FINDING EQUATIONS OF THE FAKE PROJECTIVE PLANE $(C18, p = 3, \{2I\})$

LEV BORISOV AND BOJUE WANG

Abstract. We find explicit equations of a new pair of fake projective planes, labeled by $(C18, p = 3, \{2I\})$ in the Cartwright-Steger classification. Our method involves starting with known equations of a commensurable fake projective plane $(C18, p = 3, \emptyset, d_3D_3)$ and working through a chain of cyclic covers and quotients to get to the new one.

Contents

1. Introduction.

Complex projective algebraic surfaces X are classified according to their Kodaira dimension $\kappa(X)$. The case $\kappa(X) = 2$, when X is a surface of general type, is arguably the least understood. Among such surfaces one is especially interested in those with small Hodge numbers, in particular $h^{1,0}(X) = h^{2,0}(X) = 0$, see for example the review [1]. These are further separated by the square of their canonical class K_X^2, and the extreme case $K_X^2 = 9$ occurs for the so-called fake projective planes (from here on called FPPs), which are characterized by having the same Hodge numbers as the usual projective plane $\mathbb{CP}^2$.

First example of an FPP was given by Mumford in [19], using the method of 2-adic uniformization. Research over the next several decades by multiple authors (see in particular [10, 12–16, 20]) led to the full classification of all FPPs as quotients of the complex 2-ball $\mathcal{B}^2$ by explicit finitely presented arithmetic groups in [10]. There are exactly 100 FPPs up to isomorphism, gathered into 50 pairs of complex conjugate surfaces.

Unfortunately, a ball quotient description does not directly lead to any explicit equations of an FPP in its embedding into a projective space. First such equations of an FPP in its bicanonical embedding into $\mathbb{CP}^9$ were found in [7]. Since then, L.B. and multiple coauthors have found explicit equations of ten more complex conjugate pairs of FPPs [2–5]. The current paper continues this program by finding explicit equations of an FPP labeled by $(C18, p = 3, \{2I\})$ in the Cartwright-Steger classification. Our method involves starting from known equations of a commensurable FPP and using it to produce the equations of a new FPP. Even though it is similar to the method of [4] and [5], we had to overcome significant computational difficulties that arose because of the large degree of the common Galois cover of the two FPPs.

1.1. **General description of the process.** The FPP indexed by $(C18, p = 3, \{2I\})$ is known to be commensurable to the FPP indexed by $(C18, p = 3, \emptyset, d_3D_3)$ in Cartwright-Steger list whose equations were found in [3] several years ago. For the remainder of the paper, we will adopt the following notation.

Notation 1.1. We write $\mathbb{P}^2_{fake}$ for the fake projective plane in the pair $(C18, p = 3, \emptyset, d_3D_3)$ whose equations were found in [3]. We write $\widehat{\mathbb{P}^2_{fake}}$ for the commensurable plane in the class $(C18, p = 3, \{2I\})$ whose equations we find in this paper.

The following results follow from the computations of Cartwright and Steger and additional GAP [11] calculations we did in the file GAPdataAll of [9]. The fake projective plane $\mathbb{P}^2_{fake}$ has an automorphism group $C_3 \times C_3$ and $\widehat{\mathbb{P}^2_{fake}}$ is a non-Galois degree 9 cover of $\mathbb{P}^2_{fake}/C_3 \times C_3$. In the other direction, there is a surface which we denote by $72.\mathbb{P}^2_{fake}$ which is a common Galois cover of $\mathbb{P}^2_{fake}$ and $\widehat{\mathbb{P}^2_{fake}}$.

Proposition 1.2. There is a surface $72.\mathbb{P}^2_{fake}$ with an automorphism group G_{648} of order $648 = 2^3 3^4$ which is isomorphic to the direct product of C_3 and the semidirect product of $SL(2, \mathbb{Z}/3\mathbb{Z})$ and $C_3 \times C_3$ (with the canonical action of the former on the latter)

$$G_{648} \cong C_3 \times (SL(2, \mathbb{Z}/3\mathbb{Z}) \ltimes (C_3 \times C_3)).$$

The fake projective planes $\mathbb{P}^2_{fake}$ and $\widehat{\mathbb{P}^2_{fake}}$ are the quotients of $72.\mathbb{P}^2_{fake}$ by the normal subgroup of G_{648} of order 72

$$G_{72} = \{1\} \times (Q_8 \ltimes (C_3 \times C_3))$$

and the nonnormal subgroup of G_{648} of order 72

$$\widehat{G}_{72} = C_3 \times (SL(2, \mathbb{Z}/3\mathbb{Z}) \times \{1\}),$$

respectively. Here Q_8 is the normal 2-Sylow subgroup of $SL(2, \mathbb{Z}/3\mathbb{Z})$, isomorphic to the quaternion group.

Proof. This is a result of the GAP computation in GAPdataAll, see [9]. □

In this paper we consider the following diagram of surfaces and Galois covers.

$$\begin{array}{ccccccc}
 & & & 72.\mathbb{P}^2_{fake} & & & \\
 & & \swarrow & & \searrow & & \\
 & 8.\mathbb{P}^2_{fake} & & & & 9.\widehat{\mathbb{P}^2_{fake}} & \\
 & \downarrow & \searrow & & & & \searrow \\
 & 4.\mathbb{P}^2_{fake} & & 8.\mathbb{P}^2_{fake}/C_3 & & & \widehat{\mathbb{P}^2_{fake}} \\
\swarrow & \downarrow & \searrow & & & & \\
2.\mathbb{P}^2_{fake} & 2.\mathbb{P}^2_{fake} & 2.\mathbb{P}^2_{fake} & & & & \\
\searrow & \downarrow & \swarrow & & & & \\
 & \mathbb{P}^2_{fake} & & & & &
\end{array}$$

Here the surface $8.\mathbb{P}^2_{fake}$ is the quotient of $72.\mathbb{P}^2_{fake}$ by the normal subgroup

$$\{1\} \times (\{1\} \ltimes (C_3 \times C_3))$$

of G_{648}. This surface is an unramified Q_8 Galois cover of $\mathbb{P}^2_{fake}$ and $4.\mathbb{P}^2_{fake}$ and $2.\mathbb{P}^2_{fake}$ correspond to the center and three index two subgroups of Q_8, respectively. Three different covers $2.\mathbb{P}^2_{fake} \to \mathbb{P}^2_{fake}$ correspond to 2-torsion line bundles on $\mathbb{P}^2_{fake}$ that are permuted by an order three automorphism of $\mathbb{P}^2_{fake}$. The surface $8.\mathbb{P}^2_{fake}/C_3$ is a singular quotient of $8.\mathbb{P}^2_{fake}$ by the image of the central C_3 of G_{648} (the first C_3 factor). It is used in intermediate calculations to get enough points on $72.\mathbb{P}^2_{fake}$ with high accuracy. The surface $9.\widehat{\mathbb{P}^2_{fake}}$ is the quotient of $72.\mathbb{P}^2_{fake}$ by the 2-Sylow subgroup of G_{648}

$$\{1\} \times (Q_8 \ltimes \{1\}).$$

It is a $C_3 \times C_3$ unramified Galois cover of $\widehat{\mathbb{P}^2_{fake}}$, which we used to simplify the equations of $\widehat{\mathbb{P}^2_{fake}}$.

