This notebook constructs a surface FPP/C3 with expected \pi_1=C_{13} and finds its singular points.
It is a supporting file for the Section 2 of the joint paper of L.B., A.B., and E.F.

We use the thesis of Jacob Lurie and define the Cartan cubic in 27 variables.

The variables will be lines on the Fermat cubic x0^3+...+x3^3=0.
These lines are the line through (1:a:0:0), (0:0:1:b) with a^3=b^3=-1
and its permutations.

There are 27 collections of triples are given by translates of (x0+x1+x2+x3=0)
[(1:-1:0:0),(0:0:-1:1)], [(1:0:0:-1),(0:1:-1:0)],[(1:0:-1:0),(0:1:0:-1)].
Specifically, we can multiply the variables by cube roots of 1 but the first factor can be taken
as 1. There are additional 18 collections which are translates of (x0+x1=0).
(multiply x1 by a cube root, and also 6 pairs of elements).

The 9 triples that give positive signs are x_i + a x_j = 0 with 1<i<j.

In[∘]:=
```
(* setting up the variables *)
VarsP = {P1, P2, P3, P4, P5, P6, P7, P8, P9, P10, P11, P12, P13,
   P14, P15, P16, P17, P18, P19, P20, P21, P22, P23, P24, P25, P26, P27};

(* setting up the lines *)
w = Exp[2 Pi I / 3];
Lines = {};
For[i = 1, i ≤ 3, i++,
  For[j = 1, j ≤ 3, j++,
   AppendTo[Lines, {{1, -w^i, 0, 0}, {0, 0, 1, -w^j}}];
   AppendTo[Lines, {{1, 0, -w^i, 0}, {0, 1, 0, -w^j}}];
   AppendTo[Lines, {{1, 0, 0, -w^i}, {0, 1, -w^j, 0}}];
  ]];
Length[Lines]

(* setting up the extra C3 action
   we tried permuting last three variables, it didn't work
```

```
   now just scaling the first coordinate *)
C3action = {};
Pairs = {};
For[i = 1, i ≤ 27, i++,
  For[j = 1, j ≤ 27, j++,
    line1 = Lines[[i]];
    line2 = Lines[[j]];
    line2[[1]][[1]] = w line2[[1]][[1]];
    line2[[2]][[1]] = w line2[[2]][[1]];
    If[MatrixRank[Join[line1, line2]] == 2,
     AppendTo[C3action, VarsP[[i]] → VarsP[[j]]];
     AppendTo[Pairs, {i, j}];
    ];
   ];];
{C3action, Length[C3action]}


(* setting up planes *)
PlusPlanes = {};
For[i = 1, i ≤ 3, i++,
  AppendTo[PlusPlanes, {0, 1, w^i, 0}];
  AppendTo[PlusPlanes, {0, 1, 0, w^i}];
  AppendTo[PlusPlanes, {0, 0, 1, w^i}];
 ];

Length[PlusPlanes]

MinusPlanes = {};
For[i = 1, i ≤ 3, i++,
  AppendTo[MinusPlanes, {1, w^i, 0, 0}];
  AppendTo[MinusPlanes, {1, 0, w^i, 0}];
  AppendTo[MinusPlanes, {1, 0, 0, w^i}];
 ];

For[i = 1, i ≤ 3, i++,
  For[j = 1, j ≤ 3, j++,
   For[k = 1, k ≤ 3, k++,
    AppendTo[MinusPlanes, {1, w^i, w^j, w^k}];
   ]]];
Length[MinusPlanes]

CartanCubic = 0;
(* get the polynomial *)
For[i = 1, i ≤ 9, i++,
```

```
  plane = PlusPlanes[[i]];
  mon = 1;
  For[j = 1, j ≤ 27, j++,
   If[(Simplify[plane.Lines[[j]][[1]]] == 0) &&
      (Simplify[plane.Lines[[j]][[2]]] == 0),
     mon = mon VarsP[[j]];
    ];
  ];
  CartanCubic += mon;
 ];

CartanCubic


For[i = 1, i ≤ 36, i++,
  plane = MinusPlanes[[i]];
  mon = 1;
  For[j = 1, j ≤ 27, j++,
   If[(plane.Lines[[j]][[1]] == 0) && (plane.Lines[[j]][[2]] == 0),
     mon = mon VarsP[[j]];
    ];
  ];
  CartanCubic -= mon;
 ];

CartanCubic
```

```
Out[•]= 27

Out[•]= {{P1 → P10, P2 → P11, P3 → P12, P4 → P13, P5 → P14, P6 → P15,
   P7 → P16, P8 → P17, P9 → P18, P10 → P19, P11 → P20, P12 → P21, P13 → P22,
   P14 → P23, P15 → P24, P16 → P25, P17 → P26, P18 → P27, P19 → P1, P20 → P2,
   P21 → P3, P22 → P4, P23 → P5, P24 → P6, P25 → P7, P26 → P8, P27 → P9}, 27}

Out[•]= 9

Out[•]= 36

Out[•]= P1 P10 P19 + P11 P2 P20 + P12 P21 P3 + P13 P22 P4 +
  P14 P23 P5 + P15 P24 P6 + P16 P25 P7 + P17 P26 P8 + P18 P27 P9
```

Out[∘]= - P10 P13 P16 - P11 P14 P17 - P12 P15 P18 - P16 P17 P18 + P1 P10 P19 - P1 P18 P2 +
P11 P2 P20 - P1 P14 P21 - P16 P20 P21 - P18 P22 P23 - P17 P19 P24 - P13 P2 P24 -
P14 P15 P25 - P19 P22 P25 - P12 P13 P26 - P20 P23 P26 - P10 P11 P27 - P21 P24 P27 -
P25 P26 P27 + P12 P21 P3 - P10 P23 P3 - P2 P25 P3 - P15 P20 P4 + P13 P22 P4 -
P17 P3 P4 - P12 P19 P5 + P14 P23 P5 - P27 P4 P5 - P11 P22 P6 + P15 P24 P6 - P1 P26 P6 -
P16 P5 P6 - P11 P12 P7 - P23 P24 P7 + P16 P25 P7 - P1 P4 P7 - P10 P15 P8 - P21 P22 P8 +
P17 P26 P8 - P2 P5 P8 - P13 P14 P9 - P19 P20 P9 + P18 P27 P9 - P3 P6 P9 - P7 P8 P9

In[∘]:= **Eqs27 = Table[D[CartanCubic, VarsP[[i]]], {i, 1, 27}];**

In[∘]:=
```
(* we will find the Cartan subgroup of the E_
 6 that will be used to find an element of order 13 *)
grading = {};
For[i = 1, i ≤ Length[MonomialList[CartanCubic]], i++,
  mon = MonomialList[CartanCubic][[i]];
  AppendTo[grading, Table[Exponent[mon, VarsP[[k]]], {k, 1, 27}]];
 ];
degrel = NullSpace[grading]

count = 0;
For[i1 = 0, i1 ≤ 12, i1++,
  Print[i1];
  For[i2 = 0, i2 ≤ 12, i2++,
   For[i3 = 0, i3 ≤ 12, i3++,
    For[i4 = 0, i4 ≤ 12, i4++,
     For[i5 = 0, i5 ≤ 12, i5++,
      For[i6 = 0, i6 ≤ 12, i6++,

       wt = Mod[degrel[[1]] i1 + degrel[[2]] i2 + degrel[[3]] i3 + degrel[[4]] i4
           + degrel[[5]] i5 + degrel[[6]] i6, 13];
       If[Sort[wt] == {0, 0, 0, 1, 1, 2, 2, 3, 3,
           4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9, 10, 10, 11, 11, 12, 12},
        isOK = 1;
        For[m = 1, m ≤ 27, m++,
         If[Mod[wt[[Pairs[[m]][[1]]]] - 9 wt[[Pairs[[m]][[2]]]], 13] ≠ 0,
           isOK = 0; m = 28;
          ];
        ];
        If[isOK == 1,
         count++;
         Print[{i1, i2, i3, i4, i5, i6}];
         i1 = 13; i2 = 13; i3 = 13; i4 = 13; i5 = 13; i6 = 13;
        ];];
      ]]]]]];
count

deggood = Mod[{0, 1, 2, 1, 12, 5}.degrel, 13]
Sort[deggood]

Eqs27[[9]]
Eqs27[[18]]
Eqs27[[27]]
```

Out[∘]= {{0, 0, 1, 0, -1, 0, 0, 1, -1, -1, 0, 0, 0, 1, 0, 1, -1, 0, 1, 0, -1, 0, 0, 0, -1, 0, 1},
{-1, 1, 0, 1, -1, 0, 0, 0, 0, 0, 0, 0, -1, 1, 0, 1, -1, 0, 1, -1, 0, 0, 0, 0, -1, 1, 0},
{1, -1, 1, 0, 0, -1, -1, 1, 0, -1, 1, 0, 0, 0, 0, 1, -1, 0, 0, 0, -1, 0, 0, 1, 0, 0, 0},
{0, 1, -1, 1, -1, 0, -1, 0, 1, 0, 0, 1, -1, 0, 0, 1, 0, -1, 0, -1, 0, 0, 1, 0, 0, 0, 0},
{0, 1, -1, 0, 0, 0, 0, -1, 1, 1, -1, 1, -1, 0, 0, 0, 1, -1, -1, 0, 0, 1, 0, 0, 0, 0, 0},
{1, -1, 1, -1, 1, -1, 0, 0, 0, -1, 1, -1, 1, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}

0

{0, 1, 2, 1, 12, 5}

Out[∘]= 1

Out[∘]= {6, 7, 7, 10, 3, 6, 10, 3, 0, 5, 8, 8, 4, 9, 5, 4, 9, 0, 2, 11, 11, 12, 1, 2, 12, 1, 0}

Out[∘]= {0, 0, 0, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9, 10, 10, 11, 11, 12, 12}

Out[∘]= - P13 P14 - P19 P20 + P18 P27 - P3 P6 - P7 P8

Out[∘]= - P12 P15 - P16 P17 - P1 P2 - P22 P23 + P27 P9

Out[∘]= - P10 P11 - P21 P24 - P25 P26 - P4 P5 + P18 P9

Below is an affine chart of the OP^2 in some other coordinates, obtained from another spreadsheet by modifying the description in the paper of Gross and Elkies.

```
In[•]:= VarsQ = {Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13,
          Q14, Q15, Q16, Q17, Q18, Q19, Q20, Q21, Q22, Q23, Q24, Q25, Q26, Q27}
        CubicQ = Q10 Q18 Q2 - Q11 Q2 Q20 - Q16 Q2 Q23 - Q14 Q2 Q25 - Q11 Q12 Q26 - Q10 Q13 Q26 -
           Q15 Q16 Q26 - Q14 Q17 Q26 - Q19 Q20 Q27 - Q18 Q21 Q27 - Q23 Q24 Q27 - Q22 Q25 Q27 +
           Q1 Q26 Q27 - Q12 Q18 Q3 - Q13 Q20 Q3 + Q16 Q22 Q3 - Q14 Q24 Q3 - Q10 Q19 Q4 - Q11 Q21 Q4 -
           Q17 Q23 Q4 + Q15 Q25 Q4 - Q1 Q3 Q4 - Q12 Q19 Q5 - Q1 Q2 Q5 + Q13 Q21 Q5 - Q17 Q22 Q5 -
           Q15 Q24 Q5 - Q17 Q18 Q6 - Q16 Q19 Q6 + Q11 Q24 Q6 - Q13 Q25 Q6 + Q17 Q20 Q7 -
           Q16 Q21 Q7 - Q10 Q24 Q7 - Q12 Q25 Q7 - Q15 Q18 Q8 + Q14 Q19 Q8 - Q11 Q22 Q8 -
           Q13 Q23 Q8 - Q1 Q7 Q8 - Q15 Q20 Q9 - Q14 Q21 Q9 - Q10 Q22 Q9 + Q12 Q23 Q9 - Q1 Q6 Q9;
        VarsU = {U1, U2, U3, U4, U5, U6, U7, U8, U9, U10, U11, U12, U13, U14, U15, U16}
        QtoU = {Q1 → U10^2 + U11^2 + U12^2 + U13^2 + U14^2 + U15^2 + U16^2 + U9^2,
           Q26 → U1^2 + U2^2 + U3^2 + U4^2 + U5^2 + U6^2 + U7^2 + U8^2, Q27 → 1,
           Q2 → U1 - ⅈ U4, Q5 → U1 + ⅈ U4, Q10 → -ⅈ U12 + U9, Q13 → ⅈ U12 + U9,
           Q18 → -U10 U2 - U11 U3 - U12 U4 - U13 U5 - U14 U6 - U15 U7 - U16 U8 + U1 U9 -
             ⅈ (-U1 U12 - U11 U2 + U10 U3 - U16 U5 + U15 U6 - U14 U7 + U13 U8 - U4 U9),
           Q21 → -U10 U2 - U11 U3 - U12 U4 - U13 U5 - U14 U6 - U15 U7 - U16 U8 + U1 U9 +
             ⅈ (-U1 U12 - U11 U2 + U10 U3 - U16 U5 + U15 U6 - U14 U7 + U13 U8 - U4 U9),
           Q3 → U2 - ⅈ U3, Q4 → U2 + ⅈ U3, Q11 → U10 - ⅈ U11, Q12 → U10 + ⅈ U11,
           Q19 → -U1 U10 - U12 U3 + U11 U4 - U14 U5 + U13 U6 + U16 U7 - U15 U8 - U2 U9 -
             ⅈ (-U1 U11 + U12 U2 - U10 U4 - U15 U5 - U16 U6 + U13 U7 + U14 U8 - U3 U9),
           Q20 → -U1 U10 - U12 U3 + U11 U4 - U14 U5 + U13 U6 + U16 U7 - U15 U8 - U2 U9 +
             ⅈ (-U1 U11 + U12 U2 - U10 U4 - U15 U5 - U16 U6 + U13 U7 + U14 U8 - U3 U9),
           Q6 → U5 - ⅈ U8, Q9 → U5 + ⅈ U8, Q14 → U13 - ⅈ U16, Q17 → U13 + ⅈ U16,
           Q22 → -U1 U13 + U14 U2 + U15 U3 + U16 U4 - U10 U6 - U11 U7 - U12 U8 - U5 U9 -
             ⅈ (-U1 U16 + U15 U2 - U14 U3 - U13 U4 + U12 U5 + U11 U6 - U10 U7 - U8 U9),
           Q25 → -U1 U13 + U14 U2 + U15 U3 + U16 U4 - U10 U6 - U11 U7 - U12 U8 - U5 U9 +
             ⅈ (-U1 U16 + U15 U2 - U14 U3 - U13 U4 + U12 U5 + U11 U6 - U10 U7 - U8 U9),
           Q7 → U6 - ⅈ U7, Q8 → U6 + ⅈ U7, Q15 → U14 - ⅈ U15, Q16 → U14 + ⅈ U15,
           Q23 → -U1 U14 - U13 U2 + U16 U3 - U15 U4 + U10 U5 + U12 U7 - U11 U8 - U6 U9 -
             ⅈ (-U1 U15 - U16 U2 - U13 U3 + U14 U4 + U11 U5 - U12 U6 + U10 U8 - U7 U9),
           Q24 → -U1 U14 - U13 U2 + U16 U3 - U15 U4 + U10 U5 + U12 U7 - U11 U8 - U6 U9 +
             ⅈ (-U1 U15 - U16 U2 - U13 U3 + U14 U4 + U11 U5 - U12 U6 + U10 U8 - U7 U9)};
        Expand[Table[D[CubicQ, VarsQ[[i]]] /. QtoU, {i, 1, 27}]]

Out[•]= {Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14,
         Q15, Q16, Q17, Q18, Q19, Q20, Q21, Q22, Q23, Q24, Q25, Q26, Q27}

Out[•]= {U1, U2, U3, U4, U5, U6, U7, U8, U9, U10, U11, U12, U13, U14, U15, U16}

Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

In[∘]:=
```
(* trying to match *)
ThirdP[a_, b_] :=
  Ret = Sum[k Exponent[1 + D[D[CartanCubic, VarsP[[a]]], VarsP[[b]]], VarsP[[k]]],
    {k, 1, 27}];
ThirdQ[a_, b_] :=
  Ret =
   Sum[k Exponent[1 + D[D[CubicQ, VarsQ[[a]]], VarsQ[[b]]], VarsQ[[k]]], {k, 1, 27}];
```

In[∘]:=
```
TriplesP = {};
For[i = 1, i ≤ 27, i++, For[j = i, j ≤ 27, j++, For[k = j, k ≤ 27, k++,
    If[Coefficient[CartanCubic, VarsP[[i]] × VarsP[[j]] × VarsP[[k]]] == 1,
      AppendTo[TriplesP, {i, j, k}]];
   ]]];
TriplesP
TriplesQ = {};
For[i = 1, i ≤ 27, i++, For[j = i, j ≤ 27, j++, For[k = j, k ≤ 27, k++,
    If[Coefficient[CubicQ, VarsQ[[i]] × VarsQ[[j]] × VarsQ[[k]]] == 1,
      AppendTo[TriplesQ, {i, j, k}]];
   ]]];
TriplesQ
```

Out[∘]=
```
{{1, 10, 19}, {2, 11, 20}, {3, 12, 21}, {4, 13, 22},
 {5, 14, 23}, {6, 15, 24}, {7, 16, 25}, {8, 17, 26}, {9, 18, 27}}
```

Out[∘]=
```
{{1, 26, 27}, {2, 10, 18}, {3, 16, 22}, {4, 15, 25},
 {5, 13, 21}, {6, 11, 24}, {7, 17, 20}, {8, 14, 19}, {9, 12, 23}}
```

In[•]:=
```
(* this was found by starting with the {9,10},
{18,18},{27,2} and then trial and error *)
Matches = {{1, 3}, {2, 12}, {3, 7}, {4, 1}, {5, 5}, {6, 24},
   {7, 4}, {8, 19}, {9, 10}, {10, 16}, {11, 23}, {12, 17}, {13, 26},
   {14, 13}, {15, 6}, {16, 15}, {17, 8}, {18, 18}, {19, 22}, {20, 9},
   {21, 20}, {22, 27}, {23, 21}, {24, 11}, {25, 25}, {26, 14}, {27, 2}};
For[numsteps = 1, numsteps ≤ 27, numsteps++,
  lm = Length[Matches];
  For[i = 1, i ≤ lm, i++,
   For[j = 1, j ≤ i, j++,
    nextp = ThirdP[Matches[[i]][[1]], Matches[[j]][[1]]];
    nextq = ThirdQ[Matches[[i]][[2]], Matches[[j]][[2]]];
    If[nextp > 0,
     If[nextq == 0, Print["Warning"]; numsteps = 100];
     (* check signs as well *)
     If[Coefficient[CartanCubic,
        VarsP[[nextp]] × VarsP[[Matches[[i]][[1]]]] × VarsP[[Matches[[j]][[1]]]]] ≠
       Coefficient[CubicQ, VarsQ[[nextq]] × VarsQ[[Matches[[i]][[2]]]] ×
         VarsQ[[Matches[[j]][[2]]]]],
      Print["Warning: signs"]; numsteps = 100];

