of X's and of Y's appear to have the same mirror family. Recently, Hori and Tong [8] gave a more detailed string theory argument supporting the same conclusion. If we denote by Z their common mirror, Kontsevich's Homological Mirror Symmetry conjecture predicts

$$\mathbf{D}^b_{\mathsf{coh}}(X) \cong \mathbf{Fuk}(Z) \cong \mathbf{D}^b_{\mathsf{coh}}(Y).$$

0.5. This appears to be the first example of a derived equivalence between Calabi-Yau threefolds which can be proved to be non-birational. Indeed, if X and Y were birational, they would have to differ by a sequence of flops because they are minimal in the sense of Mori theory. On the other hand, no flops are possible on either X or Y, because they have Picard number $h^{1,1} = 1$. Therefore they would have to be isomorphic. However, this can not be true, since if we denote by H_X and H_Y, respectively, the ample generators of the Picard groups of X and of Y, we have [11]

$$H_X^3 = 42, \qquad H_Y^3 = 14.$$

0.6. Our results appear to fit very well with the theory of Homological Projective Duality developed by Kuznetsov [9]. Indeed, we have a pair of varieties $\mathbf{G}$ and $\mathbf{Pf}$, embedded in dual projective spaces, whose dual hyperplane sections are derived equivalent. It would be interesting to understand this relationship further. To this end, we make some comments in Section 6 about how this example seems to fit in the general theory.

We expect a similar statement to hold true for Calabi-Yau's of dimension $2n-3$ obtained by linear cuts of $\mathbf{G}(2, 2n+1)$ and $\mathbf{Pf}(2n+1)$. We plan to address this topic in a future paper [2].

0.7. Let us now briefly describe the construction of the derived equivalence Φ. Recall [3] that, in order to give Φ, it is essentially enough to describe the image of Φ on structure sheaves of points $\mathscr{O}_y$ for $y \in Y$, and to check that the family $\{\Phi\mathscr{O}_y\}_{y\in Y}$ is an orthonormal basis for $\mathbf{D}^b_{\mathsf{coh}}(X)$.

A point y in Y can be regarded as a two-form on V of rank four. As such it has a kernel K which is a three-dimensional linear subspace of V. On the other hand, a point x in X corresponds to a two-dimensional linear subspace T of V. For a general choice of x and y we will have

$$T \cap K = 0.$$

However, for a fixed $y \in Y$, the set of points $x \in X$ for which T intersects K non-trivially is a curve C_y in X. Our choice for the functor Φ is to set

$$\Phi\mathscr{O}_y = \mathscr{I}_{C_y},$$

where $\mathscr{I}_{C_y}$ denotes the ideal sheaf of the curve C_y. The technical core of the paper consists then of showing that this choice gives rise to an orthonormal family.

It would be interesting to find out the relationship of this approach to the construction of Donaldson-Thomas moduli spaces. It seems reasonable to conjecture that Y is such a moduli space of ideal sheaves of curves on X. Perhaps other examples of derived equivalences can be obtained as Donaldson-Thomas spaces.

0.8. The paper is organized as follows. In Section 1 we set up the notation, and we recall the appropriate definitions from linear algebra. We also sketch the standard constructions of the tangent spaces to the Pfaffian and Grassmannian varieties. In Section 2 we give a purely linear-algebra argument for the fact that X is smooth if and only if Y is. In Section 3 we define the curves C_y which are parametrized by the points of Y, and we argue that they are all different and have dimension one. Section 4 contains a reduction of the computation of the orthogonality of the family to the vanishing of certain global Ext's on the Grassmannian $\mathbf{G}$, which will be checked using Macaulay. In Section 5 we define the functor Φ and we argue that it is an equivalence. For this we use Bridgeland's celebrated application of the Intersection Theorem from commutative algebra, in a slightly modified form (Theorem 5.6). Finally, in Section 6 we discuss connections with results of Kuznetsov on Homological Projective Duality. We include in Appendix A the Macaulay code for the computation of the global Ext groups on $\mathbf{G}$, with comments on the techniques used.

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1. Basic facts from linear algebra

In this section we set up the basic notation and review some standard results from linear algebra and differential geometry.

1.1. Let V be a vector space. Denote by $\mathbf{G}(k, V)$ the Grassmannian of k-dimensional planes in V. The Plücker embedding is the map

$$\mathbf{G}(k, V) \hookrightarrow \mathbf{P}(\wedge^k V),$$

which maps a point x of $\mathbf{G}(k, V)$ (i.e., a k-plane $x \subset V$) to the point

$$[e_1 \wedge e_2 \wedge \cdots \wedge e_k] \in \mathbf{P}(\wedge^k V),$$

where $e_1, \ldots, e_k$ is any basis of x.

If x is a point in the Grassmannian, the tangent space $T_{\mathbf{G}(k,V),x}$ is canonically identified with

$$\mathsf{Hom}(x, V/x).$$

Thus, if V has dimension n, the Grassmannian $\mathbf{G}(k, V)$ has dimension $k(n-k)$.

In particular, if $k = 1$, the tangent space $T_{\mathbf{P}V,x}$ to the projective space at a point x is naturally identified with $\mathsf{Hom}(x, V/x)$ which is (up to scalars) the same as V/x.

1.2. The tangent map to the Plücker embedding is the map

$$\mathsf{Hom}(x, V/x) \to \mathsf{Hom}(\wedge^k x, \wedge^k V/\wedge^k x),$$

given by

$$\varphi \mapsto \left(v_1 \wedge \cdots \wedge v_k \mapsto \sum_{i=1}^{k} v_1 \wedge v_2 \wedge \cdots \wedge v_{i-1} \wedge \varphi(v_i) \wedge v_{i+1} \wedge \cdots \wedge v_k \right).$$

1.3. An element $y \in \wedge^k V^*$ will be called a k-form on V. Since we have

$$\wedge^k V^* \cong \left(\wedge^k V\right)^*,$$

the annihilator of y, $\mathsf{Ann}(y)$, is a hyperplane $H \subset \wedge^k V$. It consists of all $x \in \wedge^k V$ such that $y(x) = 0$.

1.4. From now on we will concentrate on the case $k = 2$, and define

$$\mathbf{P} = \mathbf{P}(\wedge^2 V), \quad \mathbf{P}^* = \mathbf{P}(\wedge^2 V^*), \quad \mathbf{G} = \mathbf{G}(2, V).$$

Frequently, when regarding a point $x \in \mathbf{G}$ as a two-plane in V we will call this plane T.

If y is a two-form on V, its kernel K is defined to be the set of all $v \in V$ such that $y(v \wedge w) = 0$ for all $w \in V$. We define the rank of y by

$$\mathsf{rk}\, y = n - \dim K.$$

The rank of a two-form is always even (as it equals the rank of the corresponding skew symmetric matrix).

The kernel of a form y does not change if we multiply y by a non-zero scalar. Thus we can speak of the kernel of a point $y \in \mathbf{P}^*$, and we will frequently refer to such a point as a two-form.

1.5. Proposition. *Let y be a point in $\mathbf{P}^*$, and let H be the corresponding hyperplane in $\mathbf{P}$. Let x be a point of intersection of H and $\mathbf{G}$. Then H is tangent to $\mathbf{G}$ at x (under the Plücker embedding) if and only if $T \subset K$, where T is the two-plane in V corresponding to x, and K is the kernel of y, regarded as a two-form on V.*

Proof. First assume $T \subset K$. Let φ be a tangent vector to $\mathbf{G}$ at x, i.e., a morphism $\varphi : T \to V/T$. Then the image of φ under the differential of the Plücker map is the map $\bigwedge^2 T \to \bigwedge^2 V/\bigwedge^2 T$ given by

$$t_1 \wedge t_2 \mapsto \varphi(t_1) \wedge t_2 + t_1 \wedge \varphi(t_2).$$

Choosing t_1 and t_2 linearly independent vectors in T, we identify this (up to scalars) with the element

$$t = \varphi(t_1) \wedge t_2 + t_1 \wedge \varphi(t_2)$$

of $\bigwedge^2 V/\bigwedge^2 T$. Since $T \subset K$, t_1, t_2 are in the kernel of y, and therefore y vanishes on t. In other words, we have proven that

$$\varphi \in T_{\mathbf{G},x} \implies \varphi \in T_{H,x},$$

or that H is tangent to $\mathbf{G}$ at x.

Conversely, assume $T \not\subset K$. At least one of t_1, t_2 is not in K, assume it is t_1. There exists then $v \in V$ such that $y(t_1 \wedge v) \neq 0$. Define φ on the basis t_1, t_2 of T by setting

$$\varphi(t_1) = 0$$
$$\varphi(t_2) = v.$$

Then $y(t) \neq 0$, and therefore the tangent vector to $\mathbf{G}$ at x corresponding to φ is not in H. We conclude that H is not tangent to $\mathbf{G}$ at x. □

1.6. From now on assume that $n = \dim V$ is odd. The Pfaffian locus $\mathbf{Pf} \subset \mathbf{P}^*$ is defined to be the locus of non-zero degenerate two-forms on V. (A form is called degenerate if its rank is less than $n-1$.) If we let A be a generic $n \times n$ matrix of linear forms on V, then $\mathbf{Pf}$ is cut out by the n Pfaffians of $(n-1) \times (n-1)$ diagonal minors of A (obtained by removing the i-th row and column of A, for $i = 1, \ldots, n$).

Since a point $y \in \mathbf{Pf}$ corresponds to a two-form on V, its kernel is a subspace $K \subset V$. If V is odd-dimensional, y being degenerate implies K is at least three-dimensional. Since we've eliminated the zero form by projectivizing, if $\dim V = 7$, $\dim K$ could only be three or five (the rank of y can be either four or two).

1.7. Proposition 1.5 shows that the Pfaffian is precisely the classical dual variety to $\mathbf{G}$: indeed, $H \cap \mathbf{G}$ is singular for a hyperplane H if and only if H is tangent at some point of $\mathbf{G}$, i.e., as a point of $\mathbf{P}^*$, it corresponds to a form with kernel of dimension ≥ 3.

1.8. Unlike the Grassmannian, which is smooth, the Pfaffian is singular at the points where the rank drops further (e.g., for $\dim V = 7$, when the rank is two). At singular points $y \in \mathbf{Pf}$, we have $T_{\mathbf{Pf},y} = T_{\mathbf{P}^*,y}$. If y is smooth, the following proposition describes the tangent space $T_{\mathbf{Pf},y} \subset T_{\mathbf{P}^*,y}$.