The method of the paper is to start with the known equations of $\mathbb{P}^2_{fake}$, make our way up to $72.\mathbb{P}^2_{fake}$ and then take invariants to get down to $\widehat{\mathbb{P}^2_{fake}}$. We start with equations of $\mathbb{P}^2_{fake}$ found in [3]. They describe the image of $\mathbb{P}^2_{fake}$ in its bicanonical embedding into $\mathbb{CP}^9$ as being cut out by 84 cubic equations in 10 variables. The coefficients (in $\mathbb{Z}[\sqrt{-2}]$) are about 100 digits long. There is an explicit $C_3 \times C_3$ action on $\mathbb{CP}^9$ that gives automorphism of $\mathbb{P}^2_{fake}$.

Step 1. We find a nonreduced linear cut on $\mathbb{P}^2_{fake}$ which corresponds to the square of a 2-torsion element of the Picard group of $\mathbb{P}^2_{fake}$. This is done by first running an exhaustive search modulo 73, then lifting the nonreduced cut it to 73-adics to high enough accuracy and finally recognizing the resulting coefficients as algebraic numbers.

Step 2. We use the nonreduced linear cut above to simplify the equations of $\mathbb{P}^2_{fake}$ by picking a better basis of $H^0(\mathbb{P}^2_{fake}, 2K)$. We also choose to make it a basis of $C_3 \times C_3$ eigenvectors. The downside is that the equations now have coefficients in $\mathbb{Q}(\sqrt{-2}, \sqrt{-3})$ which will be the case for most of the process.

Step 3. We use the same cut to find equations of the double cover $2.\mathbb{P}^2_{fake}$ in its putative bicanonical embedding into $\mathbb{CP}^{19}$. It is cut out by 100 quadratic equations in 20 variables.[1] We also use the known automorphism group of $\mathbb{P}^2_{fake}$ to construct $4.\mathbb{P}^2_{fake}$. We do not try to compute the equations of its putative bicanonical embedding, as there would be too many of them, but we develop a way of constructing points of its image in $\mathbb{CP}^{39}$ with high accuracy.

Step 4. We construct the double cover $8.\mathbb{P}^2_{fake}$ of $4.\mathbb{P}^2_{fake}$. In fact, we first construct its quotient $8.\mathbb{P}^2_{fake}/C_3$ which is likely the image of the canonical map $8.\mathbb{P}^2_{fake} \to \mathbb{CP}^6$.[2] It is given by 4 cubic and 58 degree four equations in 7 variables. We also find the action of Q_8 on $8.\mathbb{P}^2_{fake}/C_3$. We then find a way of constructing points with high accuracy in what is likely the bicanonical embedding of $8.\mathbb{P}^2_{fake}$ into $\mathbb{CP}^{79}$.

Step 5. In what is arguably the most delicate part of the calculation, the $C_3 \times C_3$ cover $72.\mathbb{P}^2_{fake} \to 8.\mathbb{P}^2_{fake}$ is determined by finding a relation among certain bicanonical sections on $8.\mathbb{P}^2_{fake}$. As a result, we find a basis of the 71-dimensional space $H^0(72.\mathbb{P}^2_{fake}, K)$ in terms of algebraic functions in $H^0(8.\mathbb{P}^2_{fake}, 2K)$ and $H^0(8.\mathbb{P}^2_{fake}, K)$. We also find the values of the elements of this basis on some random points, with very high accuracy. We find the action of G_{648} on this 71-dimensional space, in particular we find the action of the subgroup

$$\widehat{G}_{72} = C_3 \times (SL(2, \mathbb{Z}/3\mathbb{Z}) \times \{1\}).$$

We compute the linear invariants of the Q_8 action on the dimension 71 space to get points in the putative canonical embedding of $9.\widehat{\mathbb{P}^2_{fake}}$ into $\mathbb{CP}^7$. We also compute the quadratic invariants of the action of G_{72}. We use $C_3 \times C_3$

[1] We do not actually verify that it is an embedding, but it is highly likely

[2] Again, we do not actually verify that this is the image, but it factors through it.

invariant products of elements of $H^0(9.\widehat{\mathbb{P}^2_{fake}}, K)$ to compute special curves on $\widehat{\mathbb{P}^2_{fake}}$.

Step 6. Looking at the intersections of the above special curves, we find a basis of $H^0(\widehat{\mathbb{P}^2_{fake}}, 2K)$ where the equations are defined over the smaller field $\mathbb{Q}(\sqrt{-2})$ and have smaller coefficients.

Step 7. We use the usual methods to verify that the equations we obtain indeed cut out a fake projective plane.

1.2. **Disclaimers, acknowledgements and further directions.** We describe most of our surfaces in terms of multiple points in the images of their maps into a projective space, computed with high accuracy (hundreds to thousands of decimal digits). Thus our intermediate calculations cannot be deemed fully rigorous, which necessitates an eventual verification that the surface we obtain is indeed an FPP in a bicanonical embedding. We have use GAP, Magma and Mathemaica [11,17,18] with most of the computations performed in the latter.

We believe that our results may allow us to eventually compute all three remaining pairs of FPPs that are commensurable to $\mathbb{P}^2_{fake}$ and $\widehat{\mathbb{P}^2_{fake}}$. They are labeled by $(C18, p = 3, \{2\}, D_3)$, $(C18, p = 3, \{2\}, (dD)_3)$ and $(C18, p = 3, \{2\}, (d^2D)_3)$ in Cartwright-Steger classification.

2. Technical details: group-theoretic calculations.

In this section we collect the results of GAP calculations and related computations of characters of finite group representations.

Proposition 2.1. The torsion subgroups of the Picard groups of the covers of fake projective planes above are given by the following table.

$\mathbb{P}^2_{fake}$	$4.\mathbb{P}^2_{fake}$	$8.\mathbb{P}^2_{fake}$	$72.\mathbb{P}^2_{fake}$	$9.\widehat{\mathbb{P}^2_{fake}}$	$\widehat{\mathbb{P}^2_{fake}}$
$C_2^2 \times C_{13}$	$C_2^8 \times C_3^2$	$C_3^2 \times C_{13}$	$C_2^8 \times C_3 \times C_{13}$	$C_2^3 \times C_3 \times C_{13}$	$C_2 \times C_3^2$

(2.1)

Proof. Computed by GAP. □

Recall that there is a divisor class H on $\mathbb{P}^2_{fake}$ so that its canonical class is $K = 3H$. We abuse the notation and use K and $3H$ interchangeably throughout the rest of the paper to denote the canonical line bundles (or divisor classes) on various surfaces. Note that the corresponding line bundles come with a natural linearization with respect to the automorphism group of the surface.

Proposition 2.2. We have the following dimensions of the spaces of global sections of the canonical and the bicanonical invertible sheaves $\mathcal{O}(3H)$ and

$\mathcal{O}(6H)$ on the surfaces mentioned above.

	$\dim_{\mathbb{C}} H^0(\cdot, 3H)$	$\dim_{\mathbb{C}} H^0(\cdot, 6H)$
$\mathbb{P}^2_{fake}, \widehat{\mathbb{P}^2_{fake}}$	0	10
$2.\mathbb{P}^2_{fake}$	1	20
$4.\mathbb{P}^2_{fake}$	3	40
$8.\mathbb{P}^2_{fake}$	7	80
$8.\mathbb{P}^2_{fake}/C_3$	7	32
$72.\mathbb{P}^2_{fake}$	71	720
$9.\widehat{\mathbb{P}^2_{fake}}$	8	90

Proof. The case of $8.\mathbb{P}^2_{fake}/C_3$ is special since this surfaces is singular, and it also does not cover a fake projective plane. We will treat it last.