     AppendTo[Matches, {nextp, nextq}];
    ];
   ]];
  Matches = DeleteDuplicates[Matches];
 ];
Sort[Matches]
Length[Matches]
```

Out[•]=
```
{{1, 3}, {2, 12}, {3, 7}, {4, 1}, {5, 5}, {6, 24}, {7, 4}, {8, 19}, {9, 10}, {10, 16},
 {11, 23}, {12, 17}, {13, 26}, {14, 13}, {15, 6}, {16, 15}, {17, 8}, {18, 18}, {19, 22},
 {20, 9}, {21, 20}, {22, 27}, {23, 21}, {24, 11}, {25, 25}, {26, 14}, {27, 2}}
```

Out[•]= 27

In[•]:=
```
(* checking substitutions *)
QtoP = Table[VarsQ[[Matches[[i]][[2]]]] → VarsP[[Matches[[i]][[1]]]], {i, 1, 27}];
PtoQ = Table[VarsP[[Matches[[i]][[1]]]] → VarsQ[[Matches[[i]][[2]]]], {i, 1, 27}];
(CubicQ /. QtoP) - CartanCubic
(CartanCubic /. PtoQ) - CubicQ
```

Out[•]= 0

Out[•]= 0

We will now write down the affine chart on OP^2 in P-variables.

In[•]:=
```
PtoU = PtoQ /. QtoU
Expand[Eqs27 /. PtoU] (* yes indeed *)
(* Let us now try to change to variables
 V with more torus symmetry of the map *)
VarsV = {V1, V2, V3, V4, V5, V6, V7, V8, V9, V10, V11, V12, V13, V14, V15, V16};
UtoV =
  {U2 → (V2 + V3) / 2, U3 → ⅈ (V2 - V3) / 2,
   U10 → (V10 + V11) / 2, U11 → ⅈ (V10 - V11) / 2,
   U6 → (V6 + V7) / 2, U7 → ⅈ (V6 - V7) / 2,
   U1 → (V1 + V4) / 2, U4 → ⅈ (V1 - V4) / 2,
   U5 → (V5 + V8) / 2, U8 → ⅈ (V5 - V8) / 2,
   U9 → (V9 + V12) / 2, U12 → ⅈ (V9 - V12) / 2,
   U14 → (V14 + V15) / 2, U15 → ⅈ (V14 - V15) / 2,
   U13 → (V13 + V16) / 2, U16 → ⅈ (V13 - V16) / 2
  };
PtoV = Simplify[PtoU /. UtoV]
Expand[Eqs27 /. PtoV](* yes indeed *)
```

Out[•]=
$$\{P1 \to U2 - i\,U3,\ P2 \to U10 + i\,U11,\ P3 \to U6 - i\,U7,\ P4 \to U10^2 + U11^2 + U12^2 + U13^2 + U14^2 + U15^2 + U16^2 + U9^2,\ P5 \to U1 + i\,U4,$$
$$P6 \to -U1\,U14 - U13\,U2 + U16\,U3 - U15\,U4 + U10\,U5 + U12\,U7 - U11\,U8 - U6\,U9 + i\,(-U1\,U15 - U16\,U2 - U13\,U3 + U14\,U4 + U11\,U5 - U12\,U6 + U10\,U8 - U7\,U9),$$
$$P7 \to U2 + i\,U3,\ P8 \to -U1\,U10 - U12\,U3 + U11\,U4 - U14\,U5 + U13\,U6 + U16\,U7 - U15\,U8 - U2\,U9 - i\,(-U1\,U11 + U12\,U2 - U10\,U4 - U15\,U5 - U16\,U6 + U13\,U7 + U14\,U8 - U3\,U9),\ P9 \to -i\,U12 + U9,$$
$$P10 \to U14 + i\,U15,\ P11 \to -U1\,U14 - U13\,U2 + U16\,U3 - U15\,U4 + U10\,U5 + U12\,U7 - U11\,U8 - U6\,U9 - i\,(-U1\,U15 - U16\,U2 - U13\,U3 + U14\,U4 + U11\,U5 - U12\,U6 + U10\,U8 - U7\,U9),$$
$$P12 \to U13 + i\,U16,\ P13 \to U1^2 + U2^2 + U3^2 + U4^2 + U5^2 + U6^2 + U7^2 + U8^2,$$
$$P14 \to i\,U12 + U9,\ P15 \to U5 - i\,U8,\ P16 \to U14 - i\,U15,\ P17 \to U6 + i\,U7,$$
$$P18 \to -U10\,U2 - U11\,U3 - U12\,U4 - U13\,U5 - U14\,U6 - U15\,U7 - U16\,U8 + U1\,U9 - i\,(-U1\,U12 - U11\,U2 + U10\,U3 - U16\,U5 + U15\,U6 - U14\,U7 + U13\,U8 - U4\,U9),$$
$$P19 \to -U1\,U13 + U14\,U2 + U15\,U3 + U16\,U4 - U10\,U6 - U11\,U7 - U12\,U8 - U5\,U9 - i\,(-U1\,U16 + U15\,U2 - U14\,U3 - U13\,U4 + U12\,U5 + U11\,U6 - U10\,U7 - U8\,U9),$$
$$P20 \to U5 + i\,U8,\ P21 \to -U1\,U10 - U12\,U3 + U11\,U4 - U14\,U5 + U13\,U6 + U16\,U7 - U15\,U8 - U2\,U9 + i\,(-U1\,U11 + U12\,U2 - U10\,U4 - U15\,U5 - U16\,U6 + U13\,U7 + U14\,U8 - U3\,U9),$$
$$P22 \to 1,\ P23 \to -U10\,U2 - U11\,U3 - U12\,U4 - U13\,U5 - U14\,U6 - U15\,U7 - U16\,U8 + U1\,U9 + i\,(-U1\,U12 - U11\,U2 + U10\,U3 - U16\,U5 + U15\,U6 - U14\,U7 + U13\,U8 - U4\,U9),$$
$$P24 \to U10 - i\,U11,\ P25 \to -U1\,U13 + U14\,U2 + U15\,U3 + U16\,U4 - U10\,U6 - U11\,U7 - U12\,U8 - U5\,U9 + i\,(-U1\,U16 + U15\,U2 - U14\,U3 - U13\,U4 + U12\,U5 + U11\,U6 - U10\,U7 - U8\,U9),$$
$$P26 \to U13 - i\,U16,\ P27 \to U1 - i\,U4\}$$

Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

Out[∘]= $\{P1 \to V2,\ P2 \to V11,\ P3 \to V6,\ P4 \to V10\ V11 + V14\ V15 + V13\ V16 + V12\ V9,\ P5 \to V4,$
$P6 \to -V1\ V15 - V16\ V3 - V12\ V7 + V11\ V8,\ P7 \to V3,\ P8 \to -V1\ V10 - V12\ V2 + V16\ V6 - V14\ V8,$
$P9 \to V9,\ P10 \to V15,\ P11 \to -V13\ V2 - V14\ V4 + V10\ V5 - V6\ V9,\ P12 \to V16,$
$P13 \to V2\ V3 + V1\ V4 + V6\ V7 + V5\ V8,\ P14 \to V12,\ P15 \to V5,\ P16 \to V14,\ P17 \to V7,$
$P18 \to -V10\ V3 + V12\ V4 - V15\ V6 - V13\ V8,\ P19 \to -V1\ V13 + V14\ V3 - V12\ V5 - V11\ V6,\ P20 \to V8,$
$P21 \to -V11\ V4 - V15\ V5 + V13\ V7 - V3\ V9,\ P22 \to 1,\ P23 \to -V11\ V2 - V16\ V5 - V14\ V7 + V1\ V9,$
$P24 \to V10,\ P25 \to V15\ V2 - V16\ V4 - V10\ V7 - V8\ V9,\ P26 \to V13,\ P27 \to V1\}$

Out[∘]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

In[∘]:=
```
(* this is one way of seeing the same chart *)
Solve[Table[0 == Eqs27[[k]], {k, 1, 27}], Complement[VarsP,
  {P22, P1, P2, P3, P5, P7, P9, P10, P12, P14, P15, P16, P17, P20, P24, P26, P27}]]
```

Out[∘]=
$$\Big\{\Big\{P11 \to \frac{P15\ P24 - P1\ P26 - P16\ P5 - P3\ P9}{P22},\ P13 \to \frac{P15\ P20 + P17\ P3 + P27\ P5 + P1\ P7}{P22},$$
$$P18 \to \frac{-P20\ P26 - P10\ P3 + P14\ P5 - P24\ P7}{P22},\ P19 \to \frac{-P14\ P15 - P26\ P27 - P2\ P3 + P16\ P7}{P22},$$
$$P21 \to \frac{-P10\ P15 + P17\ P26 - P2\ P5 - P7\ P9}{P22},\ P23 \to \frac{-P12\ P15 - P16\ P17 - P1\ P2 + P27\ P9}{P22},$$
$$P25 \to \frac{P1\ P10 - P17\ P24 - P12\ P5 - P20\ P9}{P22},\ P4 \to \frac{P10\ P16 + P2\ P24 + P12\ P26 + P14\ P9}{P22},$$
$$P6 \to \frac{-P14\ P17 + P2\ P20 - P10\ P27 - P12\ P7}{P22},\ P8 \to \frac{-P1\ P14 - P16\ P20 - P24\ P27 + P12\ P3}{P22}\Big\}\Big\}$$

We will set up a C13 equivariant cut which, for some strange reason, will also admit a C3 action.

In[∘]:=
```
Sort[Table[{deggood[[i]], i}, {i, 1, 27}]]
CoeffToFind = {d1, d2, d3, d4, d5, d6, d7, d8, d9, d10, d11, d12};
Lineqs = {P9, P18, P27, P23 + d1 P26, P19 + d2 P24,
   P5 + d3 P8, P13 + d4 P16, P10 + d5 P15, P1 + d6 P6, P2 + d7 P3,
   P11 + d8 P12, P14 + d9 P17, P4 + d10 P7, P20 + d11 P21, P22 + d12 P25};
FullSimplify[(Eqs27[[9]] + Eqs27[[18]] + Eqs27[[27]]) /.
    Solve[Table[0 == Lineqs[[k]], {k, 1, Length[Lineqs]}], VarsP][[1]]] /.
  {d12 → -1 / d1, d11 → -1 / d2, d10 → -1 / d3, d9 → -1 / d4, d8 → -1 / d5, d7 → -1 / d6};
Solve[Table[0 == Lineqs[[i]], {i, 1, 15}],
   {P1, P10, P12, P13, P17, P18, P19, P3, P21, P25, P23, P27, P7, P5, P9}][[1]] /.
 {d12 → -1 / d1, d11 → -1 / d2, d10 → -1 / d3, d9 → -1 / d4, d8 → -1 / d5, d7 → -1 / d6}
EqsCut = (Eqs27 /. Solve[Table[0 == Lineqs[[i]], {i, 1, 15}],
      {P1, P10, P12, P13, P17, P18, P19, P3, P21, P25, P23, P27, P7, P5, P9}][[1]] /.
   {d12 → -1 / d1, d11 → -1 / d2, d10 → -1 / d3, d9 → -1 / d4, d8 → -1 / d5, d7 → -1 / d6})
EqsAffine = (Lineqs /. {d12 → -1 / d1, d11 → -1 / d2, d10 → -1 / d3,
      d9 → -1 / d4, d8 → -1 / d5, d7 → -1 / d6}) /. PtoV;
PtoT = Sort[{P6 → t6, P14 → t9, P24 → t2, P26 → t1, P4 → t10, P15 → t5,
    P20 → t11, P22 → t12, P8 → t3, P16 → t4, P2 → t7, P11 → t8}];
EqsT = Table[0 == EqsCut[[k]] /. PtoT, {k, 1, 26}] /.
   {d12 → -1 / d1, d11 → -1 / d2, d10 → -1 / d3, d9 → -1 / d4, d8 → -1 / d5, d7 → -1 / d6};
VarsT = {t1, t2, t3, t4, t5, t6, t7, t8, t9, t10, t11, t12};
(* We will now set all but d1,d2 to 1, which we can do by torus symmetry *)
EqsAffine = EqsAffine /. {d3 → 1, d4 → 1, d5 → 1, d6 → 1}
EqsT = EqsT /. {d3 → 1, d4 → 1, d5 → 1, d6 → 1}
```

Out[∘]=
```
{{0, 9}, {0, 18}, {0, 27}, {1, 23}, {1, 26}, {2, 19}, {2, 24}, {3, 5}, {3, 8}, {4, 13},
 {4, 16}, {5, 10}, {5, 15}, {6, 1}, {6, 6}, {7, 2}, {7, 3}, {8, 11}, {8, 12},
 {9, 14}, {9, 17}, {10, 4}, {10, 7}, {11, 20}, {11, 21}, {12, 22}, {12, 25}}
```

Solve: Equations may not give solutions for all "solve" variables.

Out[∘]=
```
{P1 → -d6 P6, P10 → -d5 P15, P12 → d5 P11, P13 → -d4 P16,
 P17 → d4 P14, P18 → 0, P19 → -d2 P24, P3 → d6 P2, P21 → d2 P20,
 P25 → d1 P22, P23 → -d1 P26, P27 → 0, P7 → d3 P4, P5 → -d3 P8, P9 → 0}
```

Out[$\circ$]= $\{-d2\ P14\ P20 + d2\ d5\ P15\ P24 - d3\ P4^2 - P26\ P6,\ P11\ P20 - d1\ d6\ P2\ P22 + d4\ P16\ P24 + d3\ P8^2,$
$d2\ d5\ P11\ P20 - d1\ P2\ P22 - d1\ d5\ P15\ P26 - d4\ P14\ P4,$
$-d4\ d6\ P14\ P2 - P15\ P20 - d4\ P16\ P22 + d3\ d6\ P4\ P6,\ d2\ d5\ P11\ P24 - d1\ P14\ P26 - P16\ P6 - P2\ P8,$
$-P11\ P22 + P15\ P24 + d6\ P26\ P6 + d3\ P16\ P8,\ -d5\ P11^2 + d1\ P16\ P22 + d1\ P24\ P26 + d6\ P4\ P6,$
$d5\ P15^2 - d2\ P20\ P22 + d4\ P14\ P26 + d3\ P2\ P8,\ d4\ P14\ P16 + d2\ P20\ P24 - d6\ P2\ P6 - d3\ P4\ P8,$
$d4\ P16^2 + d1\ d6\ P2\ P26 + d2\ d6\ P24\ P6 - P15\ P8,\ -d4\ P14^2 + P2\ P20 - d3\ d5\ P11\ P4 - P22\ P6,$
$d2\ d6\ P2\ P20 + d4\ P16\ P26 - d3\ P11\ P4 - d2\ d3\ P24\ P8,\ d5\ P15\ P16 - P2\ P24 - d5\ P11\ P26 + P22\ P4,$
$-d4\ P11\ P14 - d1\ P15\ P22 + d2\ d6\ P20\ P6 + d1\ d3\ P26\ P8,$
$-d1\ P14\ P22 - P20\ P4 + P24\ P6 + d5\ P15\ P8,\ -d4\ d5\ P15\ P16 - d2\ P20^2 + d1\ d3\ P22\ P4 + d3\ P6\ P8,$
$-P11\ P14 + d2\ P24^2 - d6\ P2\ P4 + P26\ P8,\ -d5\ P11\ P15 - d4\ P14\ P16 + d1\ P22\ P26 + d6\ P2\ P6,$
$-d1\ P22^2 - d4\ P14\ P24 + d5\ d6\ P15\ P6 + d3\ d5\ P11\ P8,$
$P11\ P2 - d2\ P16\ P20 + d1\ P26^2 - P15\ P4,\ d5\ d6\ P11\ P2 - P16\ P20 + d6\ P14\ P6 - P22\ P8,$
$d1\ d2\ P22\ P24 - d4\ P16\ P4 - P11\ P6 - d2\ P20\ P8,\ d5\ d6\ P15\ P2 - P20\ P26 - d3\ P24\ P4 - d3\ P14\ P8,$
$d4\ P16\ P2 + d2\ d4\ P14\ P24 + d1\ d3\ P26\ P4 + P15\ P6,\ -P14\ P15 - d6\ P2^2 + d2\ P22\ P24 + d3\ P16\ P4,$
$d4\ d5\ P11\ P16 + d1\ P20\ P26 + d6\ P6^2 + d4\ P14\ P8,\ d5\ P11\ P15 - d2\ P20\ P24 - d1\ P22\ P26 + d3\ P4\ P8\}$

Out[$\circ$]= $\{V9,\ -V10\ V3 + V12\ V4 - V15\ V6 - V13\ V8,\ V1,\ d1\ V13 - V11\ V2 - V16\ V5 - V14\ V7 + V1\ V9,$
$d2\ V10 - V1\ V13 + V14\ V3 - V12\ V5 - V11\ V6,\ -V1\ V10 - V12\ V2 + V4 + V16\ V6 - V14\ V8,$
$V14 + V2\ V3 + V1\ V4 + V6\ V7 + V5\ V8,\ V15 + V5,\ -V1\ V15 + V2 - V16\ V3 - V12\ V7 + V11\ V8,\ V11 - V6,$
$-V16 - V13\ V2 - V14\ V4 + V10\ V5 - V6\ V9,\ V12 - V7,\ V10\ V11 + V14\ V15 + V13\ V16 - V3 + V12\ V9,$
$V8 - \dfrac{-V11\ V4 - V15\ V5 + V13\ V7 - V3\ V9}{d2},\ 1 - \dfrac{V15\ V2 - V16\ V4 - V10\ V7 - V8\ V9}{d1}\}$

Out[$\circ$]= $\{0 == -t10^2 + d2\ t2\ t5 - t1\ t6 - d2\ t11\ t9,\ 0 == t3^2 + t2\ t4 - d1\ t12\ t7 + t11\ t8,$
$0 == -d1\ t1\ t5 - d1\ t12\ t7 + d2\ t11\ t8 - t10\ t9,\ 0 == -t12\ t4 - t11\ t5 + t10\ t6 - t7\ t9,$
$0 == -t4\ t6 - t3\ t7 + d2\ t2\ t8 - d1\ t1\ t9,\ 0 == t3\ t4 + t2\ t5 + t1\ t6 - t12\ t8,$
$0 == d1\ t1\ t2 + d1\ t12\ t4 + t10\ t6 - t8^2,\ 0 == -d2\ t11\ t12 + t5^2 + t3\ t7 + t1\ t9,$
$0 == d2\ t11\ t2 - t10\ t3 - t6\ t7 + t4\ t9,\ 0 == t4^2 - t3\ t5 + d2\ t2\ t6 + d1\ t1\ t7,$
$0 == -t12\ t6 + t11\ t7 - t10\ t8 - t9^2,\ 0 == -d2\ t2\ t3 + t1\ t4 + d2\ t11\ t7 - t10\ t8,$
$0 == t10\ t12 + t4\ t5 - t2\ t7 - t1\ t8,\ 0 == d1\ t1\ t3 - d1\ t12\ t5 + d2\ t11\ t6 - t8\ t9,$
$0 == -t10\ t11 + t3\ t5 + t2\ t6 - d1\ t12\ t9,\ 0 == -d2\ t11^2 + d1\ t10\ t12 - t4\ t5 + t3\ t6,$
$0 == d2\ t2^2 + t1\ t3 - t10\ t7 - t8\ t9,\ 0 == d1\ t1\ t12 + t6\ t7 - t5\ t8 - t4\ t9,$
$0 == -d1\ t12^2 + t5\ t6 + t3\ t8 - t2\ t9,\ 0 == d1\ t1^2 - d2\ t11\ t4 - t10\ t5 + t7\ t8,$
$0 == -t12\ t3 - t11\ t4 + t7\ t8 + t6\ t9,\ 0 == d1\ d2\ t12\ t2 - d2\ t11\ t3 - t10\ t4 - t6\ t8,$
$0 == -t1\ t11 - t10\ t2 + t5\ t7 - t3\ t9,\ 0 == d1\ t1\ t10 + t5\ t6 + t4\ t7 + d2\ t2\ t9,$
$0 == d2\ t12\ t2 + t10\ t4 - t7^2 - t5\ t9,\ 0 == d1\ t1\ t11 + t6^2 + t4\ t8 + t3\ t9\}$