1.9. Proposition. *Let $y \in \mathbf{Pf}$ correspond to a degenerate two-form on V with kernel $K \subset V$ of dimension three. Identify, up to scalars, $T_{\mathbf{P}^*,y}$ with $\wedge^2 V^*/\langle y \rangle$, so that vectors in $T_{\mathbf{P}^*,y}$ are thought of as two-forms modulo y. Then $v \in T_{\mathbf{P}^*,y}$ is tangent to $\mathbf{Pf}$ at y if and only if K is isotropic for v, i.e.,*

$$v(k_1 \wedge k_2) = 0 \text{ for all } k_1, k_2 \in K.$$

Proof. Pick a nonzero element $y_0 \in \wedge^2 V^*$ in the cone over the Pfaffian that maps to y under projectivization. Since all rank $n-3$ skew forms on V are in the same orbit of the $GL(V)$ action, y_0 is contained in the dense open subset of the cone over the Pfaffian which is its orbit under the $GL(V)$ action. As a result, the tangent space to $\mathrm{Cone}(\mathbf{Pf})$ at y_0 can be viewed as the span of gy_0 for $g \in \mathrm{Lie}(GL(V)) = \mathrm{End}(V^*)$ under the natural action. In an appropriate basis of V^* one has $y_0 = x_1 \wedge x_2 + \ldots + x_{n-4} \wedge x_{n-3}$, and $gy_0 = gx_1 \wedge x_2 + x_1 \wedge gx_2 + \ldots + gx_{n-4} \wedge x_{n-3} + x_{n-4} \wedge gx_{n-3}$. Since $x_i \in \mathrm{Ann}(K)$, we see, after passing from $\mathrm{Cone}(\mathbf{Pf})$ to $\mathbf{Pf}$, that the tangent space to $\mathbf{Pf}$ is contained in the space of forms that make K isotropic. It is easy to see that the Pfaffian and this space are of the same dimension, which finishes the proof. □

2. Simultaneous smoothness

From now on, the space V is assumed to be seven-dimensional. In this section we argue that, for a given choice of seven-dimensional linear subspace

$$W \subset \wedge^2 V^*,$$

the hyperplane sections

$$X = \mathbf{M} \cap \mathbf{G}$$

and

$$Y = \mathbf{W} \cap \mathbf{Pf}$$

are either both smooth, or both singular, at least in the case when X and Y are of the (expected) dimension three.

2.1. In the previous section we noted that a point $x \in X$ corresponds to a plane $T \subset V$, while a point $y \in Y$, regarded as a two-form on V, has a kernel $K \subset V$, of dimension three or five.

Let $R \subset X \times Y$ denote the locus of pairs (x, y) for which $T \subset K$. Let π_X, π_Y be the projections from R to X and to Y, respectively.

2.2. Proposition. *The following statements hold:*

(a) the set of points $x \in X$ where $\dim T_{X,x} > 3$ coincides with the image $\pi_X(R)$;

(b) the set of points $y \in Y$ where $\dim T_{Y,y} > 3$ coincides with the image $\pi_Y(R)$.

Proof. (a) Let $x \in \mathbf{G}$ be a point on the Grassmannian, and y an arbitrary point in $\mathbf{P}^*$. Let H be the hyperplane in $\mathbf{P}$ corresponding to y. By Proposition 1.5, the following two statements are equivalent:

– H is tangent to $\mathbf{G}$ at x; and

– $\mathsf{T} \subset \mathsf{K} = \mathsf{Ker}(y)$.

Assume first that x is a point in X such that $\dim \mathsf{T}_{\mathsf{X},x} > 3$. Then $\mathsf{T}_{\mathsf{X},x}$, which is the intersection of $\mathsf{T}_{\mathbf{G},x}$ and $\mathbf{M}$ inside $\mathsf{T}_{\mathbf{P},x}$, has dimension higher than the expected

$$3 = 10 + 13 - 20.$$

Therefore $\mathsf{T}_{\mathbf{G},x}$ and $\mathbf{M}$ do not span all of $\mathsf{T}_{\mathbf{P},x}$, and there exists a hyperplane H inside $\mathsf{T}_{\mathbf{P},x}$ (and thus a hyperplane in $\mathbf{P}$ through x) containing both. Let y be the point in $\mathbf{P}^*$ corresponding to H.

Since H is tangent to $\mathbf{G}$ at x, $\mathsf{T} \subset \mathsf{K}$ by the claim. Therefore $y \in \mathbf{Pf}$ (because it has a kernel of dimension ≥ 2). Since we also have $\mathbf{M} \subset \mathsf{H}$, it follows that $y \in \boldsymbol{W}$, and therefore $y \in \mathsf{Y}$. The pair (x, y) is thus in R, and x is in the image $\pi_{\mathsf{X}}(\mathsf{R})$.

Conversely, assume that x in X is in the image of π_{X}, and let (x, y) be a point in R. Let H be the hyperplane in $\mathbf{P}$ corresponding to y. It then follows that H contains $\mathbf{M}$ (because $y \in \boldsymbol{W}$), and therefore $x \in \mathsf{H}$. The claim implies now that H is also tangent to $\mathbf{G}$ at x, and therefore the tangent space to H at x contains $\mathsf{T}_{\mathbf{G},x}$ and $\mathsf{T}_{\mathbf{M},x}$. Thus the intersection of these two spaces (which is $\mathsf{T}_{\mathsf{X},x}$) can not be of the expected dimension, and therefore $\dim \mathsf{T}_{\mathsf{X},x} > 3$.

(b) Let y be a point on Y. If the rank of y, regarded as a two-form on V, is two, then $\dim \mathsf{T}_{\mathsf{Y},y} > 3$ because the Pfaffian is already singular at y, and cutting it down by a codimension 14 hyperplane $\boldsymbol{W}$ will not cut down the dimension of the tangent space to three. Thus we need to show that y is in the image of R, in other words, that there exists a two dimensional space $\mathsf{T} \subset \mathsf{K}$ such that $x \in \mathsf{X}$. The kernel K is five-dimensional, and thus $\mathbf{G}(2, \mathsf{K})$ is a subvariety of $\mathbf{G}$ of dimension six, completely contained in the hyperplane in $\mathbf{P}$ corresponding to y. Completing $\{y\}$ to a basis of W gives rise to six more hyperplanes in $\mathbf{P}$, which must have a common point x in $\mathbf{G}(2, \mathsf{K})$. Thus x is in X (being on $\mathbf{G}$ and at the intersection of the seven hyperplanes corresponding to W), and $\mathsf{T} \subset \mathsf{K}$. It follows that $(x, y) \in \mathsf{R}$, thus y is in the image of π_{Y}.

We can assume thus that y is a smooth point of $\mathbf{Pf}$, the rank of y is four, and thus its kernel $\mathsf{K} \subset \mathsf{V}$ has dimension three. The tangent space $\mathsf{T}_{\mathbf{Pf},y}$ consists of all the tangent vectors in $\mathsf{T}_{\mathbf{P}^*,y}$ for which K is isotropic (Proposition 1.9). Explicitly, we have

$$\mathsf{T}_{\mathbf{P}^*,y} \cong (\wedge^2 \mathsf{V}^*)/\langle y \rangle,$$

and a two-form $\omega \in \wedge^2 \mathsf{V}^*$ will be tangent to $\mathbf{Pf}$ at y if and only if it vanishes on $\wedge^2 \mathsf{K}$. In other words, it is (modulo y) the subspace in $\wedge^2 \mathsf{V}^*$ whose annihilator is $\wedge^2 \mathsf{K}$.

The statement $\dim \mathsf{T}_{\mathsf{Y},y} > 3$ is equivalent to $\mathsf{T}_{\boldsymbol{W},y}$ and $\mathsf{T}_{\mathbf{Pf},y}$ not intersecting transversely in $\mathsf{T}_{\mathbf{P}^*,y}$, i.e., not spanning the full $\mathsf{T}_{\mathbf{P}^*,y}$. Since we have

$$\mathsf{Ann}(\mathsf{T}_{\boldsymbol{W},y}) \cap \mathsf{Ann}(\mathsf{T}_{\mathbf{Pf},y}) = \mathsf{Ann}(\langle \mathsf{T}_{\boldsymbol{W},y}, \mathsf{T}_{\mathbf{Pf},y} \rangle),$$

it follows that this is equivalent to the existence of a non-zero

$$x \in \mathsf{Ann}(\mathsf{T}_{\boldsymbol{W},y}) \cap \mathsf{Ann}(\mathsf{T}_{\mathbf{Pf},y}) \cap \mathsf{Ann}(y).$$

Being in $\mathsf{Ann}(\mathsf{T}_{\boldsymbol{W},y}) \cap \mathsf{Ann}(y)$ is equivalent to being in $\mathbf{M}$, while being in $\mathsf{Ann}(\mathsf{T}_{\mathbf{Pf},y})$ is equivalent to being a point on $\mathbf{G}$ for which $\mathsf{T} \subset \mathsf{K}$. Thus $\dim \mathsf{T}_{\mathsf{Y},y} > 3$ is equivalent to the existence of an $x \in \mathsf{X}$ such that $(x, y) \in \mathsf{R}$. $\square$

2.3. Corollary. *Assume either* X *or* Y *has dimension three. Then* X *is smooth if and only if* Y *is smooth.*

Proof. Both statements are equivalent to $R = \emptyset$. □

2.4. Remark. In [11], Rødland argues that $h^{1,1}(X) = h^{1,1}(Y) = 1$ and $h^{1,2}(X) = h^{1,2}(Y) = 50$ for generic cuts. In fact, this statement holds whenever X and Y are smooth: indeed, the family of such cuts is smooth over the appropriate open subset of $\mathbf{G}(7, \wedge^2 V^*)$, and thus all fibers are diffeomorphic.

3. A family of curves

In this section we define, for a point $y \in Y$, a curve C_y in X. (We abuse the notation slightly: C_y may not be reduced or irreducible, but it does have dimension one.) The family $\{\mathscr{I}_{C_y}\}_{y \in Y}$ of ideal sheaves of these curves is the orthogonal family which induces the equivalence of derived categories $\mathbf{D}^b_{coh}(X) \cong \mathbf{D}^b_{coh}(Y)$. We then argue that $\dim C_y = 1$ for every choice of $y \in Y$ (which is essential in proving that the family of C_y's is flat), and that

$$y_1 \neq y_2 \implies \mathrm{Hom}_X(\mathscr{I}_{C_{y_1}}, \mathscr{I}_{C_{y_2}}) = 0,$$

which is later needed for the orthogonality of the family.

Most of our statements depend on the assumption that X and Y are smooth. Therefore, for the remainder of this paper we shall assume that a space $W \subset \wedge^2 V^*$ is chosen in such a way that Y (and therefore X) is smooth.

3.1. The curve C_y corresponding to a point $y \in Y$ is defined as the scheme-theoretic intersection of a certain Schubert cycle $S_y \subset \mathbf{G}$ with X. We begin by analyzing S_y itself.