For an n-fold unramified cover X of a fake projective plane, we have $\chi(X, kH) = \frac{n}{2}(k-1)(k-2)$ by the Riemann-Roch theorem. Since $K_X = 3H$ is ample, for $k > 3$ Kodaira vanishing theorem assures that $h^0(X, kH) = \chi(X, kH)$. Thus, $h^0(X, 6H) = 10n$ which gives the values of the right column of the above table. Similarly, we have

$$\begin{aligned} h^0(X, 3H) &= \chi(3H) - h^2(X, 3H) + h^1(X, 3H) \\ &= n - h^{2,2}(X) + h^{1,2}(X) = n - 1 + h^{1,0}(X). \end{aligned}$$

Since the fundamental groups of the above surfaces have finite abelianization by Proposition 2.1, we have $h^{1,0}(X) = 0$. This implies the values in the middle column of the above table.

The dimensions of $H^0(8.\mathbb{P}^2_{fake}/C_3, 3H)$ and $H^0(8.\mathbb{P}^2_{fake}/C_3, 6H)$ are the dimensions of the spaces of invariants for the central C_3 action on $8.\mathbb{P}^2_{fake}$. Note that the action of the generator g of this C_3 on $\mathbb{P}^2_{fake}$ has 3 fixed points of type $\frac{1}{3}(1,2)$, see [14]. Since g also acts on $2.\mathbb{P}^2_{fake}$, its action on the two-point preimage of every fixed point on $\mathbb{P}^2_{fake}$ must be trivial, so g has 6 fixed points of the same type on $2.\mathbb{P}^2_{fake}$. Similarly, its action on $4.\mathbb{P}^2_{fake}$ has 12 fixed points and its action on $8.\mathbb{P}^2_{fake}$ has 24 fixed points. Each of these singular points has the same contribution to $\sum_{i=0}^2 (-1)^i \mathrm{Tr}(g, H^i(X, 3H))$ in the the Holomorphic Lefschetz formula. We know that this sum is 1 for $X = \mathbb{P}^2_{fake}$, therefore it is equal to 8 for $X = 8.\mathbb{P}^2_{fake}$, i.e.

$$\sum_{i=0}^{2} (-1)^i \mathrm{Tr}(g, H^i(8.\mathbb{P}^2_{fake}, 3H)) = 8.$$

Consequently, all sections of $H^0(8.\mathbb{P}^2_{fake}, 3H)$ are invariant with respect to the action of g, and the canonical map of $8.\mathbb{P}^2_{fake}$ factors through the quotient surface $8.\mathbb{P}^2_{fake}/C_3$. Since $\sum_{i=0}^2 (-1)^i \mathrm{Tr}(g, H^i(8.\mathbb{P}^2_{fake}, kH))$ depends only on k mod 3, we see that the action of g on the 80-dimensional space $H^0(8.\mathbb{P}^2_{fake}, 6H)$ has invariant subspace of dimension 32 and two spaces of dimension 24 each of eigenvalues $e^{\pm 2\pi \mathrm{i}/3}$. □

In what follows, it will be important for us to understand the representation of G_{648} on the dimension 71 space $H^0(72.\mathbb{P}^2_{fake}, 3H)$. According to GAP [11], the character table of G_{648} is given by

	$1a$	$3a$	$2a$	$4a$	$3b$	$3c$	$6a$	$3d$	$3e$	$6b$	$3f$	$3g$	$6c$	$12a$	$3h$	$3i$	$6d$	$3j$	$3k$	$6e$	$3l$	$3m$	$6f$	$12b$	$3n$	$3o$	$6g$	$3p$	$3q$	$6h$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	1	1	1	α	α	α	α	α	α	α	α	α	α	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$
χ_3	1	1	1	1	1	1	1	1	1	1	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	α	α	α	α	α	α	α	α	α	α
χ_4	1	1	1	1	α	α	α	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	1	1	1	1	α	α	α	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	1	1	1	1	α	α	α	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$
χ_5	1	1	1	1	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	α	α	α	1	1	1	1	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	α	α	α	1	1	1	1	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	α	α	α
χ_6	1	1	1	1	α	α	α	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	α	α	α	α	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	1	1	1	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	1	1	1	α	α	α
χ_7	1	1	1	1	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	α	α	α	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	α	α	α	1	1	1	α	α	α	α	1	1	1	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$
χ_8	1	1	1	1	α	α	α	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	1	1	1	α	α	α	α	α	α	α	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	1	1	1
χ_9	1	1	1	1	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	α	α	α	α	α	α	α	1	1	1	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	$\bar\alpha$	α	α	α	1	1	1
χ_{10}	2	2	-2	.	-1	-1	1	-1	-1	1	2	2	-2	.	-1	-1	1	-1	-1	1	2	2	-2	.	-1	-1	1	-1	-1	1
χ_{11}	2	2	-2	.	$-\alpha$	$-\alpha$	α	$-\bar\alpha$	$-\bar\alpha$	$\bar\alpha$	2	2	-2	.	$-\alpha$	$-\alpha$	α	$-\bar\alpha$	$-\bar\alpha$	$\bar\alpha$	2	2	-2	.	$-\alpha$	$-\alpha$	α	$-\bar\alpha$	$-\bar\alpha$	$\bar\alpha$
χ_{12}	2	2	-2	.	$-\bar\alpha$	$-\bar\alpha$	$\bar\alpha$	$-\alpha$	$-\alpha$	α	2	2	-2	.	$-\bar\alpha$	$-\bar\alpha$	$\bar\alpha$	$-\alpha$	$-\alpha$	α	2	2	-2	.	$-\bar\alpha$	$-\bar\alpha$	$\bar\alpha$	$-\alpha$	$-\alpha$	α
χ_{13}	2	2	-2	.	$-\alpha$	$-\alpha$	α	$-\bar\alpha$	$-\bar\alpha$	$\bar\alpha$	$\bar\beta$	$\bar\beta$	$-\bar\beta$	.	$-\bar\alpha$	$-\bar\alpha$	$\bar\alpha$	-1	-1	1	β	β	$-\beta$	.	-1	-1	1	$-\alpha$	$-\alpha$	α
χ_{14}	2	2	-2	.	$-\bar\alpha$	$-\bar\alpha$	$\bar\alpha$	$-\alpha$	$-\alpha$	α	β	β	$-\beta$	.	$-\alpha$	$-\alpha$	α	-1	-1	1	$\bar\beta$	$\bar\beta$	$-\bar\beta$	.	-1	-1	1	$-\bar\alpha$	$-\bar\alpha$	$\bar\alpha$
χ_{15}	2	2	-2	.	$-\bar\alpha$	$-\bar\alpha$	$\bar\alpha$	$-\alpha$	$-\alpha$	α	$\bar\beta$	$\bar\beta$	$-\bar\beta$	.	-1	-1	1	$-\bar\alpha$	$-\bar\alpha$	$\bar\alpha$	β	β	$-\beta$	.	$-\alpha$	$-\alpha$	α	-1	-1	1
χ_{16}	2	2	-2	.	$-\alpha$	$-\alpha$	α	$-\bar\alpha$	$-\bar\alpha$	$\bar\alpha$	β	β	$-\beta$	.	-1	-1	1	$-\alpha$	$-\alpha$	α	$\bar\beta$	$\bar\beta$	$-\bar\beta$	.	$-\bar\alpha$	$-\bar\alpha$	$\bar\alpha$	-1	-1	1
χ_{17}	2	2	-2	.	-1	-1	1	-1	-1	1	$\bar\beta$	$\bar\beta$	$-\bar\beta$	.	$-\alpha$	$-\alpha$	α	$-\alpha$	$-\alpha$	α	β	β	$-\beta$	.	$-\bar\alpha$	$-\bar\alpha$	$\bar\alpha$	$-\bar\alpha$	$-\bar\alpha$	$\bar\alpha$
χ_{18}	2	2	-2	.	-1	-1	1	-1	-1	1	β	β	$-\beta$	.	$-\bar\alpha$	$-\bar\alpha$	$\bar\alpha$	$-\bar\alpha$	$-\bar\alpha$	$\bar\alpha$	$\bar\beta$	$\bar\beta$	$-\bar\beta$	.	$-\alpha$	$-\alpha$	α	$-\alpha$	$-\alpha$	α
χ_{19}	3	3	3	-1	.	.	.	.	.	.	3	3	3	-1	.	.	.	.	.	.	3	3	3	-1	.	.	.	.	.	.
χ_{20}	3	3	3	-1	.	.	.	.	.	.	γ	γ	γ	$-\alpha$	.	.	.	.	.	.	$\bar\gamma$	$\bar\gamma$	$\bar\gamma$	$-\bar\alpha$	.	.	.	.	.	.
χ_{21}	3	3	3	-1	.	.	.	.	.	.	$\bar\gamma$	$\bar\gamma$	$\bar\gamma$	$-\bar\alpha$	.	.	.	.	.	.	γ	γ	γ	$-\alpha$	.	.	.	.	.	.
χ_{22}	8	-1	.	.	-1	2	.	-1	2	.	8	-1	.	.	-1	2	.	-1	2	.	8	-1	.	.	-1	2	.	-1	2	.
χ_{23}	8	-1	.	.	$-\bar\alpha$	β	.	$-\alpha$	$\bar\beta$	.	8	-1	.	.	$-\bar\alpha$	β	.	$-\alpha$	$\bar\beta$	.	8	-1	.	.	$-\bar\alpha$	β	.	$-\alpha$	$\bar\beta$	.
χ_{24}	8	-1	.	.	$-\alpha$	$\bar\beta$	.	$-\bar\alpha$	β	.	8	-1	.	.	$-\alpha$	$\bar\beta$	.	$-\bar\alpha$	β	.	8	-1	.	.	$-\alpha$	$\bar\beta$	.	$-\bar\alpha$	β	.
χ_{25}	8	-1	.	.	$-\bar\alpha$	β	.	$-\alpha$	$\bar\beta$	.	δ	$-\alpha$	.	.	-1	2	.	$-\bar\alpha$	β	.	$\bar\delta$	$-\bar\alpha$	.	.	$-\alpha$	$\bar\beta$	.	-1	2	.
χ_{26}	8	-1	.	.	$-\alpha$	$\bar\beta$	.	$-\bar\alpha$	β	.	$\bar\delta$	$-\bar\alpha$	.	.	-1	2	.	$-\alpha$	$\bar\beta$	.	δ	$-\alpha$	.	.	$-\bar\alpha$	β	.	-1	2	.
χ_{27}	8	-1	.	.	-1	2	.	-1	2	.	δ	$-\alpha$	.	.	$-\alpha$	$\bar\beta$	.	$-\alpha$	$\bar\beta$	.	$\bar\delta$	$-\bar\alpha$	.	.	$-\bar\alpha$	β	.	$-\bar\alpha$	β	.
χ_{28}	8	-1	.	.	-1	2	.	-1	2	.	$\bar\delta$	$-\bar\alpha$	.	.	$-\bar\alpha$	β	.	$-\bar\alpha$	β	.	δ	$-\alpha$	.	.	$-\alpha$	$\bar\beta$	.	$-\alpha$	$\bar\beta$	.
χ_{29}	8	-1	.	.	$-\alpha$	$\bar\beta$	.	$-\bar\alpha$	β	.	δ	$-\alpha$	.	.	$-\bar\alpha$	β	.	-1	2	.	$\bar\delta$	$-\bar\alpha$	.	.	-1	2	.	$-\alpha$	$\bar\beta$	.
χ_{30}	8	-1	.	.	$-\bar\alpha$	β	.	$-\alpha$	$\bar\beta$	.	$\bar\delta$	$-\bar\alpha$	.	.	$-\alpha$	$\bar\beta$	.	-1	2	.	δ	$-\alpha$	.	.	-1	2	.	$-\bar\alpha$	β	.