```
In[◦]:= Length[EqsAffine];
EqsAffine1 = Simplify[EqsAffine /. d1 → 1];
EqsSome = Complement[EqsAffine1 /. {V9 → 0, V1 → 0}, {0}]
Length[EqsSome];
sls = Solve[
    {0 == EqsSome[[2]], 0 == EqsSome[[4]], 0 == EqsSome[[6]]}, {V15, V11, V7}][[1]];
EqsSome = Complement[EqsSome /. sls, {0}]
Length[EqsSome]
NewVarsV = Complement[VarsV, {V1, V9, V7, V15, V11}]
(* Now let us try to set up singularity condition,
as a specific kind of relation on the equations at a point *)
VarsAll = Join[NewVarsV, {a1, a2, a3, a4, a5, a6, a7, a8, d2}];
AllEqsSing = EqsSome;
For[k = 1, k ≤ 11, k++,
  AppendTo[AllEqsSing, Sum[({1, 1, a1, a2, a3, a4, a5, a6, a7, a8}[[j]])
      D[EqsSome[[j]], NewVarsV[[k]]], {j, 1, 10}]];
 ];
ByteCount[AllEqsSing]
{Length[AllEqsSing], Length[VarsAll]}
```

Out[◦]= $\Big\{V10\,V11 + V14\,V15 + V13\,V16 - V3,\ V15 + V5,\ -V16 - V13\,V2 - V14\,V4 + V10\,V5,\ V11 - V6,\ d2\,V10 + V14\,V3 - V12\,V5 - V11\,V6,\ V12 - V7,\ 1 - V15\,V2 + V16\,V4 + V10\,V7,\ V13 - V11\,V2 - V16\,V5 - V14\,V7,\ \frac{V11\,V4 + V15\,V5 - V13\,V7 + d2\,V8}{d2},\ V2 - V16\,V3 - V12\,V7 + V11\,V8,\ -V10\,V3 + V12\,V4 - V15\,V6 - V13\,V8,\ -V12\,V2 + V4 + V16\,V6 - V14\,V8,\ V14 + V2\,V3 + V6\,V7 + V5\,V8\Big\}$

Out[◦]= $\Big\{-V16 - V13\,V2 - V14\,V4 + V10\,V5,\ 1 + V10\,V12 + V16\,V4 + V2\,V5,\ V13\,V16 - V3 - V14\,V5 + V10\,V6,\ V13 - V12\,V14 - V16\,V5 - V2\,V6,\ d2\,V10 + V14\,V3 - V12\,V5 - V6^2,\ \frac{-V12\,V13 - V5^2 + V4\,V6 + d2\,V8}{d2},\ -V10\,V3 + V12\,V4 + V5\,V6 - V13\,V8,\ -V12\,V2 + V4 + V16\,V6 - V14\,V8,\ V14 + V2\,V3 + V12\,V6 + V5\,V8,\ -V12^2 + V2 - V16\,V3 + V6\,V8\Big\}$

Out[◦]= 10

Out[◦]= {V10, V12, V13, V14, V16, V2, V3, V4, V5, V6, V8}

Out[◦]= 8680

Out[◦]= {21, 20}

In[•]:=
```
(* We will try to randomly look around in
 the space of solutions using the FindRoot command.
  This may need to be rerun if not lucky the first time around. *)
flag = 0;
NumTries = 100;
For[m = 1, m ≤ NumTries, m++,
  fi = Table[VarsAll[[k]] → RandomReal[1] + I RandomReal[1], {k, 1, 20}];
  If[Mod[m, 20] == 0, Print[m]];
  RemoveIndex = RandomInteger[20] + 1;
  Eqs20 = Complement[AllEqsSing, {AllEqsSing[[RemoveIndex]]}];
  fi = FindRoot[Table[0 == Eqs20[[k]], {k, 1, 20}],
    Table[{VarsAll[[n]], (VarsAll[[n]] /. fi)}, {n, 1, 20}], AccuracyGoal → 500,
    PrecisionGoal → 500, MaxIterations → 200, WorkingPrecision → 1000];
  JacMat = Table[D[EqsSome[[i]], NewVarsV[[j]]] /. fi, {i, 1, 10}, {j, 1, 11}];
  If[MatrixRank[JacMat] == 8, m = NumTries + 1; flag = 1;];
 ];
If[flag == 1, d2good = RootApproximant[d2 /. fi, 48]];
d2good
```

FindRoot: Failed to converge to the requested accuracy or precision within 200 iterations.

Out[•]= 0.255… + 3.24… i

It appears that there are d2 that are roots of some degree 7 polynomial that fit the bill. We will use the starting data to get to a nearby point.