Let $y \in \mathbf{Pf}$ be a smooth point, which is thought of as a two-form on V of rank four. Let K_y be the kernel of this form, a linear three-space in V. We define the locus $S_y \subset \mathbf{G}$ to be the set of two-planes $T \subset V$ (i.e., points $T \in \mathbf{G}$) which intersect K_y non-trivially. Note that this is precisely the Schubert cycle corresponding to the increasing sequence $(0, 3, 7)$.

3.2. Proposition. *Let* $K \subset V$ *be a linear subspace. Regard* $\wedge^2 \mathrm{Ann}\, K \subset \wedge^2 V^*$ *as a set of linear equations on* $\mathbf{P}$*. Then the set of two-planes* $T \subset V$ *such that* $T \cap K \neq 0$ *is precisely the set of closed points in* $\mathbf{G}$ *cut out by* $\wedge^2 \mathrm{Ann}\, K$*.*

Proof. We have the following sequence of equivalent statements:

- $T \cap K \neq 0$;
- the image $\overline{T}$ of T in V/K has dimension at most one;
- $\wedge^2 \overline{T} = 0$ in $\wedge^2 V/K$;
- for every $\overline{w} \in \wedge^2 (V/K)^*$ we have $\overline{w}(\wedge^2 \overline{T}) = 0$;
- for every $w \in \wedge^2 \mathrm{Ann}\, K$ we have $w(\wedge^2 T) = 0$.

(Note that the image of the map $\wedge^2 (V/K)^* \to \wedge^2 V^*$ is precisely $\wedge^2 \mathrm{Ann}\, K$.) □

3.3. For a given $y \in \mathbf{Pf}$ of rank four, apply the above proposition to $K_y \subset V$. The resulting locus S_y is then a closed subset, and we endow it with the reduced induced scheme structure.

3.4. Proposition. *For every $y \in \mathbf{Pf}$ of rank four with kernel K_y, the corresponding Schubert cycle S_y is cut out scheme-theoretically by $\bigwedge^2 \operatorname{Ann} K_y$. It is a rational variety with rational Cohen-Macaulay singularities, of codimension three in $\mathbf{G}$.*

Proof. In view of the $GL(V)$ action, we can assume without loss of generality that $y = x_1 \wedge x_2 + x_3 \wedge x_4$, where $\{x_1, \ldots, x_7\}$ is a basis of V^*. Let $\{e_1, \ldots, e_7\}$ be the dual basis of V. In the Zariski open subspace $U \cong \mathbf{C}^{10} \subseteq \mathbf{G}$ in which points T are given by

$$T = \operatorname{Span}(a_{11}e_1 + \ldots + a_{15}e_5 + e_6, a_{21}e_1 + \ldots + a_{25}e_5 + e_7),$$

the cycle S_y is characterized by the condition that the matrix $R = \{a_{ij}\}$, $i = 1, 2$, $j = 1, \ldots, 4$ has rank one. The equations from $\bigwedge^2 \operatorname{Ann} K_y$ are spanned by $x_i \wedge x_j$ which are precisely the maximal minors of the matrix R. It is a well-known fact that this determinantal variety is reduced and irreducible, and it coincides with the product of $\mathbf{C}^2$ and the cone over the Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^3$. Thus $S_y \cap U$ is cut out by $\bigwedge^2 \operatorname{Ann} K_y$ and is rational. It remains to observe that every point in $\mathbf{G}$ can be mapped inside U by an element of $GL(V)$ that fixes K_y.

The singularities of S_y are rational Cohen-Macaulay since $S_y \cap U$ is a toric variety. □

3.5. Remark. One of the equations of S_y can be taken to be y itself, since $y \in \bigwedge^2 \operatorname{Ann} K_y$.

3.6. For example, if $e_1, \ldots, e_7$ is a basis of V, $x_1, \ldots, x_7$ the dual basis of V^*, $y = x_1 \wedge x_2 + x_3 \wedge x_4$, then K is spanned by e_5, e_6, e_7, $\operatorname{Ann} K$ by x_1, x_2, x_3, x_4, and the six equations cutting out S_y from $\mathbf{G}$ are given by

$$x_i \wedge x_j = 0, \ 1 \leq i < j \leq 4.$$

(These are linear conditions on $\mathbf{P}$.)

3.7. Proposition. *There exists a subvariety $\mathbf{S}$ of $\mathbf{G} \times \mathbf{Pf}^{sm}$, flat over $\mathbf{Pf}^{sm}$, whose fiber $\mathbf{S}_y \subset \mathbf{G}$ over any closed point $y \in \mathbf{Pf}^{sm}$ is precisely S_y.*

Proof. Fix a point $y_0 \in \mathbf{Pf}^{sm}$, and let S_0 be the corresponding subvariety of $\mathbf{G}$. The group $GL(V)$ acts transitively on both $\mathbf{G}$ and $\mathbf{Pf}^{sm}$. One can take $\mathbf{S}$ to be the image of $GL(V) \times S_0$ under this action, with the reduced-induced scheme structure. The flatness statement follows from [7, Theorem 9.9], as all translates of S_0 have the same Hilbert polynomial in $\mathbf{G}$. □

3.8. Lemma. *Let $W \subset \bigwedge^2 V^*$ be a seven-subspace, and assume that $Y = \mathbf{W} \cap \mathbf{Pf}$ has dimension three. If there are points w_1, w_2 in Y whose kernels (when regarded as two-forms on V) are the same, then there exists a point $w \in Y$ of rank two. Therefore Y is then singular.*

Proof. Let K be the common kernel of w_1 and w_2. All the linear combinations of w_1 and w_2 have K contained in their kernel, so we can think of them as a one-dimensional family of forms on the four-dimensional space V/K. At some point w in this family the rank will drop to two, as this is a codimension one condition. At w the space $\mathbf{Pf}$ is singular, and therefore the regular cut Y is singular at w as well. □

3.9. Lemma. *Assume that X is smooth of dimension three. Then the space of linear forms $w \in \mathbf{P}^*$ which vanish along all of X is precisely W.*

Proof. Another way to phrase this assertion is by saying that W is the kernel of the map

$$\rho : H^0(\mathbf{P}, \mathscr{O}_{\mathbf{P}}(1)) \to H^0(X, \mathscr{O}_X(1)).$$

Since the space W is already contained in the kernel of ρ, it suffices to show that $\dim \operatorname{Ker} \rho = 7$. Factor ρ into the composition

$$\mathsf{H}^0(\mathbf{P}, \mathscr{O}_{\mathbf{P}}(1)) \to \mathsf{H}^0(\mathbf{G}, \mathscr{O}_{\mathbf{G}}(1)) \to \mathsf{H}^0(\mathsf{X}, \mathscr{O}_{\mathsf{X}}(1)),$$

where it is well-known that the first map is an isomorphism.

Consider the ideal sheaf $\mathscr{I}_{\mathsf{X}}$ of X in $\mathbf{G}$. Its twist by one has the Koszul resolution

$$0 \to \mathscr{O}_{\mathbf{G}}(-6) \to \cdots \to \mathscr{O}_{\mathbf{G}}(-1)^{21} \to \mathscr{O}_{\mathbf{G}}^7 \to \mathscr{I}_{\mathsf{X}}(1) \to 0,$$

where the sheaves $\mathscr{O}_{\mathbf{G}}(-1), \ldots, \mathscr{O}_{\mathbf{G}}(-6)$ are acyclic with no global sections. It follows that

$$\mathsf{H}^0(\mathbf{G}, \mathscr{I}_{\mathsf{X}}(1)) = \mathsf{H}^0(\mathbf{G}, \mathscr{O}_{\mathbf{G}}^7) = \mathbf{C}^7.$$

From this and the short exact sequence

$$0 \to \mathscr{I}_{\mathsf{X}}(1) \to \mathscr{O}_{\mathbf{G}}(1) \to \mathscr{O}_{\mathsf{X}}(1) \to 0$$

it follows that $\dim \operatorname{Ker} \rho = 7$. □

3.10. Let $\mathbf{C}$ be the scheme theoretic intersection of $\mathbf{S}$ with $\mathsf{X} \times \mathsf{Y}$ inside $\mathbf{G} \times \mathbf{Pf}$, regarded as a subscheme of $\mathsf{X} \times \mathsf{Y}$. The sheaf $\mathscr{I}_{\mathbf{C}}$, the ideal sheaf of $\mathbf{C}$ in $\mathsf{X} \times \mathsf{Y}$, regarded as an object of $\mathbf{D}^{\mathsf{b}}_{\mathsf{coh}}(\mathsf{X} \times \mathsf{Y})$, will be the kernel of the Fourier-Mukai transform $\mathbf{D}^{\mathsf{b}}_{\mathsf{coh}}(\mathsf{Y}) \to \mathbf{D}^{\mathsf{b}}_{\mathsf{coh}}(\mathsf{X})$.

3.11. Proposition. *The fiber of* $\mathbf{C}$ *over any closed point* $y \in \mathsf{Y}$ *has dimension one. The scheme* $\mathbf{C}$ *is flat over* Y.

Proof. By an appropriate version of Bertini's theorem, we can choose a sequence of cuts going from $\mathbf{G}$ to X so that all the intermediate members are smooth. Lefschetz's theorem then gives $\operatorname{Pic} \mathsf{X} = \mathbf{Z}$, generated by the restriction of $\mathscr{O}_{\mathbf{G}}(1)$. For a point $y \in \mathsf{Y}$, C_y is cut out from X by the same six linear equations that cut out S_y from $\mathbf{G}$, i.e., C_y is the result of intersecting X with the hyperplanes corresponding to the points of the six-dimensional linear space $\mathsf{L} = \bigwedge^2 \operatorname{Ann} \operatorname{Ker} y$. Every form $w \in \mathsf{L}$ contains $\operatorname{Ker} y$ in its kernel.

Assume by contradiction that C_y contains a divisor D. We claim that the intersection $\mathsf{H} \cap \mathsf{X}$ is either D, or all of X, for every hyperplane H in L. Indeed, any non-trivial intersection $\mathsf{H} \cap \mathsf{X}$ is the generator of $\operatorname{Pic} \mathsf{X}$, and as such it is a prime divisor on X. Therefore any such cut must equal D, since it already contains it.

Pick a point P in $\mathsf{X} \setminus \mathsf{D}$. The condition that H contain P is a single linear condition on L, thus there is a five-dimensional space R of forms in L that vanish at P. The above discussion shows that if a form in L vanishes at a point outside D, then it vanishes on all of X. Thus the forms in R vanish on all of X.

By Lemma 3.9, then, we have $\mathsf{R} \subset W$. Any two linearly-independent forms of rank four in R yield two points in Y whose kernel (as forms) is K. By Lemma 3.8, this contradicts the assumption that Y is smooth. Therefore our assumption that C_y contains a divisor must be false.