where we have $\alpha = w^2,\ \beta = 2w,\ \gamma = 3w^2,\ \delta = 8w^2$ for $w = e^{2\pi \mathrm{i}/3}$.

Proposition 2.3. The character of the representation of G_{648} on the space $H^0(72.\mathbb{P}^2_{fake}, 3H)$ is given by

$$\chi_{11} + \chi_{12} + \chi_{19} + \chi_{23} + \chi_{24} + \chi_{25} + \chi_{26} + \chi_{27} + \chi_{28} + \chi_{29} + \chi_{30}. \tag{2.2}$$

Proof. By the Holomorphic Lefschetz Formula, together with the trivial representation of G_{648} on $H^2(72.\mathbb{P}^2_{fake}, 3H)$, our representation must restrict to the regular representation both for G_{72} and for $\widehat{G}_{72}$, since every nonidentity element of this group has no fixed points. We use GAP (GAPdataAll) to compute the restrictions of the characters to these groups and then Mathematica (Dim71rep.nb) to find the unique linear combination of characters of G_{648} that has this property. □

Proposition 2.4. The dimension 7 subspace of $H^0(72.\mathbb{P}^2_{fake}, 3H)$ with character $\chi_{11} + \chi_{12} + \chi_{19}$ can be naturally identified with

$$H^0(8.\mathbb{P}^2_{fake}, 3H) \cong H^0(8.\mathbb{P}^2_{fake}/C_3, 3H).$$

Similarly, $H^0(4.\mathbb{P}^2_{fake}, 3H)$ can be identified with χ_{19}.

Proof. The map $72.\mathbb{P}^2_{fake} \to 8.\mathbb{P}^2_{fake}$ is a Galois cover, so G_{648} acts on the space of holomorphic 2-forms on $8.\mathbb{P}^2_{fake}$, and the pullback map is compatible with the action. The subgroup G_{72} is the kernel of the abelianization map $G_{648} \to C_3 \times C_3$ and is thus built from the conjugacy classes $1a$, $2a$, $3a$ and $4a$. Then χ_{11}, χ_{12} and χ_{19} are characterized by the property that they are invariant under $3a$, i.e. invariant under the normal subgroup $C_3 \times C_3$ of G_{72}. Then χ_{19} is further characterized by having trivial action of the conjugacy class $2a$, which is the central involution of Q_8. □

3. Technical details: computing the equations.

In this section we comment in more detail on the technical issues encountered in our process, as sketched in the Introduction.

3.1. **Step 1.** Let s be a nonzero element of $H^0(2.\mathbb{P}^2_{fake}, 3H)$. Then $s^2 \in H^0(2.\mathbb{P}^2_{fake}, 6H)$ is invariant under the covering involution of the double cover $2.\mathbb{P}^2_{fake} \to \mathbb{P}^2_{fake}$ and is thus a pullback of an element of $H^0(\mathbb{P}^2_{fake}, 6H)$. Moreover, this section must be invariant with respect to the action of the central C_3. So to find s^2, we looked for nonreduced linear cuts of $\mathbb{P}^2_{fake}$ in its bicanonical embedding, which are invariant under an action of a subgroup of its automorphism group (we did not a priori know which subgroup of $\mathrm{Aut}(\mathbb{P}^2_{fake})$ came from the central C_3). We first found such nonreduced cut modulo 73 by a brute force search using Magma.[3] This calculation was entirely similar to the one in [2,6,8]. As a result, we got

$$\mathrm{Cut} = Q_0 + 69(Q_1 + Q_2 + Q_3) + 7(Q_4 + Q_5 + Q_6) + 62(Q_7 + Q_8 + Q_9) \tag{3.1}$$

where Q_i were the variables of the equations of $\mathbb{P}^2_{fake}$ from [3].