```
In[◦]:= figood = fi;

        Length[EqsAffine];
        EqsAffine1 = Simplify[EqsAffine];
        EqsSome = Complement[EqsAffine1 /. {V9 → 0, V1 → 0}, {0}];
        Length[EqsSome];
        sls = Solve[
            {0 == EqsSome[[2]], 0 == EqsSome[[4]], 0 == EqsSome[[6]]}, {V15, V11, V7}][[1]];
        EqsSome = Complement[Simplify[EqsSome /. sls], {0}];
        Length[EqsSome];
        NewVarsV = Complement[VarsV, {V1, V9, V7, V15, V11}];
        VarsAll = Join[NewVarsV, {a1, a2, a3, a4, a5, a6, a7, a8, d2}];
        AllEqsSing = Simplify[EqsSome];
        For[k = 1, k ≤ 11, k++,
          AppendTo[AllEqsSing, Sum[({1, 1, a1, a2, a3, a4, a5, a6, a7, a8}[[j]])
              D[EqsSome[[j]], NewVarsV[[k]]], {j, 1, 10}]];
         ];
        ByteCount[AllEqsSing]
        {Length[AllEqsSing], Length[VarsAll]}

        PtsSingCuts = {};
        For[numpt = 0, numpt ≤ 1, numpt++,
          AllEqsSingc1 = AllEqsSing /. d1 → (1 + numpt / 10^20);
          (* we will use the same RemoveIndex as before *)
          Eqs20 = Complement[AllEqsSingc1, {AllEqsSing[[RemoveIndex]]}];
          fi = FindRoot[Table[0 == Eqs20[[k]], {k, 1, 20}], Table[
              {VarsAll[[n]], (VarsAll[[n]] /. figood)}, {n, 1, 20}], AccuracyGoal → 50000,
             PrecisionGoal → 50000, MaxIterations → 300, WorkingPrecision → 100000];
          JacMat = Table[D[EqsSome[[i]] /. d1 → 1 + numpt / 10^20, NewVarsV[[j]]] /. fi,
             {i, 1, 10}, {j, 1, 11}];
          If[MatrixRank[JacMat] == 8, AppendTo[PtsSingCuts, {1 + numpt / 10^20, d2 /. fi}]];
         ];

Out[◦]= 9672

Out[◦]= {21, 20}
```

In[•]:=
```
(* finding minimal polynomial *)
mp = MinimalPolynomial[RootApproximant[N[PtsSingCuts[[2]][[2]], 10 000], 10], t]
```

Out[•]=
```
- 1 000 000 000 000 000 000 030 000 000 000 000 000 000 300 000 000 000 000 000 001 000 000 000 ⋱
    000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 -
  67 500 000 000 000 000 002 075 000 000 000 000 000 025 500 000 000 000 000 000 134 999 999 999 ⋱
    999 999 999 674 999 999 999 999 999 991 749 999 999 999 999 999 970 000 000 000 000 000 000 ⋱
    000 000 000 000 000 000 000 t -
  85 000 000 000 000 000 000 099 999 999 999 999 999 973 499 999 999 999 999 999 855 000 000 000 ⋱
    000 000 002 300 000 000 000 000 000 029 500 000 000 000 000 000 140 000 000 000 000 000 000 ⋱
    300 000 000 000 000 000 000 t^2 -
  992 500 000 000 000 000 032 775 000 000 000 000 000 474 250 000 000 000 000 003 837 500 000 000 ⋱
    000 000 016 575 000 000 000 000 000 016 249 999 999 999 999 999 864 999 999 999 999 999 999 ⋱
    599 999 999 999 999 999 999 t^3 -
  430 000 000 000 000 000 003 399 999 999 999 999 999 884 999 999 999 999 999 998 249 999 999 999 ⋱
    999 999 994 750 000 000 000 000 000 034 000 000 000 000 000 000 265 000 000 000 000 000 000 ⋱
    900 000 000 000 000 000 002 t^4 -
  2 052 500 000 000 000 000 078 625 000 000 000 000 001 251 000 000 000 000 000 010 690 000 000 ⋱
    000 000 000 053 800 000 000 000 000 000 167 000 000 000 000 000 000 317 500 000 000 000 000 ⋱
    000 174 999 999 999 999 999 999 t^5 +
  475 000 000 000 000 000 016 700 000 000 000 000 000 242 500 000 000 000 000 001 865 000 000 000 ⋱
    000 000 008 700 000 000 000 000 000 034 500 000 000 000 000 000 140 000 000 000 000 000 000 ⋱
    300 000 000 000 000 000 000 t^6 -
  207 500 000 000 000 000 006 525 000 000 000 000 000 069 250 000 000 000 000 000 177 499 999 999 ⋱
    999 999 998 199 999 999 999 999 999 985 999 999 999 999 999 999 970 000 000 000 000 000 000 ⋱
    000 000 000 000 000 000 000 t^7 +
  40 000 000 000 000 000 001 600 000 000 000 000 000 025 000 000 000 000 000 000 190 000 000 000 ⋱
    000 000 000 700 000 000 000 000 000 001 000 000 000 000 000 000 000 000 000 000 000 000 000 ⋱
    000 000 000 000 000 000 000 t^8
```

In[•]:=
```
(* we want to see the above coefficients
 as polynomials in 10^20 with small coefficients *)
Poly = 0;
For[i = 0, i ≤ 8, i++,
  cf = SeriesCoefficient[mp, {t, 0, i}];
  nl = NullSpace[{Join[Table[10^(20 k), {k, 0, 24}], {cf}]}];
  nl = LatticeReduce[nl][[1]];
  nl = Table[-nl[[k]] / nl[[26]], {k, 1, 26}];
  Poly += Sum[t^i r^k nl[[k + 1]], {k, 0, 24}];
 ];
(Poly /. r → 10^20) - mp
```

Out[•]=
```
0
```

```
In[•]:= LocusOfSingularCuts = Numerator[Factor[Poly /. {t → d2, r → 1 / (d1 - 1)}]]
        (* now we find the potential A2 singularities *)
        LocusA2 = Solve[{LocusOfSingularCuts == 0,
            D[LocusOfSingularCuts, d1] == 0, D[LocusOfSingularCuts, d2] == 0}];
        Length[DeleteDuplicates[RootReduce[{d1, d2} /. LocusA2]]]
        LocusA2values = Sort[RootReduce[{d1, d2} /. LocusA2]]
```

Out[•]=
$$-4\,d1^3+8\,d1^4-4\,d1^5-12\,d1^2\,d2-16\,d1^3\,d2+28\,d1^4\,d2-39\,d1^5\,d2+12\,d1^6\,d2-12\,d1\,d2^2-28\,d1^2\,d2^2-54\,d1^3\,d2^2+78\,d1^4\,d2^2-34\,d1^5\,d2^2+28\,d1^6\,d2^2-12\,d1^7\,d2^2-4\,d2^3-39\,d1\,d2^3-34\,d1^2\,d2^3-277\,d1^3\,d2^3+192\,d1^4\,d2^3-277\,d1^5\,d2^3+54\,d1^6\,d2^3-16\,d1^7\,d2^3+4\,d1^8\,d2^3-8\,d2^4-28\,d1\,d2^4-78\,d1^2\,d2^4-192\,d1^3\,d2^4+192\,d1^5\,d2^4-78\,d1^6\,d2^4+28\,d1^7\,d2^4-8\,d1^8\,d2^4-4\,d2^5-16\,d1\,d2^5-54\,d1^2\,d2^5-277\,d1^3\,d2^5-192\,d1^4\,d2^5-277\,d1^5\,d2^5+34\,d1^6\,d2^5-39\,d1^7\,d2^5+4\,d1^8\,d2^5+12\,d1\,d2^6+28\,d1^2\,d2^6+34\,d1^3\,d2^6+78\,d1^4\,d2^6+54\,d1^5\,d2^6-28\,d1^6\,d2^6+12\,d1^7\,d2^6-12\,d1^2\,d2^7-39\,d1^3\,d2^7-28\,d1^4\,d2^7-16\,d1^5\,d2^7+12\,d1^6\,d2^7+4\,d1^3\,d2^8+8\,d1^4\,d2^8+4\,d1^5\,d2^8$$

Out[•]= 19

Out[•]=
```
{{0, 0}, {-i, -i}, {i, i}, {-0.439... - 0.246... i, 0.213... + 5.36... i},
 {-0.439... + 0.246... i, 0.213... - 5.36... i},
 {-7.40... × 10^-3 - 0.186... i, -0.439... + 0.246... i},
 {-7.40... × 10^-3 + 0.186... i, -0.439... - 0.246... i},
 {0.213... - 5.36... i, 1.73... + 0.969... i}, {0.213... + 5.36... i, 1.73... - 0.969... i},
 {1.73... - 0.969... i, -7.40... × 10^-3 + 0.186... i},
 {1.73... + 0.969... i, -7.40... × 10^-3 - 0.186... i},
 {-0.226... - 0.257... i, -0.0472... + 1.94... i},
 {-0.226... + 0.257... i, -0.0472... - 1.94... i},
 {-0.0472... - 1.94... i, 1.93... + 2.20... i},
 {-0.0472... + 1.94... i, 1.93... - 2.20... i},
 {0.0125... - 0.515... i, -0.226... + 0.257... i},
 {0.0125... + 0.515... i, -0.226... - 0.257... i},
 {1.93... - 2.20... i, 0.0125... + 0.515... i},
 {1.93... + 2.20... i, 0.0125... - 0.515... i}}
```

In[•]:=
```
Length[DeleteDuplicates[MinimalPolynomial[LocusA2values]]]
(* There seem to be 4 different cases: (0,0), \pm
 (i,i) and two size 8 orbits of the Galois group.
  And (0,0) is not viable, so really just three cases. *)
LocusA2valuesReduced =
 {LocusA2values[[3]], LocusA2values[[4]], LocusA2values[[19]]}
```

Out[•]= 4

Out[•]= {{i, i}, {-0.439… - 0.246… i, 0.213… + 5.36… i}, {1.93… + 2.20… i, 0.0125… - 0.515… i}}

In[•]:=
```
RootReduce[(D[D[LocusOfSingularCuts, d1], d1] × D[D[LocusOfSingularCuts, d2], d2] -
     D[D[LocusOfSingularCuts, d1], d2]^2) /.
   {d1 → LocusA2valuesReduced[[2]][[1]], d2 → LocusA2valuesReduced[[2]][[2]]}]
RootReduce[(D[D[LocusOfSingularCuts, d1], d1] × D[D[LocusOfSingularCuts, d2], d2] -
     D[D[LocusOfSingularCuts, d1], d2]^2) /.
   {d1 → LocusA2valuesReduced[[3]][[1]], d2 → LocusA2valuesReduced[[3]][[2]]}]
(* These results show that the two cases are different:
   we guess is that the 2nd one give 78 nodes while the third one gives 39 A_2
    singularities *)
```

Out[•]= $3.37\ldots \times 10^{10} + 2.69\ldots \times 10^{10}\ i$

Out[•]= 0

Just playing around with the field extensions.

In[•]:=
```
(* Let us inderstand the last solution *)
gal = ToNumberField[LocusA2valuesReduced[[3]][[2]],
   LocusA2valuesReduced[[3]][[1]]][[2]]

roots = t /. Solve[MinimalPolynomial[LocusA2valuesReduced[[3]][[1]], t] == 0]
perm = {};
For[i = 1, i ≤ 8, i++,
  For[j = 1, j ≤ 8, j++,
   (* For some reason it needs to have powers of 27 *)
   If[RootReduce[
       Sum[gal[[k]] × roots[[i]] ^ (k - 1) (27) ^ (k - 1), {k, 1, 8}]] == roots[[j]],
     AppendTo[perm, {i, j}];
    ];
  ]];
perm
perm = {};
For[i = 1, i ≤ 8, i++,
  For[j = 1, j ≤ 8, j++,
   (* For some reason it needs to have powers of 27 *)
   If[RootReduce[-1 / roots[[i]]] == roots[[j]],
     AppendTo[perm, {i, j}];
    ];
  ]];
perm
RootReduce[roots[[8]] + roots[[5]] + roots[[2]] + roots[[3]]]
RootReduce[roots[[8]] ^2 + roots[[5]] ^2 + roots[[2]] ^2 + roots[[3]] ^2]
```

Out[•]= $\left\{-\frac{5\,473\,316}{13\,109\,499}, -\frac{1\,863\,956\,045}{9\,556\,824\,771}, \frac{1\,855\,739\,033}{774\,102\,806\,451}, -\frac{786\,876\,214}{774\,102\,806\,451}, \frac{8\,944\,131\,907}{564\,320\,945\,902\,779}, -\frac{2\,465\,419\,511}{5\,078\,888\,513\,125\,011}, \frac{3\,112\,180}{564\,320\,945\,902\,779}, -\frac{274\,606}{5\,078\,888\,513\,125\,011}\right\}$

Out[•]= {-0.226… - 0.257… i, -0.226… + 0.257… i, -0.0472… - 1.94… i, -0.0472… + 1.94… i, 0.0125… - 0.515… i, 0.0125… + 0.515… i, 1.93… - 2.20… i, 1.93… + 2.20… i}

Out[•]= {{1, 4}, {2, 3}, {3, 8}, {4, 7}, {5, 2}, {6, 1}, {7, 6}, {8, 5}}

Out[•]= {{1, 7}, {2, 8}, {3, 5}, {4, 6}, {5, 3}, {6, 4}, {7, 1}, {8, 2}}

Out[•]= $\frac{5}{3}$

Out[•]= $\frac{1}{729}\left(-3761+4394\,\mathrm{i}\,\sqrt{2}\right)$

```
(* Can we have (d1,d2)→
 (-1/d2,d1) ? This would send (8,5)→(3,8)→(2,3)→(5,2)→(8,5). We will use it.
*)
```

Before continuing further, let us look at the C3 action on these surfaces. The action looks like multiplication of an index by 3 mod 13,
however, there may also be some scaling. We will experiment with it to make it work.
We will be scaling variables to make a good match.
We were successful in it, the new relations are called NewRels.

In[∘]:=
```
Unset[{r1, r2, r3, r4, r5, r6, r7, r8, r9, r10, r11, r12}];
RelsT =
  (Table[EqsCut[[k]] /. PtoT, {k, 1, 26}] /. {d12 → -1 / d1, d11 → -1 / d2, d10 → -1 / d3,
      d9 → -1 / d4, d8 → -1 / d5, d7 → -1 / d6}) /. {d3 → 1, d4 → 1, d5 → 1, d6 → 1}
ActionC3 = {t1 → t3, t3 → t9, t9 → t1, t2 → t6, t6 → t5, t5 → t2 , t4 → t12,
   t12 → t10, t10 → t4, t7 → t8, t8 → t11, t11 → t7}
ScaleT = {t1 → t1 r1, t2 → t2 r2, t3 → t3 r3, t4 → t4 r4, t5 → t5 r5, t6 → t6 r6,
    t7 → t7 r7, t8 → t8 r8, t9 → t9 r9, t10 → t10 r10, t11 → t11 r11, t12 → t12 r12};

AllMonsT = {};
For[i = 1, i ≤ 12, i++,
 For[j = 1, j ≤ i, j++, AppendTo[AllMonsT, VarsT[[i]] × VarsT[[j]]]]];