The subvarieties $\mathsf{S}_y \subset \mathbf{G}$ for $y \in \mathsf{Y}$ are all isomorphic, and they have the same Hilbert polynomial (in fact, there is a transitive $\mathsf{GL}(\mathsf{V})$-action permuting them). If we let $y, y_1, \ldots, y_6$ be a basis of W, them C_y is obtained from S_y by six cuts with hyperplanes corresponding to $y_1, \ldots, y_6$. Since C_y is a curve (codimension six in W), it follows that this is a regular sequence of cuts, and thus the Hilbert polynomial of C_y does not depend on $y \in \mathsf{Y}$. Therefore the family $\{\mathsf{C}_y\}_{y \in \mathsf{Y}}$ is flat over Y, by [7, III.9.9]. □

3.12. Proposition. *For every curve* C_y *the kernel* $H^0(\mathbf{G}, \mathscr{I}_{C_y \subset \mathbf{G}}(1))$ *of the natural restriction map*

$$\varphi : \wedge^2 V^* \cong H^0(\mathbf{G}, \mathscr{O}_{\mathbf{G}}(1)) \to H^0(\mathbf{G}, \mathscr{O}_{C_y}(1))$$

is the space of dimension 12 *given by* $W + \wedge^2 \operatorname{Ann} \operatorname{Ker} y$.

Proof. The space $W + \wedge^2 \operatorname{Ann} \operatorname{Ker} y$ is clearly in the kernel of φ. It has dimension 12, since by Lemma 3.8 $W \cap \wedge^2 \operatorname{Ann} \operatorname{Ker} y$ is one-dimensional, spanned by y. Thus, it suffices to show that φ is surjective and $\dim H^0(X, \mathscr{O}_{C_y}(1)) = 9$.

We factor φ into the composition

$$H^0(\mathbf{G}, \mathscr{O}_{\mathbf{G}}(1)) \xrightarrow{\varphi_1} H^0(S_y, \mathscr{O}(1)) \xrightarrow{\varphi_2} H^0(X, \mathscr{O}_{C_y}(1)).$$

The kernel of φ_1 is $\wedge^2 \operatorname{Ann} \operatorname{Ker} y$ by Proposition 3.2.

We will now argue that

$$\dim H^0(S_y, \mathscr{O}(1)) = 15$$

by finding an explicit resolution of singularities of S_y. Let $\tilde{S}_y$ be the set of pairs (T_1, T_2) such that

- T_i is a dimension i subspace of V;
- $T_1 \subset \operatorname{Ker} y$;
- $T_1 \subset T_2$.

This set has a natural structure of smooth algebraic variety: it is the projectivization of the rank six vector bundle $(V \otimes \mathscr{O})/\mathscr{O}(-1)$ over $\mathbf{P}(\operatorname{Ker} y)$.

We have the following diagram

$$\begin{array}{ccc} \tilde{S}_y & \xrightarrow{\mu} & \mathbf{P}(\operatorname{Ker} y) \\ \pi \big\downarrow & & \\ S_y, & & \end{array}$$

where the map π, defined by forgetting T_1, is a resolution of singularities. Since S_y has rational singularities (which can be seen explicitly from this resolution), we have

$$H^0(S_y, \mathscr{O}(1)) = H^0(\tilde{S}_y, \pi^* \mathscr{O}(1)).$$

We can calculate $H^0(\tilde{S}_y, \pi^* \mathscr{O}(1))$ as $H^0(\mathbf{P}(\operatorname{Ker} y), \mu_* \pi^* \mathscr{O}(1))$. The sheaf $\mu_* \pi^* \mathscr{O}(1)$ is seen to equal

$$\Big(V \otimes \mathscr{O}/\mathscr{O}(-1)\Big)^*(1),$$

which leads to the short exact sequence on $\mathbf{P}(\operatorname{Ker} y)$

$$0 \to \mu_* \pi^* \mathscr{O}(1) \to V^* \otimes \mathscr{O}(1) \to \mathscr{O}(2) \to 0.$$

The corresponding map on global sections $V^* \otimes (\operatorname{Ker} y)^* \to \operatorname{Sym}^2(\operatorname{Ker} y)^*$ is the dual of the multiplication map, and is thus surjective. Consequently, $H^0(\mu_* \pi^* \mathscr{O}(1))$ has dimension $7 \cdot 3 - 6 = 15$. This allows us to conclude that φ_1 is a surjective map.

Now it suffices to show that the map

$$\varphi_2 : H^0(S_y, \mathscr{O}(1)) \to H^0(C_y, \mathscr{O}(1))$$

is surjective and has kernel of dimension six. Consider the ideal sheaf $\mathscr{I} = \mathscr{I}_{C_y \subset S_y}$. From the short exact sequence

$$0 \to \mathscr{I}(1) \to \mathscr{O}_{S_y}(1) \to \mathscr{O}_{C_y}(1) \to 0,$$

it suffices to show that $h^0(S_y, \mathscr{I}(1)) = 6$ and $h^1(S_y, \mathscr{I}(1)) = 0$. Since S_y is Cohen-Macaulay of dimension seven, and C_y is obtained by six hyperplane cuts and has dimension one, we have the Koszul resolution

$$0 \to \mathscr{O}(-6) \to \mathscr{O}(-5)^6 \to \ldots \to \mathscr{O} \to \mathscr{O}_{C_y} \to 0.$$

After truncating and twisting, this gives a resolution

$$0 \to \mathscr{O}(-5) \to \mathscr{O}(-4)^6 \to \ldots \to \mathscr{O}^6 \to \mathscr{I}(1) \to 0.$$

We denote by F_i the image of the map from $\mathscr{O}(-i)^\bullet$ in the above complex. We use the fact that S_y has rational singularities and Kawamata vanishing on S_y to see that $F_5 = \mathscr{O}(-5)$ has $h^{\leq 6} = 0$. This implies that F_4 has $h^{\leq 5} = 0$, and so on. Finally, we see that F_1 has $h^0 = h^1 = h^2 = 0$. There is a long exact sequence

$$0 \to H^0(F_1) \to H^0(\mathscr{O}^6) \to H^0(\mathscr{I}(1)) \to H^1(F_1) \to H^1(\mathscr{O}^6) \to H^1(\mathscr{I}(1)) \to H^2(F_1)$$

and we get our result by observing that $H^1(\mathscr{O}) = 0$, since S_y is rational with rational singularities. □

3.13. Proposition. *Let y_1 and y_2 be two distinct points in Y. Then C_{y_1} is not a subscheme of C_{y_2} (in other words $\mathscr{I}_{C_1}$ does not contain $\mathscr{I}_{C_2}$, as ideal sheaves on X).*

Proof. Assume that $C_{y_1} \subseteq C_{y_2}$. Therefore $\mathscr{I}_{C_{y_1} \subset \mathbf{G}} \supseteq \mathscr{I}_{C_{y_2} \subset \mathbf{G}}$, and thus

$$H^0(\mathbf{G}, \mathscr{I}_{C_{y_1} \subset \mathbf{G}}) \supseteq H^0(\mathbf{G}, \mathscr{I}_{C_{y_2} \subset \mathbf{G}}).$$

By Proposition 3.12, this implies

$$W + \wedge^2 \operatorname{Ann} \operatorname{Ker} y_1 \supseteq W + \wedge^2 \operatorname{Ann} \operatorname{Ker} y_2.$$

Since both spaces have dimension 12, the above inclusion must actually be an equality. Denote this space by W_{12}. Note that

$$(\wedge^2 \operatorname{Ann} \operatorname{Ker} y_1) \cap (\wedge^2 \operatorname{Ann} \operatorname{Ker} y_2) = \wedge^2 \operatorname{Ann}(\operatorname{Ker} y_1 + \operatorname{Ker} y_2)$$

has dimension zero, one, or three depending on whether $\dim(\operatorname{Ker} y_1 \cap \operatorname{Ker} y_2)$ is zero, one, or two respectively (we can't have $\operatorname{Ker} y_1 = \operatorname{Ker} y_2$ for smooth Y by Lemma 3.8).

As a result, the space W_{12} is a sum of

$$(\wedge^2 \operatorname{Ann} \operatorname{Ker} y_1) + (\wedge^2 \operatorname{Ann} \operatorname{Ker} y_2)$$

and a space of dimension zero, one, or three in the respective cases above. It is easy to observe that the subscheme Z in $\mathbf{G}$ cut out by

$$(\wedge^2 \operatorname{Ann} \operatorname{Ker} y_1) \cup (\wedge^2 \operatorname{Ann} \operatorname{Ker} y_2)$$

has dimension at least four, five, or six in the respective cases. Indeed, in the first case, we can construct $T \in Z$ by taking an element from each $\operatorname{Ker} y_i$, which gives $\mathbf{P}^2 \times \mathbf{P}^2 \subseteq Z$. In the second case we can take $T \in Z$ to contain the intersection of $\operatorname{Ker} y_i$ to get $\mathbf{P}^5 \subseteq Z$, and in the last case we can get a $\mathbf{P}^5$ bundle over $\mathbf{P}^1$, with a small contraction by looking at T that intersect $\operatorname{Ker} y_1 \cap \operatorname{Ker} y_2$ nontrivially. The dimension of the subscheme in $\mathbf{G}$ cut out by W_{12} is then at least four, four, or three, respectively, but in fact W_{12} cuts out C, contradiction. □

3.14. Proposition. *Let y_1, y_2 be points in Y. Then we have*

$$\operatorname{Hom}_X(\mathscr{I}_{C_{y_1}}, \mathscr{I}_{C_{y_2}}) = \begin{cases} \mathbf{C} & \text{if } y_1 = y_2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. From the short exact sequence

$$0 \to \mathscr{I}_{C_2} \to \mathscr{O}_X \to \mathscr{O}_{C_2} \to 0$$

we get

$$0 \to \mathbf{C} \to \operatorname{Hom}_X(\mathscr{I}_{C_2}, \mathscr{O}_X) \to \operatorname{Ext}^1_X(\mathscr{O}_{C_2}, \mathscr{O}_X) = H^2(X, \mathscr{O}_{C_2})^* = 0.$$

Thus any non-zero homomorphism $\mathscr{I}_{C_2} \to \mathscr{O}_X$ is, up to a scalar multiple, the usual inclusion.

Let $f : \mathscr{I}_{C_1} \to \mathscr{I}_{C_2}$ be any non-zero homomorphism. Composing with the inclusion $\mathscr{I}_{C_2} \hookrightarrow \mathscr{O}_X$ we get a non-zero map $\mathscr{I}_{C_1} \to \mathscr{O}_X$ which, by the above reasoning, must be the usual inclusion of $\mathscr{I}_{C_1}$ into $\mathscr{O}_X$. Therefore f is injective. If $y_1 \neq y_2$, this is impossible by Proposition 3.13. If $y_1 = y_2$, it follows that f is a multiple of the identity map. □

4. A reduction to the Grassmannian

In this section we argue that checking the orthogonality of the family $\{\mathscr{I}_{C_y}\}_{y \in Y}$ can be reduced to a few global Ext computations on the Grassmannian $\mathbf{G}$. (These computations will be carried out using the software package Macaulay 2 [10], see Appendix A.)