Our next goal was to lift the equation of the nonreduced cut (3.1) from $\mathbb{Z}/73\mathbb{Z}$ to $\mathbb{Z}/73^k\mathbb{Z}$ for increasing powers of k. The previous method, used in the aforementioned papers was to find some points on the nonreduced cut, and enforce the condition of the cut being singular on (lifts of) the points as k grows. However, this approach was unavailable in our case because there were no points on X defined over $\mathbb{Z}/73\mathbb{Z}$. While we could have presumably worked over a finite field extension, we found the following easier alternative approach.

[3]Unfortunately, 73 was the smallest prime of the form $9k+1$ where the equations of [3] gave a reduced surface with the correct Hilbert polynomial, and it took a considerable amount of time to go through all of the cases.

We used Magma to compute the ideal of the radical of the nonreduced cut of $\mathbb{P}^2_{fake}$ modulo 73. One of the equations was

$$\begin{aligned} F &= Q_3Q_6 + 61Q_4Q_6 + 29Q_5Q_6 + 53Q_6^2 + 9Q_1Q_7 + 18Q_2Q_7 + 42Q_3Q_7 \\ &+32Q_4Q_7 + 15Q_5Q_7 + 9Q_6Q_7 + 11Q_7^2 + 25Q_1Q_8 + 3Q_3Q_8 + 13Q_4Q_8 \\ &+18Q_5Q_8 + 21Q_6Q_8 + 11Q_7Q_8 + 44Q_8^2 + 49Q_1Q_9 + 63Q_2Q_9 + 53Q_3Q_9 \\ &+12Q_4Q_9 + 26Q_5Q_9 + 12Q_6Q_9 + 51Q_7Q_9 + 68Q_8Q_9 + 44Q_9^2 \end{aligned}$$

in the variables Q_i of the bicanonical embedding of $\mathbb{P}^2_{fake}$ modulo 73. Then F^2 was in the ideal of the cut, and we wrote

$$F^2 = \sum_{i=1}^{84} H_i E_i + \mathrm{Cut}\, R \tag{3.2}$$

as polynomials in $Q_0, \ldots, Q_9$. Here E_i are the equations of $\mathbb{P}^2_{fake}$ (cubic in Q), H_i are unknown linear combinations of Q, and R is an unknown cubic polynomial in Q. We originally computed a relation (3.2) modulo 73 and then lifted it modulo 73^k for increasing powers of k. At each step $k \to k+1$ we had a system of linear equations modulo 73 on the corrections to the coefficients of H_i, F and R. We used Mathematica to solve it, and picked the initial solution which was automatically taking case of making some coefficients zero. We went up to 73^{30} which gave a good approximation to coefficients of the nonreduced cut over the complex numbers.

There is a standard way of guessing an algebraic number from its p-adic approximation. We used it to see that the nonreduced cut is given by

$$\begin{aligned} &Q_0 - \tfrac{(-773+16\,\mathrm{i}\sqrt{2})}{66449}(Q_1+Q_2+Q_3) - W\tfrac{(-50345-26294\,\mathrm{i}\sqrt{2})}{132898}(Q_4+Q_5+Q_6) \\ &-\tfrac{(50345+26294\,\mathrm{i}\sqrt{2})}{132898\,W}(Q_7+Q_8+Q_9) \end{aligned} \tag{3.3}$$

where $W = (\frac{1}{3}(2-\mathrm{i}\sqrt{2}))^{\frac{1}{3}}$. The details of the above calculation are in the Mathematica file Step1.nb.

3.2. **Step 2.** We work out Steps 2 and 3 in the Mathematica file Steps23.nb. By simply scaling Q-s by the appropriate powers of W, we arranged the cut of (3.3) to be defined over $\mathbb{Q}(\sqrt{-2})$ with equations of $\mathbb{P}^2_{fake}$ still defined over this field. However, it was convenient for us to enlarge the field to $\mathbb{Q}(\sqrt{-2}, \sqrt{-3})$ and to pick a basis of eigenvectors of the $C_3 \times C_3$ action on $H^0(\mathbb{P}^2_{fake}, 6H)$, with the new variables called $R_0, \ldots, R_9$. We made the cut to be $R_1 + R_4 + R_7$, made one of the fixed points of a C_3 action to be

$$(0:1:1:1:0:0:0:0:0:0)$$

and made the tangent space to the cone over $\mathbb{P}^2_{fake}$ at this point to be generated by

$$(0,1,1,1,0,0,0,0,0,0), (0,0,0,0,1,1,1,0,0,0), (0,0,0,0,0,0,0,1,1,1).$$

These conditions fixed the basis of R_i, and the resulting equations of $\mathbb{P}^2_{fake}$ had very small coefficients in $\mathbb{Z}[\sqrt{-2}, \sqrt{-3}]$.

3.3. **Step 3.** The torsion subgroup of the Picard group of $\mathbb{P}^2_{fake}$ is isomorphic to $C_{13} \times C_2 \times C_2$, and the nontrivial two-torsion elements are acted upon by an order 3 automorphism of $\mathbb{P}^2_{fake}$ which scales R_4, R_5, R_6 by the primitive third root of unity $w = \frac{1}{2}(-1 + \mathrm{i}\sqrt{3})$ and scales R_7, R_8, R_9 by w^2. Each 2-torsion element in the Picard group of $\mathbb{P}^2_{fake}$ gives a nonreduced cut of it in the bicanonical embedding, which are thus

$$R_1 + R_4 + R_7,\ R_1 + wR_4 + w^2R_7,\ R_1 + w^2R_4 + wR_7, \tag{3.4}$$

in the R_i coordinates.

The field of fractions of the ring $\bigoplus_{k\geq 0} H^0(4.\mathbb{P}^2_{fake}, 3kH)$ is obtained from that of $\bigoplus_{k\geq 0} H^0(\mathbb{P}^2_{fake}, 6kH)$ by attaching the square roots of the linear forms (3.4). The sections of $H^0(2.\mathbb{P}^2_{fake}, 6H)$ of the double cover can be then obtained as pullbacks of R_i and as

$$\frac{F(R)}{\sqrt{R_1 + wR_4 + w^2R_7}\sqrt{R_1 + w^2R_4 + wR_7}}$$

where $F(R)$ are quadratic polynomials in R_i with the property that they are zero on the loci of zeros of $R_1 + wR_4 + w^2R_7$ and $R_1 + w^2R_4 + wR_7$. These have been computed and given the names $U_0, \ldots, U_{19}$ where $U_i = R_i$ for $0 \leq i \leq 9$ form a basis of the subspace of invariants of the covering involution of $2.\mathbb{P}^2_{fake} \to \mathbb{P}^2_{fake}$, and the $U_{10}, \ldots, U_{19}$ form a basis of the (-1)-eigenspace. Adding the C_3 translates of the latter gave us a basis $U_0, \ldots, U_{39}$ of $H^0(4.\mathbb{P}^2_{fake}, 6H)$ in terms of R_i and the above square roots. We also extend the action of $C_3 \times C_3$ to these U_i.

Remark 3.1. We computed equations of the double cover $2.\mathbb{P}^2_{fake}$, i.e. the relations among $U_0, \ldots, U_{19}$ and got the expected dimension 100 space of these equations. We suspect that these quadratic equations cut out $2.\mathbb{P}^2_{fake}$ in its bicanonical embedding, but we did not try ascertain that (and it also may be beyond the reach of our hardware). We did not use these equations later in our computations.