RelsT = RelsT /. ScaleT;
NonZeroRelsT = Complement[RelsT, {RelsT[[9]], RelsT[[18]]}];
Length[NonZeroRelsT]
EqsSym = {};
For[i = 1, i ≤ 24, i++,
   NewRel = {NonZeroRelsT[[i]] /. ActionC3};
   grad13 = Mod[
     Sum[Exponent[MonomialList[NewRel[[1]]][[1]], VarsT[[j]]] j , {j, 1, 12}], 13];
   For[n = 1, n ≤ 24, n++,
    If[Mod[Sum[Exponent[MonomialList[NonZeroRelsT[[n]]][[1]], VarsT[[j]]] j ,
         {j, 1, 12}], 13] == grad13,
      AppendTo[NewRel, NonZeroRelsT[[n]]];
     ];
   ];
   MatMon = Table[Coefficient[NewRel[[k]], AllMonsT[[l]]],
     {k, 1, 3}, {l, 1, Length[AllMonsT]}];
   EqsSym = Union[EqsSym, DeleteDuplicates[Flatten[Minors[MatMon, 3]]]];
 ];
```

Out[∘]= {-t10^2 + d2 t2 t5 - t1 t6 - d2 t11 t9, t3^2 + t2 t4 - d1 t12 t7 + t11 t8,
 -d1 t1 t5 - d1 t12 t7 + d2 t11 t8 - t10 t9, -t12 t4 - t11 t5 + t10 t6 - t7 t9,
 -t4 t6 - t3 t7 + d2 t2 t8 - d1 t1 t9, t3 t4 + t2 t5 + t1 t6 - t12 t8,
 d1 t1 t2 + d1 t12 t4 + t10 t6 - t8^2, -d2 t11 t12 + t5^2 + t3 t7 + t1 t9,
 d2 t11 t2 - t10 t3 - t6 t7 + t4 t9, t4^2 - t3 t5 + d2 t2 t6 + d1 t1 t7,
 -t12 t6 + t11 t7 - t10 t8 - t9^2, -d2 t2 t3 + t1 t4 + d2 t11 t7 - t10 t8,
 t10 t12 + t4 t5 - t2 t7 - t1 t8, d1 t1 t3 - d1 t12 t5 + d2 t11 t6 - t8 t9,
 -t10 t11 + t3 t5 + t2 t6 - d1 t12 t9, -d2 t11^2 + d1 t10 t12 - t4 t5 + t3 t6,
 d2 t2^2 + t1 t3 - t10 t7 - t8 t9, d1 t1 t12 + t6 t7 - t5 t8 - t4 t9,
 -d1 t12^2 + t5 t6 + t3 t8 - t2 t9, d1 t1^2 - d2 t11 t4 - t10 t5 + t7 t8,
 -t12 t3 - t11 t4 + t7 t8 + t6 t9, d1 d2 t12 t2 - d2 t11 t3 - t10 t4 - t6 t8,
 -t1 t11 - t10 t2 + t5 t7 - t3 t9, d1 t1 t10 + t5 t6 + t4 t7 + d2 t2 t9,
 d2 t12 t2 + t10 t4 - t7^2 - t5 t9, d1 t1 t11 + t6^2 + t4 t8 + t3 t9}

Out[∘]= {t1 → t3, t3 → t9, t9 → t1, t2 → t6, t6 → t5, t5 → t2,
 t4 → t12, t12 → t10, t10 → t4, t7 → t8, t8 → t11, t11 → t7}

Out[∘]= 24

In[∘]:=
```
Length[EqsSym]

VarsR = {r1, r2, r3, r4, r5, r6, r7, r8, r9, r10, r11, r12};
Solve[Join[Table[0 == EqsSym[[20 k]], {k, 1, 17}],
   Table[VarsR[[i]] ≠ 0, {i, 1, 12}]], VarsR];
```

Out[∘]= 385

Solve: Equations may not give solutions for all "solve" variables.

In[∘]:= DeleteDuplicates[

EqsSym /. $\left\{r9 \to \frac{d2^{1/13}\, r3}{d1^{2/13}},\ r1 \to \frac{d2^{4/13}\, r3}{d1^{8/13}},\ r5 \to -\frac{d2^{7/13}\, r2}{d1^{1/13}},\ r10 \to d1^{5/13}\, d2^{4/13}\, r12,\ r6 \to d1^{3/13}\, d2^{5/13}\, r2,\ r4 \to d1^{7/13}\, d2^{3/13}\, r12,\ r7 \to -\frac{d2^{2/13}\, r8}{d1^{4/13}},\ r11 \to -\frac{r8}{d1^{1/13}\, d2^{6/13}}\right\}$]

Out[∘]= {0}

In[∘]:=
```
Simplify[Numerator[
  Factor[RelsT /. {r9 → d2^(1/13) r3 / d1^(2/13), r1 → d2^(4/13) r3 / d1^(8/13), r5 → - d2^(7/13) r2 / d1^(1/13), r10 → d1^(5/13) d2^(4/13) r12,
      r6 → d1^(3/13) d2^(5/13) r2, r4 → d1^(7/13) d2^(3/13) r12, r7 → - d2^(2/13) r8 / d1^(4/13),
      r11 → - r8 / (d1^(1/13) d2^(6/13))} /. r2 → r8 d1^(2/13) d2^(-1/13) /.
    r3 → r8 d1^(6/13) d2^(10/13) /. r12 → r8 d1^(-10/13) d2^(-8/13) /. r8 → 1]]]
```

Out[∘]=
```
{-t10^2 - d1 d2^2 (t2 t5 + t1 t6 - t11 t9), d1 d2^2 t3^2 + t2 t4 + t12 t7 - t11 t8,
 d1 d2^2 t1 t5 + t12 t7 - d2 (t11 t8 + t10 t9), -t12 t4 + d1 d2 (-t11 t5 + t10 t6 + d2 t7 t9),
 -d1^(2/13) (t4 t6 + d2 (-t3 t7 - t2 t8 + d1 d2 t1 t9)), d1 d2 (t3 t4 - t2 t5 + d2 t1 t6) - t12 t8,
 d1 d2^2 t1 t2 + t12 t4 + d2 t10 t6 - d2 t8^2, t11 t12 + d1 d2 (t5^2 - t3 t7 + d2 t1 t9),
 d1^(1/13) d2^(6/13) (-t11 t2 - t10 t3 + t6 t7 + t4 t9), t4^2 + d1 d2^2 (t3 t5 + t2 t6 - t1 t7),
 -t12 t6 + t11 t7 - t10 t8 - d1 d2^2 t9^2, -d1 d2^2 t2 t3 + d2 t1 t4 + d2 t11 t7 - t10 t8,
 t10 t12 - d1 d2 (t4 t5 - t2 t7 + d2 t1 t8), d1^(4/13) (d1 d2^2 t1 t3 + t12 t5 - d2 (t11 t6 + t8 t9)),
 t10 t11 - d1 d2 (d2 t3 t5 - t2 t6 + t12 t9), -d2 t11^2 + t10 t12 + d2 t4 t5 + d1 d2^2 t3 t6,
 t10 t7 + d1 d2 (t2^2 + d2 t1 t3 - t8 t9), d1^(1/13) d2^(6/13) (t1 t12 - t6 t7 + t5 t8 - t4 t9),
 -t12^2 - d1 d2^2 (t5 t6 - t3 t8 + t2 t9), d2^(2/13) (d1 d2^2 t1^2 + t11 t4 + t10 t5 - t7 t8),
 t11 t4 - d2 (t12 t3 + t7 t8) + d1 d2^2 t6 t9, -t10 t4 + d1 d2 (t12 t2 + d2 t11 t3 - t6 t8),
 d2 t1 t11 - t10 t2 + d2 t5 t7 - d1 d2^2 t3 t9, -t4 t7 + d1 d2 (t1 t10 - t5 t6 + d2 t2 t9),
 d2 t12 t2 + t10 t4 - d2 t7^2 + d1 d2^2 t5 t9, t4 t8 + d1 d2 (-t1 t11 + t6^2 + d2 t3 t9)}
```

In[∘]:=
```
NewRels = {-t10^2 - d1 d2^2 (t2 t5 + t1 t6 - t11 t9), d1 d2^2 t3^2 + t2 t4 + t12 t7 - t11 t8,
   d1 d2^2 t1 t5 + t12 t7 - d2 (t11 t8 + t10 t9), -t12 t4 + d1 d2 (-t11 t5 + t10 t6 + d2 t7 t9),
   t4 t6 + d2 (-t3 t7 - t2 t8 + d1 d2 t1 t9), d1 d2 (t3 t4 - t2 t5 + d2 t1 t6) - t12 t8,
   d1 d2^2 t1 t2 + t12 t4 + d2 t10 t6 - d2 t8^2, t11 t12 + d1 d2 (t5^2 - t3 t7 + d2 t1 t9),
   -t11 t2 - t10 t3 + t6 t7 + t4 t9, t4^2 + d1 d2^2 (t3 t5 + t2 t6 - t1 t7),
   -t12 t6 + t11 t7 - t10 t8 - d1 d2^2 t9^2, -d1 d2^2 t2 t3 + d2 t1 t4 + d2 t11 t7 - t10 t8,
   t10 t12 - d1 d2 (t4 t5 - t2 t7 + d2 t1 t8), d1 d2^2 t1 t3 + t12 t5 - d2 (t11 t6 + t8 t9),
   t10 t11 - d1 d2 (d2 t3 t5 - t2 t6 + t12 t9), -d2 t11^2 + t10 t12 + d2 t4 t5 + d1 d2^2 t3 t6,
   t10 t7 + d1 d2 (t2^2 + d2 t1 t3 - t8 t9), t1 t12 - t6 t7 + t5 t8 - t4 t9,
   -t12^2 - d1 d2^2 (t5 t6 - t3 t8 + t2 t9), d1 d2^2 t1^2 + t11 t4 + t10 t5 - t7 t8,
   t11 t4 - d2 (t12 t3 + t7 t8) + d1 d2^2 t6 t9, -t10 t4 + d1 d2 (t12 t2 + d2 t11 t3 - t6 t8),
   d2 t1 t11 - t10 t2 + d2 t5 t7 - d1 d2^2 t3 t9, -t4 t7 + d1 d2 (t1 t10 - t5 t6 + d2 t2 t9),
   d2 t12 t2 + t10 t4 - d2 t7^2 + d1 d2^2 t5 t9, t4 t8 + d1 d2 (-t1 t11 + t6^2 + d2 t3 t9)};
(* checking that the action is OK for the C13 invariant equations *)
{(NewRels[[9]] /. ActionC3) - NewRels[[18]],
 (NewRels[[18]] /. ActionC3) + NewRels[[9]] + NewRels[[18]]}
```

Out[∘]=
```
{0, 0}
```

Now we want to understand the symmetry of i -> 13-i for the indices. There may be a

substitution that sends d1->-1/d2, d2->-1/d1 or something like that.
Some trial and error gives the following, which works.

In[•]:=
```
NewRelsTwist = Numerator[Factor[
    NewRels /. {t1 → t12, t2 → -t11, t3 → t10, t4 → -t9, t5 → -t8, t6 → -t7, t7 → t6,
       t8 → t5, t9 → t4, t10 → -t3, t11 → t2, t12 → -t1} /. {d1 → -1 / d1, d2 → -1 / d2}]];
count = 0;
For[i = 1, i ≤ 26, i++,
  For[j = 1, j ≤ 26, j++,
   If[Simplify[NewRels[[i]] + NewRelsTwist[[j]]] == 0, count++];
   If[Simplify[NewRels[[i]] - NewRelsTwist[[j]]] == 0, count++];
  ]];
count
```

Out[•]= 26

Let us now try to understand i -> Mod[5 i ,13].

In[•]:=
```
(* The idea is to match one equation and have the
 square of the transformation match the previous one. *)
VarsT /. {t1 → -t12, t2 → t11, t3 → t10, t4 → -t9, t5 → -t8, t6 → -t7, t7 → t6,
   t8 → t5, t9 → t4, t10 → -t3, t11 → t2, t12 → -t1} /. {d1 → -1 / d1, d2 → -1 / d2}
ChangeVars = {t1 → t5 d1^a1 d2^b1, t2 → t10 d1^a2 d2^b2,
   t3 → t2 d1^a3 d2^b3 , t4 → t7 d1^a4 d2^b4 , t5 → t12 d1^a5 d2^b5 ,
   t6 → t4 d1^a6 d2^b6 , t7 → t9 d1^a7 d2^b7, t8 → t1 d1^a8 d2^b8,
   t9 → t6 d1^a9 d2^b9 , t10 → t11 d1^a10 d2^b10 ,
   t11 → t3 d1^a11 d2^b11, t12 → t8 d1^a12 d2^b12, d2 → d1, d1 → -1 / d2};
Tchch = Expand[VarsT /. ChangeVars /. ChangeVars]
EqsToSolve =
 Join[Table[Exponent[Tchch[[k]], d1] == Exponent[Tchch[[1]], d1], {k, 1, 12}],
  Table[Exponent[Tchch[[k]], d2] == Exponent[Tchch[[1]], d2], {k, 1, 12}]]

neweq = Numerator[Factor[NewRels[[17]] /. ChangeVars]];
cmp1 = Expand[-neweq / Coefficient[neweq, t10^2]]
cmp2 = Expand[NewRels[[1]]]
ListMons = {t2 t5, t1 t6, t11 t9};
For[i = 1, i ≤ 3, i++,
  AppendTo[EqsToSolve, Exponent[Coefficient[cmp1, ListMons[[i]]], d1] ==
    Exponent[Coefficient[cmp2, ListMons[[i]]], d1]]];
For[i = 1, i ≤ 3, i++, AppendTo[EqsToSolve, Exponent[Coefficient[cmp1, ListMons[[i]]],
     d2] == Exponent[Coefficient[cmp2, ListMons[[i]]], d2]]];

neweq = Numerator[Factor[NewRels[[16]] /. ChangeVars]];
cmp1 = Expand[d1 d2^2 neweq / Coefficient[neweq, t3^2]]
cmp2 = Expand[NewRels[[2]]]
```