4.1. We assume that we have chosen W so that X and Y are smooth of dimension three. Let y_1 and y_2 be two distinct points in Y. Regarded as smooth points of the Pfaffian $\mathbf{Pf}$, y_1 and y_2 give rise to Schubert cycles S_1 and S_2 in $\mathbf{G}$ as explained in (3.1). These are rational, integral, codimension three subschemes of $\mathbf{G}$, whose singularities are rational and Cohen-Macaulay.

Let C_1 and C_2 be the corresponding curves in X obtained by taking scheme-theoretic intersections of S_1 and S_2 with X, respectively, as defined in Section 3.

4.2. Notation. Throughout this section, if S is a subscheme of a scheme Z, we shall denote by $\mathscr{I}_S$ the ideal sheaf of S, regarded as a coherent sheaf on Z. We shall always make sure to be precise, if Z is itself a subscheme of another scheme Z', which scheme is S regarded as a subscheme of, Z or Z'.

4.3. Now assume further that y_1 and y_2 are in generic or subgeneric position, i.e., we have

$$\dim(\operatorname{Ker} y_1 \cap \operatorname{Ker} y_2) \leq 1.$$

The main computational result which we shall obtain using Macaulay 2 is the fact that, under this assumption, we have

$$\operatorname{Ext}^{j+2}_{\mathbf{G}}(\mathscr{I}_{S_1}, \mathscr{I}_{S_2}(-j-1)) = 0 \text{ for } 0 \leq j \leq 5.$$

Note that this is a computation which has nothing to do with X and Y themselves. It only depends on the choice of two smooth points on $\mathbf{Pf}$ in generic or subgeneric position.

As a shorthand, we shall say that $\mathbf{G}$-vanishing holds if the above Ext groups vanish.

4.4. Proposition. *Assuming* $\mathbf{G}$*-vanishing holds, then we have*

$$\operatorname{Ext}^1_X(\mathscr{I}_{C_1}, \mathscr{I}_{C_2}) = 0.$$

4.5. Before we begin the proof of Proposition 4.4 we need several intermediate vanishing results.

4.6. Lemma. *For* $0 \leq j \leq 5$ *we have*

$$H^{j+1}(\mathbf{G}, \mathscr{I}_{S_2}(-j)) = 0.$$

Proof. Consider the short exact sequence

$$0 \to \mathscr{I}_{S_2} \to \mathscr{O}_{\mathbf{G}} \to \mathscr{O}_{S_2} \to 0.$$

The cohomology groups

$$H^{j+1}(\mathbf{G}, \mathscr{O}_{\mathbf{G}}(-j))$$

vanish for $0 \leq j \leq 6$: for $j = 0$ this is a consequence of the rationality of $\mathbf{G}$, and for $1 \leq j \leq 6$ this is Kodaira vanishing. Since S_2 is irreducible, we conclude that

$$H^1(\mathbf{G}, \mathscr{I}_{S_2}) = 0,$$

which is the $j = 0$ case, and for $1 \leq j \leq 5$ we have

$$H^{j+1}(\mathbf{G}, \mathscr{I}_{S_2}(-j)) = H^j(S_2, \mathscr{O}_{S_2}(-j)).$$

Let $\pi : \tilde{S}_2 \to S_2$ be a resolution of singularities of S_2. Since the singularities of S_2 are rational, we have

$$H^j(S_2, \mathscr{O}_{S_2}(-j)) = H^j(\tilde{S}_2, \pi^* \mathscr{O}_{S_2}(-j)).$$

By the Kawamata vanishing theorem ($\pi^* \mathscr{O}_{\tilde{S}_2}(1)$ is nef and big) these groups vanish for $1 \leq j \leq 5$. $\square$

4.7. We now divide the proof of Proposition 4.4 into several steps, corresponding to decreasing subvarieties of $\mathbf{G}$ obtained by successive cuts with hyperplanes of increasing codimension, corresponding to subspaces of $\boldsymbol{W}$ of increasing dimension.

To begin, let H_1 be the hyperplane in $\mathbf{P}$ corresponding to y_1, and let Z_1 be the scheme-theoretic intersection of $\mathbf{G}$ and H_1. It is a hypersurface of $\mathbf{G}$ whose singularities are locally isomorphic to the product of $\mathbf{C}^2$ and an ordinary double point of dimension seven. Consider L_1 and L_2 which are the intersection of S_1 and S_2 with H_1, regarded as subschemes of Z_1. While H_1 cuts down the dimension of S_2 by one, it already contains S_1. (The fact that H_1 cuts down the dimension of S_2 follows from Proposition 3.11.) Thus L_1 and L_2 have codimensions two and three, respectively, in Z_1.

4.8. Proposition. *Assuming* $\mathbf{G}$*-vanishing, we also have* Z_1*-vanishing:*

$$\mathrm{Ext}^{j+1}_{Z_1}(\mathscr{I}_{L_1}, \mathscr{I}_{L_2}(-j)) = 0 \text{ for } 0 \le j \le 5.$$

Proof. Let $r : Z_1 \to \mathbf{G}$ denote the natural embedding. Since L_2 is obtained by a transversal cut of S_2 by H_1, we have $\mathscr{I}_{L_2} = r^*\mathscr{I}_{S_2}$. By an easy form of Grothendieck duality it follows that

$$\begin{aligned}\mathrm{Ext}^{j+1}_{Z_1}(\mathscr{I}_{L_1}, \mathscr{I}_{L_2}(-j)) &= \mathrm{Ext}^{j+1}_{Z_1}(\mathscr{I}_{L_1}, r^*\mathscr{I}_{S_2}(-j)) \\ &= \mathrm{Ext}^{j+1}_{\mathbf{G}}(r_!\mathscr{I}_{L_1}, \mathscr{I}_{S_2}(-j)) \\ &= \mathrm{Ext}^{j+2}_{\mathbf{G}}(r_*\mathscr{I}_{L_1}, \mathscr{I}_{S_2}(-j-1)),\end{aligned}$$

since the embedding $Z_1 \subset \mathbf{G}$ has relative dualizing complex $\mathscr{O}_{Z_1}(1)[-1]$.

Consider the short exact sequence of sheaves on $\mathbf{G}$

$$0 \to \mathscr{I}_{Z_1} = \mathscr{O}_{\mathbf{G}}(-1) \to \mathscr{I}_{S_1} \to r_*\mathscr{I}_{L_1} \to 0.$$

Writing down the long exact sequence of Ext's we get the result from $\mathbf{G}$-vanishing and the vanishing of

$$\mathrm{Ext}^{j+1}_{\mathbf{G}}(\mathscr{I}_{Z_1}, \mathscr{I}_{S_2}(-j-1)) = H^{j+1}(\mathbf{G}, \mathscr{I}_{S_2}(-j)),$$

which is Lemma 4.6. □

4.9. Lemma. *For* $0 \le j \le 5$ *we have*

$$H^{j+2}(Z_1, \mathscr{I}_{L_1}(-j)) = 0.$$

Proof. The proof is essentially the same as that of Lemma 4.6 and will be left to the reader. One needs to use the fact that Z_1 is rational with rational singularities. □

4.10. Lemma. *Let* Z *be a scheme,* $\mathscr{E}$ *and* $\mathscr{F}$ *coherent sheaves on* Z*, and assume given a long exact sequence*

$$0 \to \mathscr{F}^n \to \mathscr{F}^{n-1} \to \cdots \to \mathscr{F}^0 \to \mathscr{F} \to 0$$

such that

$$\mathrm{Ext}^{i+j}_{Z}(\mathscr{E}, \mathscr{F}^j) = 0$$

for a fixed value of i *and for* $0 \le j \le n$. *Then we have*

$$\mathsf{Ext}^i_Z(\mathscr{E}, \mathscr{F}) = 0.$$

Proof. Follows by splitting the long exact sequence of $\mathscr{F}$'s into short exact sequences, and writing down the associated long exact sequence of Ext's for each one. □

4.11. We can now complete the proof of Proposition 4.4. Let H_2 be the hyperplane in $\mathbf{P}$ corresponding to y_2, and let $Z_{1,2} = \mathbf{G} \cap H_1 \cap H_2$. Let D_1, D_2 be the intersections of S_1 and S_2 with $Z_{1,2}$, regarded as subschemes of $Z_{1,2}$. Note that since $S_2 \subset H_2$, we have $D_2 = L_2$; both D_1 and D_2 are now codimension two in $Z_{1,2}$.

Let W_5 be any linear subspace of W which, together with the one-dimensional subspaces corresponding to y_1 and y_2, spans W. Because of dimension reasons, X, C_1, and C_2 are obtained from $Z_{1,2}$, D_1, and D_2 by five successive transversal cuts with five hyperplanes corresponding to a basis of W_5.

The strategy of the proof is to look at the successive embeddings

$$X \xrightarrow{g} Z_{1,2} \xrightarrow{h} Z_1 \xrightarrow{r} \mathbf{G},$$

and to use the vanishing of Ext groups on each one to conclude the vanishing of other Ext groups on the previous one. We have already argued (Proposition 4.8) that $\mathbf{G}$-vanishing implies Z_1-vanishing. The passage from Z_1-vanishing to the required vanishing on X will be done by an appropriate resolution of $h_* g_* \mathscr{I}_{C_2}$ on Z_1.