3.4. **Step 4.** Step 4 takes a lot of work, and it is done in Step4.nb.

We first recall that $8.\mathbb{P}^2_{fake}$ is acted upon by

$$C_3 \times SL(2, \mathbb{Z}/3\mathbb{Z})$$

so that the quotient by the normal 2-Sylow subgroup Q_8 of $SL(2, \mathbb{Z}/3\mathbb{Z})$ induces the automorphism action of $C_3 \times C_3$ on $\mathbb{P}^2_{fake}$. By our construction of Step 3, we have also lifted the action of $C_3 \times C_3$ to act on $4.\mathbb{P}^2_{fake}$, which is the quotient of $8.\mathbb{P}^2_{fake}$ by the central involution $\sigma \in Q_8$. Holomorphic Lefschetz formula allows one to figure out the action of $C_3 \times SL(2, \mathbb{Z}/3\mathbb{Z})$ on $H^0(8.\mathbb{P}^2_{fake}, 3H)$ and there exist, unique up to scaling, two elements s_1 and s_2 of $H^0(8.\mathbb{P}^2_{fake}, 3H)$ with the following properties.

- Both s_i are (-1)-eigenfunctions for the covering involution σ of $8.\mathbb{P}^2_{fake} \to 4.\mathbb{P}^2_{fake}$.
- Both s_i are invariant with respect to the central C_3.
- Both s_i are eigenfunctions for the action of the other C_3, one with weight w and the other with weight w^2.

Consequently, $f_0 = s_1 s_2$, $f_1 = s_1^2$ and $f_2 = s_2^2$ are invariant with respect to σ and are pullbacks of elements of $H^0(4.\mathbb{P}^2_{fake}, 6H)$ (i.e. linear combinations of $U_0, \ldots, U_{39}$) that satisfy

$$f_0^2 = f_1 f_2 \tag{3.5}$$

and have prescribed weights with respect to the $C_3 \times C_3$ action on U_i. This resulted in a system of 26 quadratic equations on 16 unknown coefficients of f_i. After a fortunate choice of two additional scaling equations (since s_i are only up to scaling, we can scale two out of three f_i), Mathematica readily solved the resulting equations in 14 variables numerically and then recognized the results as good approximations to algebraic numbers in $\mathbb{Q}(\sqrt{-2}, \sqrt{-3})$.

There is a two-dimensional subspace of $H^0(8.\mathbb{P}^2_{fake}, 3H)$ which is anti-invariant with respect to σ and is invariant with respect to both C_3 groups. For an element v of it, we knew where v^2 and vs_1 were, which allowed us to find it. This gave us seven linearly independent elements of $H^0(8.\mathbb{P}^2_{fake}, 3H)$, namely the square roots of (3.4), and four other of the form $F_i(U)/\sqrt{f_1}$ for a solution f_1 of (3.5) and a linear function F_i of $U_0, \ldots, U_{39}$. We denoted this basis by $V_0, \ldots, V_6$.

By Proposition 2.2 sections $V_0, \ldots, V_6$ of $H^0(8.\mathbb{P}^2_{fake}, 3H)$ are all invariant with respect to the action of the central C_3 (this can also be seen by direct examination). As a consequence, they are pullbacks of the sections of the canonical line bundle on the singular surface $8.\mathbb{P}^2_{fake}/C_3$ with 24 singularities of type A_2. We found equations on V_i, namely a 4-dimensional space of cubic equations and a 58-dimensional space of quartic equations. These equations allowed us to find points on the canonical image of $8.\mathbb{P}^2_{fake}$ (or $8.\mathbb{P}^2_{fake}/C_3$) with high accuracy.

The next step was to lift the action of $C_2 \times C_2$ on $4.\mathbb{P}^2_{fake}$ to an action of Q_8 on $8.\mathbb{P}^2_{fake}$. Specifically, this meant finding an order 4 automorphism which lifted the order 2 automorphism of $4.\mathbb{P}^2_{fake}$. We also knew that its action on the dimension four subspace spanned $V_3, \ldots, V_6$ was traceless, and that the action permuted the points with $V_1 = V_2 = 0$. Taken together, this information allowed us to find the desired order 4 automorphism.

Our next goal was to understand the space $H^0(8.\mathbb{P}^2_{fake}, 6H)$. We have $\dim H^0(8.\mathbb{P}^2_{fake}, 6H) = 80$, so it would be rather useless to try to compute equations among these, since solving systems of nonlinear equations in 80 variables is well beyond the capabilities of our available hardware. Instead, we had to settle for being able to compute a lot of points in the image

$8.\mathbb{P}^2_{fake} \to \mathbb{CP}^{79}$ with high accuracy. The approach we took was to first compute points on $8.\mathbb{P}^2_{fake}/C_3$ where we do have equations and then compute the values of elements $H^0(8.\mathbb{P}^2_{fake}, 6H)$ on them.

We observed that R_0, R_1, R_4, R_7 can be easily written as degree two polynomials in V_i. In contrast, R_2 is not invariant under the central C_3 and can therefore not be written as a rational function in V_i. However, R_2^3 can be written as a rational function in R_0, R_1, R_4, R_7 (namely as a ratio of a degree 12 polynomial and a degree 9 polynomial), and we computed it. Similarly, we computed rational functions in these variables for $R_2R_3, R_2R_6, R_2R_9, R_2^2R_5, R_2^2R_8$. Recall that U_i for $0 \le i \le 39$ are written as rational functions in R_i and V_0, V_1, V_2. Therefore, for given (high accuracy) values of V_i, we can find three values for R_2 and then find values of the rest of R_i and $U_0, \ldots, U_{39}$ for each of three values of R_2.

We then computed the subspace of $H^0(8.\mathbb{P}^2_{fake}, 6H)$ of sections that are anti-invariant with respect to the covering involution σ. We did this by considering rational functions in V-s and R_2, of total V-degree 2, which are zero on the curves $V_0+V_1+V_2 = 0$ and $V_3 = 0$, divided by $(V_0+V_1+V_2)V_3$. We first got a database of points on these two curves, and then computed vanishing conditions. The calculation was performed in Step4.nb and is split into three cases according to the character of the central C_3. Specifically, for the trivial character, we looked for degree four polynomials in V_i which vanish at the aforementioned curves. For the other characters, we looked for linear combinations of products of quadratic polynomials in V with some sections of $H^0(4.\mathbb{P}^2_{fake}, 6H)$ with the same central weight.

Afterwards, we computed the action of the two C_3-s (the central one and the chosen subgroup of $SL(2, \mathbb{Z}/3\mathbb{Z})$) on the space $H^0(4.\mathbb{P}^2_{fake}, 6H)$ of dimension 80. We picked an eigenbasis of it, denoted by $\tilde{U}_0, \ldots, \tilde{U}_{79}$. Finally, we computed points on $8.\mathbb{P}^2_{fake}$ with accuracy of several thousand digits, in preparation for the next step.

3.5. **Step 5.** Naturally, this is the trickiest step of the whole paper, worked out in Step5.nb.