```
ListMons = {t2 t4, t12 t7, t11 t8};
For[i = 1, i ≤ 3, i++,
  AppendTo[EqsToSolve, Exponent[Coefficient[cmp1, ListMons[[i]]], d1] ==
    Exponent[Coefficient[cmp2, ListMons[[i]]], d1]]];
For[i = 1, i ≤ 3, i++, AppendTo[EqsToSolve, Exponent[Coefficient[cmp1, ListMons[[i]]],
     d2] == Exponent[Coefficient[cmp2, ListMons[[i]]], d2]]];

neweq = Numerator[Factor[NewRels[[19]] /. ChangeVars]];
cmp1 = Expand[-d2 neweq / Coefficient[neweq, t8^2]]
cmp2 = Expand[NewRels[[7]]]
ListMons = {t1 t2, t12 t4, t10 t6};
For[i = 1, i ≤ 3, i++,
  AppendTo[EqsToSolve, Exponent[Coefficient[cmp1, ListMons[[i]]], d1] ==
    Exponent[Coefficient[cmp2, ListMons[[i]]], d1]]];
For[i = 1, i ≤ 3, i++, AppendTo[EqsToSolve, Exponent[Coefficient[cmp1, ListMons[[i]]],
     d2] == Exponent[Coefficient[cmp2, ListMons[[i]]], d2]]];
```

Out[∘]= $\{-t12, t11, t10, -t9, -t8, -t7, t6, t5, t4, -t3, t2, -t1\}$

Out[∘]= $\left\{d1^{a5+b1}\left(-\frac{1}{d2}\right)^{a1} d2^{b5}\, t12,\ d1^{a10+b2}\left(-\frac{1}{d2}\right)^{a2} d2^{b10}\, t11,\ d1^{a2+b3}\left(-\frac{1}{d2}\right)^{a3} d2^{b2}\, t10,\ d1^{a7+b4}\left(-\frac{1}{d2}\right)^{a4} d2^{b7}\, t9,\ d1^{a12+b5}\left(-\frac{1}{d2}\right)^{a5} d2^{b12}\, t8,\ d1^{a4+b6}\left(-\frac{1}{d2}\right)^{a6} d2^{b4}\, t7,\ d1^{a9+b7}\left(-\frac{1}{d2}\right)^{a7} d2^{b9}\, t6,\ d1^{a1+b8}\left(-\frac{1}{d2}\right)^{a8} d2^{b1}\, t5,\ d1^{a6+b9}\left(-\frac{1}{d2}\right)^{a9} d2^{b6}\, t4,\ d1^{a11+b10}\left(-\frac{1}{d2}\right)^{a10} d2^{b11}\, t3,\ d1^{a3+b11}\left(-\frac{1}{d2}\right)^{a11} d2^{b3}\, t2,\ d1^{a8+b12}\left(-\frac{1}{d2}\right)^{a12} d2^{b8}\, t1\right\}$

Out[∘]= $\{True, a10+b2 == a5+b1, a2+b3 == a5+b1, a7+b4 == a5+b1, a12+b5 == a5+b1, a4+b6 == a5+b1, a9+b7 == a5+b1, a1+b8 == a5+b1, a6+b9 == a5+b1, a11+b10 == a5+b1, a3+b11 == a5+b1, a8+b12 == a5+b1, True, -a2+b10 == -a1+b5, -a3+b2 == -a1+b5, -a4+b7 == -a1+b5, -a5+b12 == -a1+b5, -a6+b4 == -a1+b5, -a7+b9 == -a1+b5, -a8+b1 == -a1+b5, -a9+b6 == -a1+b5, -a10+b11 == -a1+b5, -a11+b3 == -a1+b5, -a12+b8 == -a1+b5\}$

Out[∘]= $-t10^2 - d1^{1+a1-2\,a2+a3}\, d2^{b1-2\,b2+b3}\, t2\, t5 + d1^{-2\,a2+a8+a9}\, d2^{-2\,b2+b8+b9}\, t1\, t6 + d1^{-1+a10-2\,a2+a7}\, d2^{1+b10-2\,b2+b7}\, t11\, t9$

Out[∘]= $-t10^2 - d1\, d2^2\, t2\, t5 - d1\, d2^2\, t1\, t6 + d1\, d2^2\, t11\, t9$

Out[∘]= $d1\, d2^2\, t3^2 + d1^{2-2\,a11+a3+a6}\, d2^{1-2\,b11+b3+b6}\, t2\, t4 - d1^{1-2\,a11+a4+a5}\, d2^{2-2\,b11+b4+b5}\, t12\, t7 - d1^{a10-2\,a11+a12}\, d2^{2+b10-2\,b11+b12}\, t11\, t8$

Out[∘]= $d1\, d2^2\, t3^2 + t2\, t4 + t12\, t7 - t11\, t8$

Out[∘]= $-d1^{2-2\,a12+a3+a8}\, d2^{-2\,b12+b3+b8}\, t1\, t2 + d1^{2-2\,a12+a5+a6}\, d2^{-2\,b12+b5+b6}\, t12\, t4 + d1^{2-2\,a12+a2+a9}\, d2^{-2\,b12+b2+b9}\, t10\, t6 - d2\, t8^2$

Out[•]= $d1\, d2^2\, t1\, t2 + t12\, t4 + d2\, t10\, t6 - d2\, t8^2$

```
In[•]:= sls = Solve[EqsToSolve][[1]]
Simplify[Tchch /. sls /. {a1 → 0, b1 → 0}]
```

Out[•]= {a10 → 1 + a1, a11 → 1 + a1, a12 → 1 + a1, a2 → a1, a3 → a1, a4 → 1 + a1, a5 → a1, a6 → a1, a7 → 1 + a1, a8 → 1 + a1, a9 → a1, b10 → -1 + b1, b11 → b1, b12 → -1 + b1, b2 → -1 + b1, b3 → b1, b4 → -1 + b1, b5 → -1 + b1, b6 → -1 + b1, b7 → b1, b8 → b1, b9 → b1}

Out[•]= $\left\{\frac{t12}{d2}, \frac{t11}{d2}, \frac{t10}{d2}, -\frac{t9}{d2}, \frac{t8}{d2}, \frac{t7}{d2}, -\frac{t6}{d2}, -\frac{t5}{d2}, \frac{t4}{d2}, -\frac{t3}{d2}, -\frac{t2}{d2}, -\frac{t1}{d2}\right\}$

```
In[•]:= (* Now let's match signs *)
ChangeVars = {t1 → t5 d1^a1 d2^b1, t2 → t10 d1^a2 d2^b2,
    t3 → t2 d1^a3 d2^b3 , t4 → t7 d1^a4 d2^b4 , t5 → t12 d1^a5 d2^b5 ,
    t6 → t4 d1^a6 d2^b6 , t7 → t9 d1^a7 d2^b7, t8 → t1 d1^a8 d2^b8,
    t9 → t6 d1^a9 d2^b9 , t10 → t11 d1^a10 d2^b10 , t11 → t3 d1^a11 d2^b11,
    t12 → t8 d1^a12 d2^b12, d2 → d1, d1 → -1 / d2} /. sls /. {a1 → 0, b1 → 1}
VarsT /. {t1 → t12, t2 → -t11, t3 → t10, t4 → -t9, t5 → -t8, t6 → -t7, t7 → t6,
   t8 → t5, t9 → t4, t10 → -t3, t11 → t2, t12 → -t1} /. {d1 → -1 / d1, d2 → -1 / d2}
```

$$\text{ChangeVars} = \left\{t1 \to t5,\ t2 \to \frac{t10}{d2},\ t3 \to -t2,\ t4 \to \frac{d1\, t7}{d2},\ t5 \to -\frac{t12}{d2},\ t6 \to \frac{t4}{d2},\ t7 \to d1\, t9,\ t8 \to d1\, t1,\ t9 \to t6,\ t10 \to \frac{d1\, t11}{d2},\ t11 \to d1\, t3,\ t12 \to \frac{d1\, t8}{d2},\ d2 \to d1,\ d1 \to -\frac{1}{d2}\right\};$$

$$\text{ChangeVars} = \left\{t1 \to d2\, t5\, c1,\ t2 \to t10\, c2,\ t3 \to d2\, t2\, c3,\ t4 \to d1\, t7\, c4,\ t5 \to t12\, c5,\ t6 \to t4\, c6,\ t7 \to d1\, d2\, t9\, c7,\ t8 \to d1\, d2\, t1\, c8,\ t9 \to d2\, t6\, c9,\ t10 \to d1\, t11\, c10,\ t11 \to d1\, d2\, t3\, c11,\ t12 \to d1\, t8\, c12,\ d2 \to d1,\ d1 \to -\frac{1}{d2}\right\} /.$$

```
   {c1 → c2, c10 → -c2, c11 → -c2, c12 → -c2, c3 → c2, c4 → -c2,
    c5 → c2, c6 → c2, c7 → -c2, c8 → -c2, c9 → c2} /. c2 → 1
Tchch = (VarsT /. ChangeVars /. ChangeVars)

EqsToSolve = {}

neweq = Numerator[Factor[NewRels[[17]] /. ChangeVars]];
cmp1 = Expand[-neweq / Coefficient[neweq, t10^2]]
cmp2 = Expand[NewRels[[1]]]
ListMons = {t2 t5, t1 t6, t11 t9};
For[i = 1, i ≤ 3, i++, AppendTo[EqsToSolve,
   Coefficient[cmp1, ListMons[[i]]] == Coefficient[cmp2, ListMons[[i]]]]];

neweq = Numerator[Factor[NewRels[[16]] /. ChangeVars]];
```

$\text{cmp1} = \text{Expand}\left[d1\, d2^2\, \text{neweq} / \text{Coefficient}[\text{neweq}, t3\text{^}2]\right]$

```
cmp2 = Expand[NewRels[[2]]]
```

```
ListMons = {t2 t4, t12 t7, t11 t8};
For[i = 1, i ≤ 3, i++, AppendTo[EqsToSolve,
   Coefficient[cmp1, ListMons[[i]]] == Coefficient[cmp2, ListMons[[i]]]]];

neweq = Numerator[Factor[NewRels[[19]] /. ChangeVars]];
cmp1 = Expand[-d2 neweq / Coefficient[neweq, t8^2]]
cmp2 = Expand[NewRels[[7]]]
ListMons = {t1 t2, t12 t4, t10 t6};
For[i = 1, i ≤ 3, i++, AppendTo[EqsToSolve,
   Coefficient[cmp1, ListMons[[i]]] == Coefficient[cmp2, ListMons[[i]]]]];

neweq = Numerator[Factor[NewRels[[20]] /. ChangeVars]];
cmp1 = Expand[d1 d2 neweq / Coefficient[neweq, t5^2]]
cmp2 = Expand[NewRels[[8]]]
ListMons = {t11 t12, t3 t7, t1 t9};
For[i = 1, i ≤ 3, i++, AppendTo[EqsToSolve,
   Coefficient[cmp1, ListMons[[i]]] == Coefficient[cmp2, ListMons[[i]]]]];


EqsToSolve = EqsToSolve /. {d1 → 1, d2 → 1}
Solve[EqsToSolve]

neweq = Numerator[Factor[NewRels[[17]] /. ChangeVars]];
cmp1 = Expand[-neweq / Coefficient[neweq, t10^2]];
cmp2 = Expand[NewRels[[1]]];
cmp1 - cmp2

neweq = Numerator[Factor[NewRels[[16]] /. ChangeVars]];
cmp1 = Expand[d1 d2^2 neweq / Coefficient[neweq, t3^2]];
cmp2 = Expand[NewRels[[2]]];
cmp1 - cmp2

neweq = Numerator[Factor[NewRels[[19]] /. ChangeVars]];
cmp1 = Expand[-d2 neweq / Coefficient[neweq, t8^2]];
cmp2 = Expand[NewRels[[7]]];
cmp1 - cmp2

neweq = Numerator[Factor[NewRels[[20]] /. ChangeVars]];
cmp1 = Expand[d1 d2 neweq / Coefficient[neweq, t5^2]];
cmp2 = Expand[NewRels[[8]]];
cmp1 - cmp2
```

Out[ ]= $\{t1 \to d2\, t5,\ t2 \to t10,\ t3 \to d2\, t2,\ t4 \to d1\, t7,\ t5 \to t12,\ t6 \to t4,\ t7 \to d1\, d2\, t9,$
$t8 \to d1\, d2\, t1,\ t9 \to d2\, t6,\ t10 \to d1\, t11,\ t11 \to d1\, d2\, t3,\ t12 \to d1\, t8,\ d2 \to d1,\ d1 \to -\frac{1}{d2}\}$

Out[ ]= $\{t12, -t11, t10, -t9, -t8, -t7, t6, t5, t4, -t3, t2, -t1\}$

Out[ ]= $\{t1 \to d2\, t5,\ t2 \to t10,\ t3 \to d2\, t2,\ t4 \to -d1\, t7,$
$t5 \to t12,\ t6 \to t4,\ t7 \to -d1\, d2\, t9,\ t8 \to -d1\, d2\, t1,\ t9 \to d2\, t6,$
$t10 \to -d1\, t11,\ t11 \to -d1\, d2\, t3,\ t12 \to -d1\, t8,\ d2 \to d1,\ d1 \to -\frac{1}{d2}\}$

Out[ ]= $\{d1\, t12, -d1\, t11, d1\, t10, -d1\, t9, -d1\, t8, -d1\, t7, d1\, t6, d1\, t5, d1\, t4, -d1\, t3, d1\, t2, -d1\, t1\}$

Out[ ]= $\{\}$

Out[ ]= $-t10^2 - d1\, d2^2\, t2\, t5 - d1\, d2^2\, t1\, t6 + d1\, d2^2\, t11\, t9$

Out[ ]= $-t10^2 - d1\, d2^2\, t2\, t5 - d1\, d2^2\, t1\, t6 + d1\, d2^2\, t11\, t9$

Out[ ]= $d1\, d2^2\, t3^2 + t2\, t4 + t12\, t7 - t11\, t8$

Out[ ]= $d1\, d2^2\, t3^2 + t2\, t4 + t12\, t7 - t11\, t8$

Out[ ]= $d1\, d2^2\, t1\, t2 + t12\, t4 + d2\, t10\, t6 - d2\, t8^2$

Out[ ]= $d1\, d2^2\, t1\, t2 + t12\, t4 + d2\, t10\, t6 - d2\, t8^2$

Out[ ]= $t11\, t12 + d1\, d2\, t5^2 - d1\, d2\, t3\, t7 + d1\, d2^2\, t1\, t9$

Out[ ]= $t11\, t12 + d1\, d2\, t5^2 - d1\, d2\, t3\, t7 + d1\, d2^2\, t1\, t9$

Out[ ]= {True, True, True, True, True, True, True, True, True, True, True, True}

Out[ ]= $\{\{\}\}$

Out[ ]= 0

Out[ ]= 0

Out[ ]= 0

Out[ ]= 0

In[∘]:=
```
(* Now making sure it works *)
ChangeVars = {t1 → d2 t5, t2 → t10, t3 → d2 t2, t4 → -d1 t7,
   t5 → t12, t6 → t4, t7 → -d1 d2 t9, t8 → -d1 d2 t1, t9 → d2 t6,
   t10 → -d1 t11, t11 → -d1 d2 t3, t12 → -d1 t8, d2 → d1, d1 → -1/d2};
Tchch = (VarsT /. ChangeVars /. ChangeVars) / d1
count = 0;
For[i = 1, i ≤ 26, i++,
  For[j = 1, j ≤ 26, j++,
   cmp1 = Numerator[Factor[NewRels[[j]] /. ChangeVars]];
   cmp2 = NewRels[[i]];
   If[ByteCount[Simplify[cmp1 / cmp2]] < 200, Print[{i, j, Simplify[cmp1 / cmp2]}];
    count++];
  ]];
count
(* it works *)
```

Out[∘]=
```
{t12, -t11, t10, -t9, -t8, -t7, t6, t5, t4, -t3, t2, -t1}
```

$\{1, 17, d1\}$

$\{2, 16, -d1^2\}$

$\{3, 13, -d1^2\}$

$\{4, 24, -d1\}$

$\{5, 21, -d1^2\}$

$\{6, 14, d1\}$

$\{7, 19, d1^2\}$

$\{8, 20, -d1\}$

$\{9, 9, -d1\, d2\}$

$\{10, 26, -d1\}$

$\{11, 25, d1^2\}$

$\{12, 22, -d1^2\}$

$\{13, 6, d1\}$

$\{14, 3, -d1^2\}$

$\{15, 23, d1\}$

$\{16, 1, d1^2\}$

$\{17, 2, -d1\}$

$\{18, 18, -d1\, d2\}$

$\{19, 8, d1\}$

$\{20, 7, -d1^2\}$

$\{21, 4, d1^2\}$

$\{22, 15, d1\}$

$\{23, 12, -d1^2\}$

$\{24, 5, d1\}$

$\{25, 10, -d1^2\}$

$\{26, 11, d1\}$

Out[∘]= 26

In[∘]:=
```
ChangeVars = {t1 → d2 t5, t2 → t10, t3 → d2 t2, t4 → -d1 t7,
   t5 → t12, t6 → t4, t7 → -d1 d2 t9, t8 → -d1 d2 t1, t9 → d2 t6,
   t10 → -d1 t11, t11 → -d1 d2 t3, t12 → -d1 t8, d2 → d1, d1 → -1/d2};
VarsT /. ChangeVars /. ChangeVars
VarsT /. ChangeVars /. ChangeVars /. ChangeVars
Join[VarsT, {d1, d2}] /. ChangeVars /. ChangeVars /. ChangeVars /. ChangeVars
```

Out[∘]= {d1 t12, -d1 t11, d1 t10, -d1 t9, -d1 t8, -d1 t7, d1 t6, d1 t5, d1 t4, -d1 t3, d1 t2, -d1 t1}

Out[∘]= $\left\{\frac{d1\, t8}{d2}, -d1\, t3, \frac{d1\, t11}{d2}, t6, -d1\, t1, -d1\, t9, -\frac{t4}{d2}, -\frac{t12}{d2}, \frac{d1\, t7}{d2}, t2, -\frac{t10}{d2}, t5\right\}$

Out[∘]= {t1, t2, t3, t4, t5, t6, t7, t8, t9, t10, t11, t12, d1, d2}

We will now construct quotients by C13. The (i,i) was found to have a curve of singular points (by a separate Magma calculation), so we can ignore it. We will focus on the third point. We will look at degree 2 and degree 3 invariants.

It is important to apply the symmetry to get the smaller field of definition.

In[∘]:=
```
VarsT = {t1, t2, t3, t4, t5, t6, t7, t8, t9, t10, t11, t12};
NumCut = 3;
EqsCover = NewRels;
GB = GroebnerBasis[EqsCover, VarsT, MonomialOrder → DegreeReverseLexicographic,
   CoefficientDomain → RationalFunctions];
VarsS = {S1, S2, S3, S4, S5, S6, S7, S8, S9, S10, S11, S12, S13, S14};
degS = {2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3};