We have

$$\mathscr{I}_{C_1} = g^* \mathscr{I}_{D_1},$$

thus

$$\mathsf{Ext}^1_X(\mathscr{I}_{C_1}, \mathscr{I}_{C_2}) = \mathsf{Ext}^1_X(g^* \mathscr{I}_{D_1}, \mathscr{I}_{C_2}) = \mathsf{Ext}^1_{Z_{1,2}}(\mathscr{I}_{D_1}, g_* \mathscr{I}_{C_2}),$$

and

$$g_* \mathscr{I}_{C_2} = \mathscr{I}_{D_2} \otimes_{Z_{1,2}} \mathscr{O}_X.$$

Because the intersection of D_2 and X inside $Z_{1,2}$ is transversal, there are no higher Tor's in the above tensor product. We conclude that $g_* \mathscr{I}_{C_2}$ is quasi-isomorphic to the complex

$$\begin{aligned} &\mathscr{I}_{D_2} \otimes \mathsf{Koszul}^{Z_{1,2}}(W_5) = \\ &\qquad (0 \to \mathscr{I}_{D_2}(-5) \otimes \wedge^5 W_5 \to \cdots \to \mathscr{I}_{D_2}(-1) \otimes W_5 \to \mathscr{I}_{D_2} \to 0). \end{aligned}$$

Now let us move on to the embedding $h : Z_{1,2} \to Z_1$. Again,

$$\mathsf{Ext}^1_{Z_{1,2}}(\mathscr{I}_{D_1}, g_* \mathscr{I}_{C_2}) = \mathsf{Ext}^1_{Z_{1,2}}(h^* \mathscr{I}_{L_1}, g_* \mathscr{I}_{C_2}) = \mathsf{Ext}^1_{Z_1}(\mathscr{I}_{L_1}, h_* g_* \mathscr{I}_{C_2}).$$

We use the previous resolution of $g_* \mathscr{I}_{C_2}$ to compute a resolution of $h_* g_* \mathscr{I}_{C_2}$ on Z_1. The main difference is that $\mathscr{I}_{D_2}$ is not obtained from $\mathscr{I}_{L_2}$ by a transversal intersection. In fact, L_2 is already contained in H_2, so we have a short exact sequence on Z_1

$$0 \to \mathscr{I}_{Z_{1,2}} = \mathscr{O}_{Z_1}(-1) \to \mathscr{I}_{L_2} \to h_* \mathscr{I}_{D_1} \to 0,$$

and $h_*\mathscr{I}_{D_1}$ is quasi-isomorphic to the two-term complex

$$0 \to \mathscr{O}_{Z_1}(-1) \to \mathscr{I}_{L_2} \to 0.$$

Observe that the Koszul complex restricts from Z_1 to $Z_{1,2}$, because the five cuts from W_5 are transversal to Z_1 and $Z_{1,2}$. In other words,

$$h^* \operatorname{Koszul}^{Z_1}(W_5) = \operatorname{Koszul}^{Z_{1,2}}(W_5).$$

The projection formula for derived categories implies that

$$\begin{aligned} h_* g_* \mathscr{I}_{C_2} &= h_*(\operatorname{Koszul}^{Z_{1,2}}(W_5) \otimes \mathscr{I}_{D_1}) = h_*(h^* \operatorname{Koszul}^{Z_1}(W_5) \otimes \mathscr{I}_{D_1}) \\ &= \operatorname{Koszul}^{Z_1}(W_5) \otimes h_* \mathscr{I}_{D_1} = \operatorname{Koszul}^{Z_1}(W_5) \otimes (\mathscr{O}_{Z_1}(-1) \to \mathscr{I}_{L_2}). \end{aligned}$$

All operations above are to be understood as derived. The computations of derived tensor product, and derived pull-back are correct, as the Koszul complex is locally free. Since g and h are embeddings, the left hand side is just the one sheaf complex $h_* g_* \mathscr{I}_{C_2}$.

Thus $h_* g_* \mathscr{I}_{C_2}$ is quasi-isomorphic to the total complex of

$$\operatorname{Koszul}^{Z_1}(W_5) \otimes (\mathscr{O}_{Z_1}(-1) \to \mathscr{I}_{L_2}).$$

By Lemma 4.10, in order to conclude that

$$\operatorname{Ext}^1_X(\mathscr{I}_{C_1}, \mathscr{I}_{C_2}) = 0$$

it suffices to prove that, for $0 \le j \le 5$, we have

$$\operatorname{Ext}^{j+1}_{Z_1}(\mathscr{I}_{L_1}, \mathscr{I}_{L_2}(-j)) = 0$$

and

$$\operatorname{Ext}^{j+2}_{Z_1}(\mathscr{I}_{L_1}, \mathscr{O}_{Z_1}(-j-1)) = 0.$$

The first statement is precisely Z_1-vanishing, which is implied by $\mathbf{G}$-vanishing by Proposition 4.8. The second statement follows by Serre duality: we have

$$\operatorname{Ext}^{j+2}_{Z_1}(\mathscr{I}_{L_1}, \mathscr{O}_{Z_1}(-j-1)) = H^{7-j}(Z_1, \mathscr{I}_{L_1}(j-5))^*$$

which vanishes for $0 \le j \le 5$ by Lemma 4.9. □

5. The derived equivalence

In this section we define the integral transform Φ, and we argue that it gives an equivalence of derived categories $\mathbf{D}^b_{coh}(Y) \cong \mathbf{D}^b_{coh}(X)$ by verifying the criterion of [5, Theorem 6.1].

5.1. Let $\mathbf{C}$ be the dimension four subscheme of $X \times Y$ defined in (3.10). We take its ideal sheaf $\mathscr{I}_{\mathbf{C}}$ in $X \times Y$ as the kernel of an integral transform

$$\Phi : \mathbf{D}^b_{coh}(Y) \to \mathbf{D}^b_{coh}(X) \qquad \Phi(\mathscr{E}) = \mathbf{R}\pi_{X,*}(\pi_Y^*(\mathscr{E}) \overset{\mathbf{L}}{\otimes} \mathscr{I}_{\mathbf{C}}).$$

(Here, π_X and π_Y are the projections from $X \times Y$ to X and Y, respectively.)

5.2. Theorem. *The functor Φ is a Fourier-Mukai transform, i.e., it is an equivalence of categories $\mathbf{D}^b_{\mathsf{coh}}(\mathsf{Y}) \cong \mathbf{D}^b_{\mathsf{coh}}(\mathsf{X})$.*

5.3. As it was originally explained by Bondal-Orlov [1], and then expanded by Bridgeland [3], in order to check that Φ is an equivalence it is enough to check that the family $\{\Phi\mathscr{O}_y\}_{y\in\mathsf{Y}}$ is *orthonormal* (see [3] for a complete set of properties that need to be verified). Observe that since $\mathbf{C}$ is flat over Y, $\Phi\mathscr{O}_y$ is precisely the ideal sheaf $\mathscr{I}_{C_y}$ on X. Thus we can think of $\{\mathscr{I}_{C_y}\}_{y\in\mathsf{Y}}$ as a family of sheaves on X, parametrized by the points of Y. Moreover, this family satisfies the following properties:

1. $\mathsf{Hom}_{\mathsf{X}}(\mathscr{I}_{C_{y_1}}, \mathscr{I}_{C_{y_2}}) = 0$ for $y_1 \neq y_2 \in \mathsf{Y}$ (Proposition 3.14);
2. $\mathscr{I}_{C_y}$ is a simple sheaf on X for $y \in \mathsf{Y}$ (Proposition 3.14);
3. $\mathsf{Ext}^1_{\mathsf{X}}(\mathscr{I}_{C_{y_1}}, \mathscr{I}_{C_{y_2}}) = 0$ if y_1, y_2 are points in Y such that $\mathsf{dim}(\mathsf{Ker}\, y_1 \cap \mathsf{Ker}\, y_2) \leq 1$ (Proposition 4.4 and Appendix A);
4. X is Calabi-Yau, therefore $\mathscr{I}_{C_y} \otimes \omega_{\mathsf{X}} \cong \mathscr{I}_{C_y}$.

By Serre duality, Properties 1 and 3 above imply that, if $\mathsf{dim}(\mathsf{Ker}\, y_1 \cap \mathsf{Ker}\, y_2) \leq 1$,

$$\mathsf{Ext}^i_{\mathsf{X}}(\mathscr{I}_{C_{y_1}}, \mathscr{I}_{C_{y_2}}) = 0 \text{ for all } i,$$

5.4. Lemma. *Let U be the set of pairs of forms (y_1, y_2) in $\mathsf{Y} \times \mathsf{Y}$ such that $\mathsf{Ker}(y_1)$ is not isotropic for y_2, and vice-versa. Then the complement of U has codimension at least two.*

Proof. It is enough to show that, for each $y_1 \in \mathsf{Y}$, the reduced subscheme D of Y defined by

$$\mathsf{D} = \{y_2 \in \mathsf{Y} \ : \ \mathsf{K}_1 = \mathsf{Ker}(y_1) \text{ is isotropic for } y_2\}$$

has dimension at most one.

Consider the three-dimensional space $\wedge^2\mathsf{K}_1$, whose elements can be regarded as linear equations on $\mathbf{P}^*$ (whose vanishing give hyperplanes in $\mathbf{P}^*$). Every form y_2 in D vanishes on $\wedge^2\mathsf{K}_1$, and therefore D is contained in every hyperplane cut out by a point in $\wedge^2\mathsf{K}_1$.

Rephrasing the above, we have restriction maps

$$\wedge^2\mathsf{K}_1 \hookrightarrow \mathsf{H}^0(\mathbf{P}^*, \mathscr{O}_{\mathbf{P}^*}(1)) \to \mathsf{H}^0(\mathsf{Y}, \mathscr{O}_{\mathsf{Y}}(1)) \to \mathsf{H}^0(\mathsf{D}, \mathscr{O}_{\mathsf{D}}(1))$$

and the composite

$$\wedge^2\mathsf{K}_1 \to \mathsf{H}^0(\mathsf{D}, \mathscr{O}_{\mathsf{D}}(1))$$

is zero.

Assume that D contains a divisor $\mathscr{O}_{\mathsf{Y}}(k)$. Then $\mathscr{I}_{\mathsf{D}}$ is contained in $\mathscr{O}_{\mathsf{Y}}(-k)$, and thus

$$\mathsf{H}^0(\mathsf{Y}, \mathscr{I}_{\mathsf{D}}(1)) \subseteq \mathsf{H}^0(\mathsf{Y}, \mathscr{O}_{\mathsf{Y}}(1-k)),$$

whose dimension is ≤ 1. Thus the kernel $\mathsf{H}^0(\mathsf{Y}, \mathscr{I}_{\mathsf{D}}(1))$ of the restriction map $\mathsf{H}^0(\mathsf{Y}, \mathscr{O}_{\mathsf{Y}}(1)) \to \mathsf{H}^0(\mathsf{D}, \mathscr{O}_{\mathsf{D}}(1))$ has dimension at most one. Since $\wedge^2\mathsf{K}_1$ maps into this kernel, and its dimension is three, the map

$$\wedge^2\mathsf{K}_1 \to \mathsf{H}^0(\mathsf{Y}, \mathscr{O}_{\mathsf{Y}}(1))$$

is not injective. Therefore there exists $x \in \wedge^2 K_1$ such that the hyperplane defined by x in $\mathbf{P}^*$ contains all of Y.

From the resolution of $\mathscr{O}_Y$ inside $\boldsymbol{W}$ written by Rødland in [11] (which works for any dimension three cut, not only for generic ones), the space of forms on $\mathbf{P}^*$ which vanish on Y is precisely $M = \operatorname{Ann} W$. Thus $x \in M$. On the other hand, every form in $\wedge^2 K_1$ is decomposable (as $G(2,3) \cong \mathbf{P}^2$), so $x \in \mathbf{G}$. We conclude that $x \in X$. On the other hand, $T_x \subset K_1 = K_{y_1}$ by the definition of x (recall, from Section 2, that T_x is the plane in V spanned by k_1 and k_2 if $x = k_1 \wedge k_2$, $k_i \in K_1$). Thus the pair (x, y_1) lies in the set R defined in (2.1), and thus X and Y can not be smooth by Corollary 2.3, contradiction. Thus D can not contain a divisor, and

$$\dim(Y \times Y \setminus U) \leq 4.$$

□

5.5. In order to finish the proof of Theorem 5.2, we apply the following modification of [5, Theorem 6.1], corrected in [4], to the family $\mathscr{P}_y = \mathscr{I}_{C_y}$ of sheaves on X.