The map $72.\mathbb{P}^2_{fake} \to 8.\mathbb{P}^2_{fake}$ is a Galois cover with the covering group $C_3 \times C_3$, and we have a good understanding of $H^0(72.\mathbb{P}^2_{fake}, 3H)$ by Proposition 2.3. In what follows, we will denote the corresponding subspaces of $H^0(72.\mathbb{P}^2_{fake}, 3H)$ as $H^0(72.\mathbb{P}^2_{fake}, 3H)_{11}, \ldots, H^0(72.\mathbb{P}^2_{fake}, 3H)_{30}$, according to the index of the irreducible character. Note that each of the 8-dimensional irreps of $H^0(72.\mathbb{P}^2_{fake}, 3H)_i$ for $23 \le i \le 30$ has one-dimensional eigenspaces for all non-trivial characters of the covering group. Indeed, all nonzero elements of this group are in the conjugacy class $3a$ and thus have trace (-1). We also observe that each of these representations is acted upon by the central involution σ of Q_8 which permutes $C_3 \times C_3$ eigenspaces by inverting eigenvalues, because it corresponds to $(-\mathrm{Id})$ in $SL(2, \mathbb{Z}/3\mathbb{Z})$. The following proposition is the key to our approach.

Proposition 3.2. Consider an order 3 element h of $SL(2, \mathbb{Z}/3\mathbb{Z})$ and its action on the 3-torsion subgroup $C_3 \times C_3$ of $\mathrm{Pic}(8.\mathbb{P}^2_{fake})$. Suppose that the character $(w, 1)$ of the covering $C_3 \times C_3$ corresponds to the eigenvector of h in $C_3 \times C_3$. Let f_1 and $f_2 = \sigma(f_1)$ be a $(w, 1)$-eigenvector and a $(w^2, 1)$-eigenvector for the covering $C_3 \times C_3$ in the space $H^0(72.\mathbb{P}^2_{fake}, 3H)_{29}$, respectively. Likewise, let g_1 and $g_2 = \sigma(g_1)$ be an $(w, 1)$- and $(w^2, 1)$-eigenvectors in the space $H^0(72.\mathbb{P}^2_{fake}, 3H)_{30}$. Then $s_1 = f_1 f_2$, $s_2 = f_1 g_2$, $s_3 = g_1 f_2$, $s_4 = g_1 g_2$ are invariant under the covering group and can be thought of as elements of $H^0(8.\mathbb{P}^2_{fake}, 6H)$. These sections s_i have the following properties.

- $s_1 s_4 = s_2 s_3$
- $\sigma(s_2) = s_3$
- Sections s_1, s_2, s_3, s_4 have weights $(w, 1, 1, w^2)$ respectively for the central C_3 action on $H^0(8.\mathbb{P}^2_{fake}, 6H)$.
- The weights of s_1, s_2, s_3, s_4 for the action of $h \in SL(2, \mathbb{Z}/3\mathbb{Z})$ are $(w^{2a}, w^{a+b}, w^{a+b}, w^{2b})$ for some a and b in $\mathbb{Z}/3\mathbb{Z}$.

Proof. The first two statements are immediate from the construction. To prove the third statement, observe that the generator of the central C_3 has trace $8w^2$ in χ_{29} and $8w$ in χ_{30} (after an appropriate choice of generator or a switch of χ_{29} and χ_{30}). Thus f_i have eigenvalues w^2 and g_i have eigenvalues w.

The last statement is the most delicate. Since h preserves the corresponding element of the Picard group, its action preserves the corresponding eigenspaces of $H^0(72.\mathbb{P}^2_{fake}, 3H)_{29}$ and $H^0(72.\mathbb{P}^2_{fake}, 3H)_{30}$. Thus f_i and g_i are eigenvectors for its action, with eigenvalues w^a and w^b for some a and b. $\square$

Remark 3.3. There is nothing particularly special about using χ_{29} and χ_{30} in Proposition 3.2. In fact, 29 can be replaced by 25 or 27 and 30 can be replaced by 26 or 28. Since we do not know which values of a and b correspond to which subrepresentations, as we get a solution $(s_1, \ldots, s_4)$ we will not know exactly which subrepresentations they come from.

For each pair of values (a, b), the conditions of Proposition 3.2 can be translated into a system of polynomial equations on the coefficients of s_i in the bases of the corresponding subspaces of $H^0(8.\mathbb{P}^2_{fake}, 6H)$. The number of variables is generally under 20, and we were able to solve one the systems. Specifically, we solved it modulo 4363, which is a large prime for which both (-2) and (-3) are quadratic residues, then lifted the solution to powers of 4363 and finally used this p-adic approximation of solutions to realize them as algebraic numbers.

Getting an equation of the form $s_1 s_4 = s_2 s_3$ is indicative of some additional divisor classes, given by (s_1, s_3) and (s_1, s_4). We computed the corresponding divisors and found that a third power can be written as a section of $H^0(8.\mathbb{P}^2_{fake}, 9H)$. Specifically, we were able to write it as a degree 3 polynomial (called *goodrr* in the Mathematica file Step5.nb) in $V_0, \ldots, V_6$, since a third power is also invariant with respect to the central C_3. In view of Remark 3.3, we do not know precisely which irreducible subrepresentation the corresponding section $f_1 \in H^0(72.\mathbb{P}^2_{fake}, 3H)$ lies in, but it is not important to us. Indeed, we know from Proposition 2.1 that unramified triple covers of $8.\mathbb{P}^2_{fake}$ come from $72.\mathbb{P}^2_{fake}$, and we know that by adding f_1, and its Q_8-translates to the function field of the cone over $8.\mathbb{P}^2_{fake}$, we will get the function field of the cone over $72.\mathbb{P}^2_{fake}$.

More precisely, we computed a basis of a dimension 8 subspace of elements of $H^0(8.\mathbb{P}^2_{fake}, 6H)$ which vanish on $f_1 = 0$ and were thus able to describe a set of 8 linearly independent sections in $H^0(72.\mathbb{P}^2_{fake}, 3H)$ as R/f_1 for R in this subspace. Then Q_8 translates of these forms, together with (pullbacks of) $V_i \in H^0(8.\mathbb{P}^2_{fake}, 3H)$ gave the basis of $H^0(72.\mathbb{P}^2_{fake}, 3H)$. To be able to really compute values of elements of $H^0(72.\mathbb{P}^2_{fake}, 3H)$ on points of $72.\mathbb{P}^2_{fake}$ we needed to be careful in identifying values of Q_8-translates f_i of f_1. While we knew their cubes, it was not clear which cubic roots had to be taken. This issue was solved by computing products $f_i f_j f_k$ which lie in $H^0(8.\mathbb{P}^2_{fake}, 9H)$ and using the values of the products to pick correct values of all but two f_i (first two f_i can be taking arbitrarily, each choice giving one of the preimage points of $C_3 \times C_3$ cover $72.\mathbb{P}^2_{fake} \to 8.\mathbb{P}^2_{fake}$).

In order to construct the surfaces $9.\widehat{\mathbb{P}^2_{fake}}$ and $\widehat{\mathbb{P}^2_{fake}}$ we found a lift of the action of $C_3 \times SL(2, \mathbb{Z}/3\mathbb{Z})$ from $8.\mathbb{P}^2_{fake}$ to $72.\mathbb{P}^2_{fake}$ by picking lifts of the generators. We then averaged over Q_8 to get values of sections of $H^0(9.\widehat{\mathbb{P}^2_{fake}}, 3H)$, called $W_1, \ldots W_8$. We similarly averaged over $C_3 \times SL(2, \mathbb{Z}/3\mathbb{Z})$ to get a basis of $H^0(\widehat{\mathbb{P}^2_{fake}}, 6H)$, called $Z_0, \ldots, Z_9$. We computed equations on Z_i, which were $\dim 84$ space of cubics in Z_i, with coefficients in $\mathbb{Q}(\sqrt{-2}, \sqrt{-3})$. In fact, we had to assume that the equations would lie in this field and still had to use several thousand digits of accuracy in our computation of points. We also computed the values of four pairwise products of W_i which lie in $H^0(\widehat{\mathbb{P}^2_{fake}}, 6H)$ which gave natural reducible linear cuts of $\widehat{\mathbb{P}^2_{fake}}$ in its bicanonical embedding. These were used in the next step.