StoT = {S1 → t1 t12 + t3 t10 + t9 t4, S2 → t2 t11, S3 → t6 t7, S4 → t5 t8,
   S5 → t1 t2 t10, S6 → t3 t6 t4, S7 → t9 t5 t12, S8 → t1 t4 t8, S9 → t3 t12 t11,
   S10 → t9 t10 t7, S11 → t1 t5 t7, S12 → t3 t2 t8, S13 → t9 t6 t11,
   S14 → t1 t3 t9};

ChangeVars = {t1 → d2 t5, t2 → t10, t3 → d2 t2, t4 → -d1 t7,
   t5 → t12, t6 → t4, t7 → -d1 d2 t9, t8 → -d1 d2 t1, t9 → d2 t6,
   t10 → -d1 t11, t11 → -d1 d2 t3, t12 → -d1 t8, d2 → d1, d1 → -1/d2};
(* symmetrize *)
StoT = Simplify[
   Table[VarsS[[k]] → (VarsS[[k]] /. StoT) + (VarsS[[k]] /. StoT /. ChangeVars) +
```

```
      (VarsS[[k]] /. StoT /. ChangeVars /. ChangeVars) +
      (VarsS[[k]] /. StoT /. ChangeVars /. ChangeVars /. ChangeVars),
    {k, 1, Length[VarsS]}]];

EqsFPPC3withrr = {};

VarsR = {R1, R2, R3, R4, R5, R6, R7, R8, R9, R10, R11, R12, R13, R14};

For[deg = 5, deg ≤ 6, deg++,

  solvemons = Solve[Append[Table[VarsR[[i]] ≥ 0, {i, 1, 14}],
     Sum[VarsR[[i]] × degS[[i]], {i, 1, 14}] == deg], VarsR, Integers];
  AllMons = Table[Product[VarsS[[i]] ^ (VarsR[[i]] /. solvemons[[k]]), {i, 1, 14}],
    {k, 1, Length[solvemons]}];

  AllMonsT = {};
  AllRem = {};
  For[i = 1, i ≤ Length[AllMons], i++,
   If[Mod[i, 20] == 0, Print[i];];
   AppendTo[AllRem, PolynomialReduce[AllMons[[i]] /. StoT,
      GB, VarsT, MonomialOrder → DegreeReverseLexicographic][[2]] ];
   newrem = AllRem[[i]];
   monlist = MonomialList[newrem /. {d1 → LocusA2valuesReduced[[NumCut]][[1]],
       d2 → LocusA2valuesReduced[[NumCut]][[2]]}];
   For[k = 1, k ≤ Length[monlist], k++,
    monlist[[k]] =
      Product[VarsT[[j]] ^ Exponent[monlist[[k]], VarsT[[j]]], {j, 1, 12}];
   ];
   AllMonsT = Union[AllMonsT, monlist];
  ];
  RemMatrix = Table[Coefficient[AllRem[[i]], AllMonsT[[j]]],
    {j, 1, Length[AllMonsT]}, {i, 1, Length[AllMons]}];
  RtRm = RootReduce[RemMatrix /. {d1 → LocusA2valuesReduced[[3]][[1]],
      d2 → LocusA2valuesReduced[[3]][[2]]}];
  blah = RowReduce[NullSpace[N[RtRm, 10 000]]];
  Print[Accuracy[blah]];
  goodnull = RootApproximant[Chop[blah], 3];
  For[i = 1, i ≤ Length[goodnull], i++, For[j = 1, j ≤ Length[goodnull[[i]]], j++,
    If[Exponent[MinimalPolynomial[goodnull[[i]][[j]], t], t] == 3,
      Print["Warning: not enough accuracy"]];
   ]];
  goodnullwithrr = Table[Rationalize[Re[goodnull[[i]][[j]]]] +
     Rationalize[Im[goodnull[[i]][[j]]] / (2 ^ (1 / 2))] rr,
```

```
      {i, 1, Length[goodnull]}, {j, 1, Length[goodnull[[1]]]}];
    Print[DeleteDuplicates[Simplify[(goodnullwithrr /. rr -> ⅈ √2) - goodnull]]];
    (* should be zeros *)
    EqsFPPC3withrr = Join[EqsFPPC3withrr, goodnullwithrr.AllMons];
    EqsFPPC3withrr =
     Table[Numerator[Factor[EqsFPPC3withrr[[i]]]], {i, 1, Length[EqsFPPC3withrr]}];
    Print[Table[ByteCount[EqsFPPC3withrr[[i]]], {i, 1, Length[EqsFPPC3withrr]}]];

  ];
```

```
20

40

9997.37
```

N: Internal precision limit $MaxExtraPrecision = 50.` reached while evaluating
(1745112117623942218726363224770079545789463319246135818208251302 −
6497473836090552585382138628110456459723401524502342277089344480 ⅈ √2)/
393191027700942320455890352432987020468711999897998417768226657.

N: Internal precision limit $MaxExtraPrecision = 50.` reached while evaluating
(1745112117623942218726363224770079545789463319246135818208251302 +
6497473836090552585382138628110456459723401524502342277089344480 ⅈ √2)/
393191027700942320455890352432987020468711999897998417768226657.

N: Internal precision limit $MaxExtraPrecision = 50.` reached while evaluating
(−550107013630993822791940332858658100953088576150632821417080182857654252470 −
225044065740359959104824681203542577551013093548508095519554207736667209565 ⅈ √2)
/37757134001267150024450176390522074670528001174700263579072449692454674726.

General: Further output of N::meprec will be suppressed during this calculation.

```
{{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}

{8912, 9392, 9392, 8888, 8888, 8888, 9344, 9344, 9376}

20

40

60

9995.99

{{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}

{8912, 9392, 9392, 8888, 8888, 8888, 9344, 9344, 9376, 15 976, 20 256, 20 320, 20 256, 20 256,
 20 320, 20 256, 20 256, 20 256, 21 056, 20 256, 20 256, 20 256, 20 256, 20 320, 20 256, 20 256,
 20 256, 20 256, 20 256, 20 320, 20 256, 20 256, 20 256, 20 256, 20 256, 20 256, 20 320, 16 184}
```

In[◦]:=
```
(* The name of the file needs to be
 adjusted according to your situation. *)
fname = "Desktop/FPP2020/EqsFPPC3withrr";
OpenWrite[fname];
Write[fname, {VarsS, EqsFPPC3withrr}];
Close[fname];
```

The process of finding the A2 singular points was rather difficult. Specifically, we computed a degree 12 polynomial equation in S1,...,S4, then found singular points on hyperplane cuts, then excluded them to find isolated singular points. If one doesn’t want to repeat these calculation, it is a good idea to scroll all the way down.

In[◦]:=
```
(* Let us find points on this surface *)
pts = {};
Binomial[3 + 12, 3]
For[i = 1, i ≤ 300, i++,
  If[Mod[i, 20] == 5, Print[i];];
  sls = NSolve[Join[Table[0 == EqsFPPC3withrr[[k]] /. rr → ⅈ √2 , {k, 1, 38}],
     {1 == S1, S2 == i , S4 == RandomInteger[1000] / 1000}], 500];
  AppendTo[pts, {S1, S2, S3, S4} /. sls[[1]]];
  AppendTo[pts, {S1, S2, S3, S4} /. sls[[7]]];
 ];
```

Out[◦]= 455

```
5
25
45
65
85
105
125
145
165
185
205
225
245
265
285
```

In[∘]:=
```
AllMons12 = {};
For[i1 = 0, i1 ≤ 12, i1++,
  For[i2 = 0, i2 ≤ 12 - i1, i2++, For[i3 = 0, i3 ≤ 12 - i2 - i1, i3++,
    i4 = 12 - i1 - i2 - i3;
    AppendTo[AllMons12, S1^i1 S2^i2 S3^i3 S4^i4];
   ]]];
Length[AllMons12]
```

Out[∘]=
```
455
```

In[∘]:=
```
ns12 = NullSpace[Table[AllMons12[[i]] /. Table[VarsS[[k]] → pts[[j]][[k]], {k, 1, 4}],
    {j, 1, Length[pts]}, {i, 1, 455}]];
```

In[∘]:=
```
Accuracy[ns12]
Length[ns12]
ns12 = RowReduce[ns12];
```

Out[∘]=
```
433.002
```

Out[∘]=
```
1
```

In[∘]:=
```
Eq12 = Numerator[Factor[Block[{$MaxExtraPrecision = 400},
     RootApproximant[Chop[ns12], 2][[1]].AllMons12]]];
```

In[∘]:=
```
(* The name of the file needs to be
 adjusted according to your situation. *)
fname = "Desktop/FPP2020/Eq12";
OpenWrite[fname];
Write[fname, Numerator[Factor[Eq12]]];
Close[fname];
```

In[∘]:=
```
(* finding singularities on curves *)
ptssing = {};
Eq12 = Numerator[Factor[Eq12]];
For[mm = 1, mm ≤ 60, mm++,
  If[Mod[mm, 20] == 0, Print[mm]];

  sls = NSolve[{D[Eq12, S1] == 0, D[Eq12, S2] == 0, D[Eq12, S3] == 0, D[Eq12, S4] == 0,
     S1 == 1, RandomInteger[100] S2 + RandomInteger[100] S3 == 1}, 600];
  AppendTo[ptssing, sls[[1]]];
  AppendTo[ptssing, sls[[2]]];
  AppendTo[ptssing, sls[[3]]];
  AppendTo[ptssing, sls[[4]]];
  AppendTo[ptssing, sls[[5]]];
  AppendTo[ptssing, sls[[6]]];
 ];
```

```
20
40
60
```

In[∘]:=
```
deg = 9;
Mons = Flatten[Table[S1^i1 S2^i2 S3^i3 S4^(deg - i1 - i2 - i3),
    {i1, 0, deg}, {i2, 0, deg - i1}, {i3, 0, deg - i1 - i2}]];
{Length[Mons], Length[ptssing]}
MatrixRank[
 Table[Mons[[k]] /. ptssing[[l]], {k, 1, Length[Mons]}, {l, 1, Length[ptssing]}]]
```

Out[∘]= {220, 360}

Out[∘]= 210

In[∘]:=
```
(* finding an equation through the these curves *)
ns = RowReduce[NullSpace[Table[Mons[[k]] /. ptssing[[l]],
     {l, 1, Length[ptssing]}, {k, 1, Length[Mons]}]]];
Accuracy[
 ns]
```

Out[∘]= 465.397

In[ ]:=
```
EqExtra = RootApproximant[Chop[ns[[1]], 10^(-400)], 2].Mons;
```

```
N: Internal precision limit $MaxExtraPrecision = 50.` reached while evaluating
  (155082612571389869143479709020665808884953191288167662077807582086752870877813917409\
    12539891652916392735721725929980378371760235470841579798157975221040292889287900\
    8118173874068155610201351068662486169308395305326750606326549438053355254989440\
    83279783615784505935 − ≪1≫)/185151181578380895481644606048975195021436935103094254763\
  795500471918729122333375906530524976569805296779989147533824261193271108475078324\
  4315435552020068517451619457339763009946729441657904990309178691163880143022894024\
  7854250336180126197597531572912372082031 25.
```

```
N: Internal precision limit $MaxExtraPrecision = 50.` reached while evaluating
  (155082612571389869143479709020665808884953191288167662077807582086752870877813917409\
    12539891652916392735721725929980378371760235470841579798157975221040292889287900\
    8118173874068155610201351068662486169308395305326750606326549438053355254989440\
    83279783615784505935 + ≪1≫)/185151181578380895481644606048975195021436935103094254763\
  795500471918729122333375906530524976569805296779989147533824261193271108475078324\
  4315435552020068517451619457339763009946729441657904990309178691163880143022894024\
  7854250336180126197597531572912372082031 25.
```

```
N: Internal precision limit $MaxExtraPrecision = 50.` reached while evaluating
  (3926952035957400056880732330603653232614437960793076260426935905905033998887947081\
    9409998711234587894769065850181062130992840632258195139129073733327482280241038\
    45774269856735760226291700166938940084553589829550957206216844436321461608678710\
    82009798847459277569 40 − ≪1≫)/18515118157838089548164460604897519502143693510309425\
  4763795500471918729122333375906530524976569805296779989147533824261193271108475078\
  3244315435552020068517451619457339763009946729441657904990309178691163880143022894\
  02478542503361801261975975315729123720820312 5.
```

```
General: Further output of N::meprec will be suppressed during this calculation.
```

In[ ]:=
```
For[numtry = 1, numtry ≤ 1000, numtry++,
  If[Mod[numtry, 50] == 25, Print[numtry]];
  fi = FindRoot[{D[Eq12, S1], D[Eq12, S2], D[Eq12, S3], EqExtra - 1},
    {{S1, RandomReal[10] + I RandomReal[10]}, {S2, RandomReal[10] + I RandomReal[10]},
     {S3, RandomReal[10] + I RandomReal[10]}, {S4, RandomReal[10] + I RandomReal[10]}},
    WorkingPrecision → 800, AccuracyGoal → 300, MaxIterations → 200,
    PrecisionGoal → 300, DampingFactor → 2];
  If[Abs[N[D[Eq12, S4] /. fi, 200]] < 10^(-100), numtry = 5000;
  ]
 ];
```