5.6. Theorem. *Let X be a non-singular projective variety of dimension n, and let $\{\mathscr{P}_y\}_{y \in Y}$ be a family of simple sheaves on X parametrized by a smooth projective scheme Y of dimension n. Suppose that*

$$\operatorname{Hom}_X(\mathscr{P}_{y_1}, \mathscr{P}_{y_2}) = 0$$

for any distinct points y_1, $y_2 \in Y$, and that the closed subscheme

$$\Gamma(P) = \{(y_1, y_2) \in Y \times Y \ : \ \operatorname{Ext}^i_X(\mathscr{P}_{y_1}, \mathscr{P}_{y_2}) \neq 0 \text{ for some } i \in \mathbf{Z}\}$$

of $Y \times Y$ has dimension at most $n+1$. Suppose also that $\mathscr{P}_y \otimes \omega_X \cong \mathscr{P}_y$ for all $y \in Y$. Then the functor $\Phi : \mathbf{D}^b_{coh}(Y) \to \mathbf{D}^b_{coh}(X)$ is an equivalence of categories.

Proof. The main change from the original theorem of Bridgeland-Maciocia is the removal of the assumption that the family $\mathscr{P}_y$ is complete, at the expense of the assumption that Y is smooth. The proof goes through as before, by noting that we can get the fact that

$$\operatorname{Ext}^i_X(\mathscr{P}_{y_1}, \mathscr{P}_{y_2}) = 0 \text{ for all } y_1 \neq y_2 \in Y, i \in \mathbf{Z}$$

without having to use completeness of the family. (Completeness of the family is used in order to prove smoothness of Y.) We can then apply [5, Theorem 3.1] to finish the proof. □

6. Connections with existing work

In this section we put the example in this paper in the context of Kuznetsov's Homological Projective Duality.

6.1. The derived equivalence that we obtain appears to be a particular case of Homological Projective Duality, as explained by Kuznetsov [9]. Let us recall the setting of this theory.

Let $X \subset \mathbf{P}(V)$ be a smooth projective variety, and assume that its derived category admits a semi-orthogonal decomposition of the form

$$\mathbf{D}^b_{coh}(X) = \langle \mathscr{A}_0, \mathscr{A}_1(1), \ldots, \mathscr{A}_{i-1}(i-1) \rangle$$

where

$$0 \subseteq \mathscr{A}_0 \subseteq \mathscr{A}_1 \subseteq \cdots \subseteq \mathscr{A}_{i-1}$$

are full subcategories of $\mathbf{D}^b_{coh}(X)$, and (i) denotes twisting by $\mathscr{O}_X(i)$. If H is a hyperplane in $\mathbf{P}(V)$, let X_H be the corresponding hyperplane section of X. Then it is easy to see that

$$\langle \mathscr{A}_1(1), \mathscr{A}_2(2), \ldots, \mathscr{A}_{i-1}(i-1) \rangle$$

is a semiorthogonal collection in $\mathbf{D}^b_{coh}(X_H)$. In general there is no reason to expect this collection to generate $\mathbf{D}^b_{coh}(X_H)$. Let $\mathscr{C}_H$ denote the orthogonal in $\mathbf{D}^b_{coh}(X_H)$ of the subcategory generated by the above collection.

We can think of $\{\mathscr{C}_H \ : \ H \in \mathbf{P}(V^*)\}$ as a family of triangulated categories, parametrized by $H \in \mathbf{P}(V^*)$. In certain cases we can find a smooth variety Y, together with a morphism $Y \to \mathbf{P}(V^*)$, such that $\{\mathbf{D}^b_{coh}(Y_H)\}_{H \in \mathbf{P}(V^*)}$ is essentially the family of $\mathscr{C}_H$'s (for details, see [9]). In this situation Y is called the Homological Projective Dual of X.

The main theorem of [9] is the statement that Y then admits a semiorthogonal decomposition of a similar form, and if L is a linear subspace of V^*, $L^\perp \subset V$ its annihilator, then the linear sections

$$X_L = X \times_{\mathbf{P}(V)} \mathbf{P}(L^\perp), \quad Y_L = Y \times_{\mathbf{P}(V^*)} \mathbf{P}(L)$$

have closely related derived categories $\mathbf{D}^b_{coh}(X_L)$, $\mathbf{D}^b_{coh}(Y_L)$. Explicitly, their categories will have several trivial components (for X_L, arising from the semiorthogonal decomposition of X, for Y_L from that of Y), as well as a component which is *the same* in X_L and Y_L. In particular, if the dimension of L is chosen properly, the trivial components will disappear, and we get

$$\mathbf{D}^b_{coh}(X_L) \cong \mathbf{D}^b_{coh}(Y_L).$$

Another important property of this setup is that X_L is smooth precisely when Y_L is, and thus Y is closely related to the classical projective dual of X.

6.2. In our setting, we have the smooth projective variety $\mathbf{G} \subset \mathbf{P}$, its projective dual $\mathbf{Pf}$, and linear cuts X and Y of them by dual linear spaces. The varieties X and Y are simultaneously smooth, and when they are, their derived categories are equivalent. It is very tempting, in this context, to conjecture the following.

6.3. Conjecture. *There exists a Lefschetz decomposition of the derived category of* $\mathbf{G}$*, and a smooth projective variety* $\overline{\mathbf{Pf}} \to \mathbf{P}^*$*, mapping to* $\mathbf{Pf}$*, which is homologically projectively dual to* $\mathbf{G}$ *with respect to the decomposition of* $\mathbf{D}^b_{coh}(\mathbf{G})$*. Furthermore, Theorem 0.3 is a direct application of the main result of [9].*

6.4. We close this discussion with the remark that $\mathbf{Pf}$ possesses a very simple and natural resolution of singularities. Consider the space

$$\overline{\mathbf{Pf}} = \{(y, K_3) \subset \mathbf{Pf} \times \mathbf{G}(3, V) \ : \ K_3 \subset \operatorname{Ker} y\}$$

consisting of pairs (degenerate form y, three-dimensional subspace K_3 of its kernel). The first projection is obviously a birational morphism to $\mathbf{Pf}$. The second projection exhibits $\overline{\mathbf{Pf}}$ as a $\mathbf{P}^5$-bundle over $\mathbf{G}(3, V)$, thus $\overline{\mathbf{Pf}}$ is smooth (and hence the second projection is a resolution $\overline{\mathbf{Pf}} \to \mathbf{Pf}$).

A. Appendix: Macaulay computations

In this section we verify that we have $\mathbf{G}$-vanishing, i.e.,

$$\mathsf{Ext}^{j+2}_{\mathbf{G}}(\mathscr{O}_{S_{y_1}}, \mathscr{O}_{S_{y_2}}(-j-1)) = 0$$

for $0 \leq j \leq 5$, as long as

$$\dim(\mathsf{Ker}\, y_1 \cap \mathsf{Ker}\, y_2) \leq 1.$$

As a matter of terminology, if this dimension is zero, one, or two, we say that y_1 and y_2 are in generic, subgeneric, or subsubgeneric position, respectively. Note that by Lemma 3.8 the intersection can not have dimension three.

Let y_1, y_2 be smooth points in $\mathbf{Pf}$, and let S_1 and S_2 be the corresponding Schubert cycles (3.1). By their definition, these cycles only depend on $\bigwedge^2 \mathsf{Ann}\,\mathsf{Ker}\, y_1$ and $\bigwedge^2 \mathsf{Ann}\,\mathsf{Ker}\, y_2$. By an action of $\mathsf{GL}(V)$, we can fix a basis $x_0, \ldots, x_6$ of V^* such that

$$\bigwedge^2 \mathsf{Ann}\,\mathsf{Ker}\, y_1 = \langle x_0 \wedge x_1, x_0 \wedge x_2, x_0 \wedge x_3, x_1 \wedge x_2, x_1 \wedge x_3, x_2 \wedge x_3 \rangle,$$

and, according to the generic, subgeneric, and subsubgeneric cases,

$$\begin{aligned}
\bigwedge^2 \mathsf{Ann}\,\mathsf{Ker}\, y_2 &= \langle x_3 \wedge x_4, x_3 \wedge x_5, x_3 \wedge x_6, x_4 \wedge x_5, x_4 \wedge x_6, x_5 \wedge x_6 \rangle, \\
\bigwedge^2 \mathsf{Ann}\,\mathsf{Ker}\, y_2 &= \langle x_2 \wedge x_3, x_2 \wedge x_4, x_2 \wedge x_5, x_3 \wedge x_4, x_3 \wedge x_5, x_4 \wedge x_5 \rangle, \\
\bigwedge^2 \mathsf{Ann}\,\mathsf{Ker}\, y_2 &= \langle x_1 \wedge x_2, x_1 \wedge x_3, x_1 \wedge x_4, x_2 \wedge x_3, x_2 \wedge x_4, x_3 \wedge x_4 \rangle.
\end{aligned}$$

A.1. Because of difficulties of technical order, the best way to verify $\mathbf{G}$-vanishing is to proceed as follows. Let y_1 and y_2 be two distinct forms on V of rank 4, in generic or sub-generic position.