3.6. **Step 6.** At this point we had putative equations of $\widehat{\mathbb{P}^2_{fake}}$ but the coefficients were large and were defined over $\mathbb{Q}(\sqrt{-2}, \sqrt{-3})$. Both of these features made working with this surface difficult. We followed a rather ad hoc process which somewhat surprisingly allowed us to take care of both issues.

First or all, for each pair of reducible cuts, found in Step 5, we computed their 36 intersection points on $\widehat{\mathbb{P}^2_{fake}}$. We speculated that Z_0 had to be defined over $\mathbb{Q}(\sqrt{-2})$, in the sense that there is a model of $\widehat{\mathbb{P}^2_{fake}}$ over this field where Z_0 is defined over it. We normalized the 36 points of intersections to have $Z_0 = 1$. Then we separated these 36 points according to their field of definition. We added these points to get linear combinations of the basis dual to Z_i with coefficients in the field $\mathbb{Q}(\sqrt{-2}, \sqrt{-3})$. As the pairs varied, we ended up picking 21 such points. We also speculated that Z_1 should be defined over $\mathbb{Q}(\sqrt{-2})$ and used natural linear combinations of the above 21 points to get 13 natural points in $\mathbb{C}^{10}$. We then picked 10 linearly independent ones and used a linear change of variables so that new sections were a dual basis. The resulting equations were indeed defined over $\mathbb{Q}(\sqrt{-2})$, but the coefficients were up to 8×10^3 digits long. The process was further refined by picking a small $\mathbb{Q}(\sqrt{-2})$-linear combination of the above 10 points defined over $\mathbb{Q}(\sqrt{-2}, \sqrt{-3})$. That led to equations in the new variables $Y_0, \ldots Y_9$ which were in $\mathbb{Q}(\sqrt{-2})$ and had coefficients a few hundred digits long. Finally, we traded the number of nonzero terms for the size of the coefficients by picking linear combinations of the equations via a lattice reduction algorithm. This led to the final output where the equations were only 20 to 30 decimal digits long, in $\mathbb{Z}[\sqrt{-2}]$. It seems plausible that one can reduce the coefficients further by picking a better basis of $H^0(\widehat{\mathbb{P}^2_{fake}}, 6H)$, but we were unable to do so.

The details are in the file Step6.nb.

3.7. **Step 7.** The techniques of the previous steps used probabilistic approaches and approximate calculations, and the overall complexity of the code was also formidable. Fortunately, it is possible to verify that the surface we obtained is a fake projective plane by doing exact and relatively short calculations in Magma. We can then confidently identify it as $(C18, p = 3, \{2I\})$. The method of verification that the surface is an FPP has not changed much since [4]. Specifically, we first observed that the surface S in question has the correct Hilbert polynomial. Then we showed that it is smooth by picking three random minors of the Jacobian matrix and checking that adding them to the equations gives zero Hilbert polynomial over a finite field. For better or for worse, we used the same minors as in [8], and it worked. We also computed the dimension of the cohomology spaces of the structure sheaf and the first cohomology space of the cotangent bundle. This allowed us to conclude that the surface is an FPP. Then it suffices to compute, as in [7], that $h^2(X, 2K_X(-1)) = 0$ to show that our embedding is precisely the bicanonical one. As a slight improvement over previous approaches, we did the calculations entirely in Magma, as opposed to a mix of Magma and Macaulay2. The details are in Step7Magma (Hilbert polynomial of $\widehat{\mathbb{P}^2_{fake}}$ over the number field) and Step7Magma4363 (the rest).

References

[1] I. Bauer, F. Catanese, R. Pignatelli, *Surfaces of general type with geometric genus zero: a survey. Complex and differential geometry*, 1–48, Springer Proc. Math., 8, Springer, Heidelberg, 2011.

[2] L. Borisov, *On equations of fake projective planes with automorphism group of order 21.* Épijournal Géom. Algébrique 7 (2023), Art. 17, 19 pp.

[3] L. Borisov, A. Buch, E. Fatighenti, *A journey from the octonionic* $\mathbb{P}^2$ *to a fake* $\mathbb{P}^2$, Proc. Amer. Math. Soc. 150 (2022), no. 4, 1467–1475.

[4] L. Borisov, E. Fatighenti, *New explicit constructions of surfaces of general type*, preprint arXiv:2004.02637, to appear in Journal of Algebraic Geometry.

[5] L. Borisov, M. Ji, Y. Li, *Explicit equations of the fake projective plane (a=7,p=2,∅,D3X7).* Comm. Algebra 53 (2025), no. 8, 3477–3485.

[6] L. Borisov, M. Ji, Y. Li, S. Mondal, *On the Geometry of a Fake Projective Plane with 21 Automorphisms.* preprint arXiv:2308.10429, to appear in Involve.

[7] L. Borisov, J. Keum, *Explicit equations of a fake projective plane*, Duke Math. J. 169 (2020), no. 6, 1135–1162.

[8] L. Borisov, Z. Lihn, *Realizing a Fake Projective Plane as a Degree 25 Surface in* $\mathbb{P}^5$. J. Korean Math. Soc. 61 (2024), no. 4, 683–692.

[9] Auxiliary files attached to the current paper. *GAPdataAll, Dim71.nb, Step1Magma, Step1.nb, Steps23.nb, Step4.nb, Step5.nb, Step6.nb, Step7Magma, Step7Magma4363, BetterNewFPPrr.*

[10] D. Cartwright, T. Steger, *Enumeration of the 50 fake projective planes*, C. R. Acad. Sci. Paris, Ser. I **348** (2010), 11–13.

[11] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.13.1 of 2024-06-11. (http://www.gap-system.org)

[12] M.-N. Ishida, *An elliptic surface covered by Mumford's fake projective plane*, Tohoku Math.J. (2) **40** no. 3 (1988), 367–396.

[13] J. Keum, *A fake projective plane with an order 7 automorphism*, Topology **45** (2006), 919–927.

[14] J. Keum, *Quotients of fake projective planes*, Geom. Topol. **12** (2008), no. 4, 2497–2515.

[15] V. S. Kharlamov, V. M. Kulikov, *On real structures on rigid surfaces*, Izv. Russ. Akad. Nauk. Ser. Mat. **66**, no. 1, (2002), 133-152; Izv. Math. **66**, no. 1, (2002), 133–150.

[16] B. Klingler, *Sur la rigidité de certains groupes fondamentaux, l'arithméticité des réseaux hyperboliques complexes, et les "faux plans projectifs"*, Invent. Math. **153** (2003), 105–143.

[17] W. Bosma, J. Cannon, C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput., 24 (1997), 235–265.

[18] Wolfram Research, Inc., Mathematica, Version 14.1.0, 2024.

[19] D. Mumford, *An algebraic surface with K ample,* $K^2 = 9$, $p_g = q = 0$, Amer. J. Math. **101** (1979) 233–244.

[20] G. Prasad, S.-K. Yeung, *Fake projective planes*, Invent. Math. **168** (2007), 321–370; Addendum, **182** (2010), 213–227.

Department of Mathematics, Rutgers University, Piscataway, NJ 08854
E-mail address: `borisov@math.rutgers.edu`

Department of Mathematics, Rutgers University, Piscataway, NJ 08854
E-mail address: `bw391@math.rutgers.edu`