We will use the local-to-global spectral sequence, and verify the vanishing of

$$H^{j+2-i}(\mathbf{G}, \underline{\mathsf{Ext}}^i_{\mathbf{G}}(\mathscr{I}_{S_1}, \mathscr{I}_{S_2}(-j-1))), \ 0 \leq j \leq 5, \ i \in \mathbf{Z}.$$

A.2. Lemma. *For $y_1 \neq y_2$ in generic or subgeneric position we have*

$$\underline{\mathsf{Ext}}^i_{\mathbf{G}}(\mathscr{I}_{S_1}, \mathscr{I}_{S_2}) = 0,$$

unless i is zero or two. We have

$$\underline{\mathsf{Hom}}_{\mathbf{G}}(\mathscr{I}_{S_1}, \mathscr{I}_{S_2}) = \mathscr{I}_{S_2}.$$

Proof. Observe that S_1 is Cohen-Macaulay of codimension three, and $\mathbf{G}$ is smooth, therefore

$$\mathsf{Ext}^i_{\mathbf{G}}(\mathscr{O}_{S_1}, \mathscr{O}_{\mathbf{G}}) = 0$$

unless $i = 3$. From the long-exact sequence of $\underline{\mathsf{Ext}}$'s associated to the short exact sequence

$$0 \to \mathscr{I}_{S_2} \to \mathscr{O}_{\mathbf{G}} \to \mathscr{O}_{S_2} \to 0$$

and the fact that $\underline{\mathsf{Hom}}_{\mathbf{G}}(\mathscr{O}_{S_1}, \mathscr{O}_{S_2}) = 0$ (because S_1, S_2 are irreducible of the same dimension), we get that

$$\underline{\mathsf{Ext}}^1_{\mathbf{G}}(\mathscr{O}_{S_1}, \mathscr{I}_{S_2}) = 0.$$

Now from the long exact sequence associated to

$$0 \to \mathscr{I}_{S_1} \to \mathscr{O}_{\mathbf{G}} \to \mathscr{O}_{S_1} \to 0$$

we conclude that

$$\mathsf{Hom}_{\mathbf{G}}(\mathscr{I}_{S_1}, \mathscr{I}_{S_2}) = \mathscr{I}_{S_2},$$

and for $i \geq 1$,

$$\underline{\mathsf{Ext}}^i_{\mathbf{G}}(\mathscr{I}_{S_1}, \mathscr{I}_{S_2}) = \underline{\mathsf{Ext}}^{i+1}_{\mathbf{G}}(\mathscr{O}_{S_1}, \mathscr{I}_{S_2}).$$

In $\mathbf{D}^b_{\mathsf{coh}}(\mathbf{G})$ rewrite

$$\mathbf{R}\underline{\mathsf{Hom}}_{\mathbf{G}}(\mathscr{O}_{S_1}, \mathscr{I}_{S_2}) = \mathscr{O}^*_{S_1} \overset{\mathbf{L}}{\otimes} \mathscr{I}_{S_2},$$

where the dual is taken in the derived sense. Since S_1 is Cohen-Macaulay and $\mathbf{G}$ is smooth, $\mathscr{O}^*_{S_1}$ equals $\mathscr{F}[-3]$ for some sheaf $\mathscr{F}$ supported on S_1. The tensor product is right exact, therefore

$$\mathscr{O}^*_{S_1} \overset{\mathbf{L}}{\otimes} \mathscr{I}_{S_2} = (\mathscr{F} \overset{\mathbf{L}}{\otimes} \mathscr{I}_{S_2})[-3]$$

is supported in cohomological degrees 3 or less. Therefore

$$\underline{\mathsf{Ext}}^i_{\mathbf{G}}(\mathscr{I}_{S_1}, \mathscr{I}_{S_2}) = 0$$

for $i \geq 3$ (recall the shift by one from above). The only thing left to check is that

$$\underline{\mathsf{Ext}}^1_{\mathbf{G}}(\mathscr{I}_{S_1}, \mathscr{I}_{S_2}) = 0,$$

which can be done either by a laborious computation by hand (which we suppress), or by a Macaulay computation below. □

A.3. Lemma. *For $y_1 \neq y_2$ in generic or subgeneric position we have*

$$\underline{\mathsf{Ext}}^1_{\mathbf{G}}(\mathscr{I}_{S_1}, \mathscr{I}_{S_2}) = 0.$$

Proof. See the Macaulay code below:

```
i1 : R = ZZ/101[apply(subsets(7, 2), i -> p_i)];
i2 : I = Grassmannian(1, 6, R);
i3 : RG = R/I;
i4 : IS1 = ideal(p_{0,1}, p_{0,2}, p_{0,3}, p_{1,2}, p_{1,3}, p_{2,3});
i5 : IS2 = ideal(p_{3,4}, p_{3,5}, p_{3,6}, p_{4,5}, p_{4,6}, p_{5,6});
i6 : Ext^1(IS1, IS2)

o6 = 0

o6 : RG-module

i7 : IS2 = ideal(p_{2,3}, p_{2,4}, p_{2,5}, p_{3,4}, p_{3,5}, p_{4,5});
i8 : Ext^1(IS1, IS2)

o8 = 0

o8 : RG-module
```

□

A.4. Remark. The above computation is done over the finite field $\mathbf{Z}/101\mathbf{Z}$. Since we have obtained the vanishing statement over this field, a standard semi-continuity argument gives the same vanishing over $\mathbf{Q}$, and thus over $\mathbf{C}$. The same technique will be used in all further Macaulay computations.

A.5. Lemma. *For $2 \leq i \leq 7$ and $0 \leq j \leq 5$ we have*

$$H^i(\mathbf{G}, \mathscr{I}_{S_2}(-j-1)) = 0.$$

Proof. From the short exact sequence

$$0 \to \mathscr{I}_{S_2} \to \mathscr{O}_{\mathbf{G}} \to \mathscr{O}_{S_2} \to 0$$

and the vanishing of $H^{i-1}(\mathbf{G}, \mathscr{O}_{\mathbf{G}}(-j-1))$ and $H^i(\mathbf{G}, \mathscr{O}_{\mathbf{G}}(-j-1))$ (by Kodaira vanishing), we get

$$H^i(\mathbf{G}, \mathscr{I}_{S_2}(-j-1)) \cong H^{i-1}(S_2, \mathscr{O}_{S_2}(-j-1)).$$

This latter group vanishes from a combination of the rationality of the singularities of S_2 and Kawamata vanishing. □

A.6. Proposition. (G-vanishing). *For $y_1 \neq y_2$ in generic or subgeneric position, $0 \leq j \leq 5$ we have*

$$\mathrm{Ext}^{j+2}_{\mathbf{G}}(\mathscr{O}_{S_1}, \mathscr{O}_{S_2}(-j-1)) = 0.$$

Proof. There are two cases to consider, $0 \leq j \leq 4$, and the special case $j = 5$. Assume first that $0 \leq j \leq 4$. By Lemmas A.2 and A.5, we have

$$H^{j+2}(\mathbf{G}, \underline{\mathrm{Ext}}^0_{\mathbf{G}}(\mathscr{I}_{S_1}, \mathscr{I}_{S_2}(-j-1))) = H^{j+2}(\mathbf{G}, \mathscr{I}_{S_2}(-j-1)) = 0,$$

thus these groups do not contribute to the local-to-global spectral sequence.

Now we need to check that for $0 \leq j \leq 4$ we have

$$H^j(\mathbf{G}, \underline{\mathrm{Ext}}^2_{\mathbf{G}}(\mathscr{I}_{S_1}, \mathscr{I}_{S_2}(-j-1))) = 0.$$

This is done using the Macaulay code below:

```
i1 : R = ZZ/101[apply(subsets(7, 2), i -> p_i)];
i2 : I = Grassmannian(1, 6, R);
i3 : RG = R/I;
i4 : IS1 = ideal(p_{0,1}, p_{0,2}, p_{0,3}, p_{1,2}, p_{1,3}, p_{2,3});
i5 : IS2 = ideal(p_{3,4}, p_{3,5}, p_{3,6}, p_{4,5}, p_{4,6}, p_{5,6});
i6 : E2 = Ext^2(IS1, IS2);
i7 : J = coker lift(presentation E2, R);
i8 : (gens I) % (ann J)

o8 : 0

             1       35
o8 : Matrix R  <--- R
```

```
i9 : resJ = resolution J;
i10 : betti resJ

o11 = total: 25 342 2177 8554 23205 46046 69069 79795 71650 50071 27062 11137 3388 721 96 6
          0: 25 342 2177 8554 23205 46046 69069 79794 71644 50056 27042 11122 3382 720 96 6
          1:  .   .    .    .     .     .     .     1     6    15    20    15    6   1  . .
```

Let us explain the above computation line by line. Having computed $\mathsf{E}_2 = \underline{\mathsf{Ext}}^2_{\mathbf{G}}(\mathscr{I}_{\mathsf{S}_1}, \mathscr{I}_{\mathsf{S}_2})$, as a sheaf on $\mathbf{G}$, we push it forward to get a sheaf J on $\mathbf{G}$ (line `i7`). We do this by lifting a presentation of E_2, from the homogeneous ring of the Grassmannian RG to that of $\mathbf{P}$, R. We verify that this is correct indeed, in line `i8`: J is annihilated by the generators of I.

We then compute a resolution of J over R. From the `betti` command readout we see that this resolution has the form

$$0 \to \mathscr{O}_{\mathbf{P}}(-15)^6 \to \mathscr{O}_{\mathbf{P}}(-14)^{96} \to \cdots \to \mathscr{O}_{\mathbf{P}}^{25} \to \mathsf{J} \to 0.$$

Thus, after twisting by $(-\mathsf{j}-1)$, we only see line bundles of the form $\mathscr{O}_{\mathbf{P}}(-1), \ldots, \mathscr{O}_{\mathbf{P}}(-20)$ in the resolution of $\mathsf{J}(-\mathsf{j}-1)$. These sheaves have no cohomology on $\mathbf{P}$, so a standard argument similar to that of Lemma 4.10 shows that

$$\mathsf{H}^*(\mathbf{G}, \underline{\mathsf{Ext}}^2_{\mathbf{G}}(\mathscr{I}_{\mathsf{S}_1}, \mathscr{I}_{\mathsf{S}_2}(-\mathsf{j}-1))) = \mathsf{H}^*(\mathbf{P}, \mathsf{J}(-\mathsf{j}-1)) = 0 \text{ for } 0 \le \mathsf{j} \le 4.$$

This takes care of all the cases except $\mathsf{j} = 5$. For $\mathsf{j} = 5$, note that by Serre duality,

$$\mathsf{Ext}^7_{\mathbf{G}}(\mathscr{I}_{\mathsf{S}_1}, \mathscr{I}_{\mathsf{S}_2}(-6)) = \mathsf{Ext}^3_{\mathbf{G}}(\mathscr{I}_{\mathsf{S}_2}, \mathscr{I}_{\mathsf{S}_1}(-1)).$$

The local-to-global spectral sequence for this latter Ext group has two (potentially) non-zero terms,

$$\mathsf{H}^3(\mathbf{G}, \mathscr{I}_{\mathsf{S}_1}(-1)),$$

and

$$\mathsf{H}^1(\mathbf{G}, \underline{\mathsf{Ext}}^2_{\mathbf{G}}(\mathscr{I}_{\mathsf{S}_2}, \mathscr{I}_{\mathsf{S}_1}(-1))).$$

The first one vanishes by Lemma A.5, while the second one vanishes from the computation above of the resolution of $\underline{\mathsf{Ext}}^2_{\mathbf{G}}(\mathscr{I}_{\mathsf{S}_2}, \mathscr{I}_{\mathsf{S}_1})$.

The subgeneric case is treated the same, replacing line `i5` in the above computation with

```
i5 : IS2 = ideal(p_{2,3}, p_{2,4}, p_{2,5}, p_{3,4}, p_{3,5}, p_{4,5});
```

The resolution has the same exact Betti numbers, and all the remaining arguments apply. □

A.7. In fact, we can completely avoid using the Intersection Theorem 5.6 by checking G-vanishing for the subsubgeneric case. This can be done in Macaulay again, but the computation is more difficult (for example, Lemma A.3 fails, and we have to deal with more terms in the local-to-global spectral sequence). We omit the details.

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