```
FindRoot: The line search decreased the step size to within tolerance specified by AccuracyGoal and
  PrecisionGoal but was unable to find a sufficient decrease in the merit function. You may need more than
  800.` digits of working precision to meet these tolerances.
```

```
FindRoot: The line search decreased the step size to within tolerance specified by AccuracyGoal and
  PrecisionGoal but was unable to find a sufficient decrease in the merit function. You may need more than
  800.` digits of working precision to meet these tolerances.
```

··· FindRoot: Encountered a singular Jacobian at the point {S1, S2, S3, S4} =
{−1.1706013561110973608751456341783543898084096795523 −
7.9394458569051048365857953600983049481750579643703 i, −
0.21828246621740572580857054814870612634694055311450 +
2.7799135272287311014207029798282520357774321746333 i,
0.44064471436051208297874086432044086086533273209772 + ≪67≫ i,
4.3643901570195389183481893269345164299011230468750 +
7.6519648402548980925530486274510622024536132812500 i}. Try perturbing the
initial point(s).

··· FindRoot: Encountered a singular Jacobian at the point {S1, S2, S3, S4} =
{−5.1990703364265393396520442694312491744412670543271 +
6.0426939452068321735166637698290302092799404288079 i,
2.1913997989323695342722665970897792617229446451757 −
4.0369774723583114443322459210769854287344167570719 i, −
0.84485810526972188041660318706127047699177661781177 − ≪68≫ i,
7.8820170459465472845295153092592954635620117187500 +
1.8018553140469411033564028912223875522613525390625 i}. Try perturbing the initial
point(s).

··· FindRoot: Failed to converge to the requested accuracy or precision within 200 iterations.

··· FindRoot: The line search decreased the step size to within tolerance specified by AccuracyGoal and PrecisionGoal but was unable to find a sufficient decrease in the merit function. You may need more than 800.` digits of working precision to meet these tolerances.

··· General: Further output of FindRoot::lstol will be suppressed during this calculation.

··· FindRoot: Encountered a singular Jacobian at the point {S1, S2, S3, S4} =
{−4.6396733051303123889353097447822875628357420868213 −
6.4264807322290270010531203027060396430094446856372 i,
1.0739263455208910894798634446674544807568561031288 +
2.5360661248673378346458403449328162399472154720208 i,
0.96273892653637208650888448952320656929941016383845 + ≪68≫ i,
7.3083241389684125977055373368784785270690917968750 +
4.7208577183346989158962969668209552764892578125000 i}. Try perturbing the
initial point(s).

··· General: Further output of FindRoot::jsing will be suppressed during this calculation.

··· FindRoot: Failed to converge to the requested accuracy or precision within 200 iterations.

In[∘]:=
```
(* we may have gotten lucky here. *)
(* we can find the symmetric functions nicely *)
RootApproximant[N[((S2 + S3 + S4) / S1) /. fi, 100], 2]
RootApproximant[N[((S2^2 + S3^2 + S4^2) / S1^2) /. fi, 100], 2]
RootApproximant[N[((S2 S3 S4) / S1^3) /. fi, 100], 2]
```

Out[∘]= $$\frac{-6634 - 1690\, i \sqrt{2}}{4689}$$

Out[∘]= $$\frac{19\,047\,874 + 9\,436\,960\, i \sqrt{2}}{21\,986\,721}$$

Out[∘]= $$\frac{-31\,580\,371\,658 - 55\,821\,877\,592\, i \sqrt{2}}{927\,861\,612\,921}$$

The code below will find these singularities, which are presumably A2.

```
In[•]:= slsA2 = NSolve[{D[Eq12, S1] == 0, D[Eq12, S2] == 0,
      D[Eq12, S3] == 0, D[Eq12, S4] == 0, S2 + S3 + S4 == ((-6634 - 1690 i Sqrt[2])/4689) S1,
      S2^2 + S3^2 + S4^2 == ((19047874 + 9436960 i Sqrt[2])/21986721) S1^2,
      (S2 S3 S4) / S1^3 == (-31580371658 - 55821877592 i Sqrt[2])/927861612921, S1 == 1}, 10000];

In[•]:= Length[slsA2]
     RootApproximant[{S1, S2, S3, S4} /. slsA2[[1]], 6]

Out[•]= 3

Out[•]= {1, -0.946… - 0.287… i, -0.305… + 0.0312… i, -0.164… - 0.254… i}

In[•]:= (* Let us try to see what Eq12 looks like in the neighborhood *)
     (* The Hessian has the correct rank, so it's at least A2 singularity *)
     Block[{$MaxExtraPrecision = 400},
      MatrixRank[D[Eq12 /. S1 → 1, {{S2, S3, S4}, 2}] /. slsA2[[1]]]]

Out[•]= 2

In[•]:= slsA2all =
      NSolve[Union[Table[0 == EqsFPPC3withrr[[k]] /. rr → (-2)^(1/2), {k, 1, 38}],
        {S1 == 1, S2 == -0.946… - 0.287… i, S3 == -0.305… + 0.0312… i,
         S4 == -0.164… - 0.254… i}], 10000];

In[•]:= Length[slsA2all]
     (* the two points are really the same in the weighted projective space,
     i.e. the degree 3 coordinates differ by sign *)
     OneSingPt = RootApproximant[VarsS /. slsA2all[[1]], 12]

Out[•]= 2

Out[•]= {1, -0.946… - 0.287… i, -0.305… + 0.0312… i,
      -0.164… - 0.254… i, -0.0449… + 0.0861… i, -0.182… - 0.154… i,
      -6.32… × 10^-3 + 0.149… i, 0.137… + 0.0533… i,
      0.198… + 0.0608… i, 0.0108… - 0.103… i, -0.0910… - 0.191… i,
      0.0967… - 0.449… i, 0.256… - 0.0269… i, 0.780… - 0.739… i}
```

```
In[∘]:= (* Verifying the symmetries *)
EqsFPPC3 = EqsFPPC3withrr /. rr → (-2) ^ (1 / 2);
SymC3 = {S2 → S3, S3 → S4, S4 → S2, S5 → S6, S6 → S7,
   S7 → S5, S8 → S9, S9 → S10, S10 → S8, S11 → S12, S12 → S13, S13 → S11};
AllMons5 =
 {};
For[i = 1, i ≤ 4, i++, For[j = 5, j ≤ 14, j++,
  AppendTo[AllMons5, VarsS[[i]] × VarsS[[j]]];]];
MatrixRank[N[Join[Table[Coefficient[EqsFPPC3[[k]], AllMons5[[j]]],
    {k, 1, 9}, {j, 1, Length[AllMons5]}],
   Table[Coefficient[EqsFPPC3[[k]] /. SymC3, AllMons5[[j]]],
    {k, 1, 9}, {j, 1, Length[AllMons5]}]], 400]]
AllMons6 =
 {};
For[i = 5, i ≤ 14, i++, For[j = 5, j ≤ i, j++,
  AppendTo[AllMons6, VarsS[[i]] × VarsS[[j]]];]];
For[i = 1, i ≤ 4, i++, For[j = 1, j ≤ i, j++,
   For[k = 1, k ≤ j, k++, AppendTo[AllMons6, VarsS[[i]] × VarsS[[j]] × VarsS[[k]]];]]];
MatrixRank[N[Join[Table[Coefficient[EqsFPPC3[[k]], AllMons6[[j]]],
    {k, 10, 38}, {j, 1, Length[AllMons6]}],
   Table[Coefficient[EqsFPPC3[[k]] /. SymC3, AllMons6[[j]]],
    {k, 10, 38}, {j, 1, Length[AllMons6]}]], 400]]

Out[∘]= 9

Out[∘]= 29
```

In[∘]:=
```
(* this is one of the singular points *)
SingPt = {1, -0.946… - 0.287… i,
   -0.305… + 0.0312… i, -0.164… - 0.254… i, -0.0449… + 0.0861… i,
   -0.182… - 0.154… i, -6.32… × 10^-3 + 0.149… i, 0.137… + 0.0533… i,
   0.198… + 0.0608… i, 0.0108… - 0.103… i, -0.0910… - 0.191… i,
   0.0967… - 0.449… i, 0.256… - 0.0269… i, 0.780… - 0.739… i};
Block[{$MaxExtraPrecision = 10 000},
 Chop[N[EqsFPPC3 /. Table[VarsS[[l]] → SingPt[[l]], {l, 1, 14}], 1000]]]
```

```
N: Internal precision limit $MaxExtraPrecision = 10000.` reached while evaluating
  {22832877025328764567999225007667087066898851314506577610009876182744246627838 Root[
      53258647842318380176 – 4964124530200959864 Power[≪2≫] + 39941399978020282041 Power[
        ≪2≫] &, 3, 0] +
    13274831787496711315532020413942923153981633011947800971818956640019211461020 i √2
     Root[53258647842318380176 – 4964124530200959864 Power[≪2≫] + 39941399978020282041
       Power[≪2≫] &, 3, 0] + ≪72≫ + ≪13≫, ≪1≫, ≪34≫, ≪1≫,
   60225378966219394576853282263678128194477296772782021605295677334976536335221968 –
    13570950078956567960611795509337909260021474389821128151055534476873224285784336 i
     √2 + ≪64≫ + ≪41≫}.
```

Out[∘]=
```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Let us find a set of points on this surface, which we will use to change to new coordinates.

In[∘]:=
```
PtsOnXinS = {};
For[numpt = 1, numpt ≤ 50, numpt++,
  If[Mod[numpt, 10] == 2, Print[numpt]];
  sls = NSolve[Join[Table[0 == EqsFPPC3[[k]], {k, 1, 38}],
     {S1 == 1, S2 == RandomInteger[100] / 100 + I RandomInteger[100] / 100,
      S3 == RandomInteger[100] / 100 + I RandomInteger[100] / 100}], 2000];
  AppendTo[PtsOnXinS, sls[[1]]];
  AppendTo[PtsOnXinS, sls[[5]]];
 ];
```

```
2
12
22
32
42
```

In[◦]:=
```
MatrixRank[{{SingPt[[2]], SingPt[[3]], SingPt[[4]]},
  {SingPt[[3]], SingPt[[4]], SingPt[[2]]}, {SingPt[[4]], SingPt[[2]], SingPt[[3]]}}]
```

Out[◦]= 3

```
In[•]:= (* making a variable change *)
VarsZ = {Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10, Z11, Z12, Z13, Z14};
StoZ4 = {S1 -> Z1 + Z2 + Z3 + Z4, S2 → SingPt[[2]] Z2 + SingPt[[4]] Z3 + SingPt[[3]] Z4,
   S3 → SingPt[[2]] Z3 + SingPt[[4]] Z4 + SingPt[[3]] Z2,
   S4 → SingPt[[2]] Z4 + SingPt[[4]] Z2 + SingPt[[3]] Z3};

k = 3; l = 3;
StoZ = Join[StoZ4,
   {VarsS[[2 + k]] → SingPt[[2 + l]] × VarsZ[[2 + k]] +
      SingPt[[4 + l]] × VarsZ[[3 + k]] + SingPt[[3 + l]] × VarsZ[[4 + k]],
    VarsS[[3 + k]] → SingPt[[2 + l]] × VarsZ[[3 + k]] +
      SingPt[[4 + l]] × VarsZ[[4 + k]] + SingPt[[3 + l]] × VarsZ[[2 + k]],
    VarsS[[4 + k]] → SingPt[[2 + l]] × VarsZ[[4 + k]] +
      SingPt[[4 + l]] × VarsZ[[2 + k]] + SingPt[[3 + l]] × VarsZ[[3 + k]]}];

k = 6; l = 6;
StoZ = Join[StoZ,
   {VarsS[[2 + k]] → SingPt[[2 + l]] × VarsZ[[2 + k]] +
      SingPt[[4 + l]] × VarsZ[[3 + k]] + SingPt[[3 + l]] × VarsZ[[4 + k]],
    VarsS[[3 + k]] → SingPt[[2 + l]] × VarsZ[[3 + k]] +
      SingPt[[4 + l]] × VarsZ[[4 + k]] + SingPt[[3 + l]] × VarsZ[[2 + k]],
    VarsS[[4 + k]] → SingPt[[2 + l]] × VarsZ[[4 + k]] +
      SingPt[[4 + l]] × VarsZ[[2 + k]] + SingPt[[3 + l]] × VarsZ[[3 + k]]}];

k = 9; l = 9;
StoZ = Join[StoZ,
   {VarsS[[2 + k]] → SingPt[[2 + l]] × VarsZ[[2 + k]] +
      SingPt[[4 + l]] × VarsZ[[3 + k]] + SingPt[[3 + l]] × VarsZ[[4 + k]],
    VarsS[[3 + k]] → SingPt[[2 + l]] × VarsZ[[3 + k]] +
      SingPt[[4 + l]] × VarsZ[[4 + k]] + SingPt[[3 + l]] × VarsZ[[2 + k]],
    VarsS[[4 + k]] → SingPt[[2 + l]] × VarsZ[[4 + k]] +
      SingPt[[4 + l]] × VarsZ[[2 + k]] + SingPt[[3 + l]] × VarsZ[[3 + k]]}];

AppendTo[StoZ, S14 → Z14 SingPt[[14]]];
ZtoS = Solve[Table[VarsS[[k]] == (VarsS[[k]] /. StoZ), {k, 1, 14}], VarsZ][[1]];
PtsOnXinZ = {};
For[j = 1, j ≤ Length[PtsOnXinS], j++,
  AppendTo[PtsOnXinZ,
    N[Table[VarsZ[[k]] → (VarsZ[[k]] /. ZtoS) /. PtsOnXinS[[j]], {k, 1, 14}], 2000]];
 ];
```

In[∘]:=
```
(* finding new equations *)
AllMons5 = {};
For[i = 1, i ≤ 4, i++,
 For[j = 5, j ≤ 14, j++, AppendTo[AllMons5, VarsZ[[i]] × VarsZ[[j]]];]];
AllMons6 = {};
For[i = 5, i ≤ 14, i++,
 For[j = 5, j ≤ i, j++, AppendTo[AllMons6, VarsZ[[i]] × VarsZ[[j]]];]];
For[i = 1, i ≤ 4, i++, For[j = 1, j ≤ i, j++,
   For[k = 1, k ≤ j, k++, AppendTo[AllMons6, VarsZ[[i]] × VarsZ[[j]] × VarsZ[[k]]];]]];

ns = RowReduce[NullSpace[Table[AllMons5[[k]] /. PtsOnXinZ[[l]],
     {l, 1, Length[PtsOnXinZ]}, {k, 1, Length[AllMons5]}]]];
ns = Chop[ns, 10^(-Round[Accuracy[ns]] + 5)];
Accuracy[ns]
ns = Block[{$MaxExtraPrecision = 400}, RootApproximant[ns, 2]];
EqsFPPC3Zdeg5 = ns.AllMons5;
ns = RowReduce[NullSpace[Table[AllMons6[[k]] /. PtsOnXinZ[[l]],
     {l, 1, Length[PtsOnXinZ]}, {k, 1, Length[AllMons6]}]]];
ns = Chop[ns, 10^(-Round[Accuracy[ns]] + 5)];
Accuracy[ns]
ns = Block[{$MaxExtraPrecision = 400}, RootApproximant[ns, 2]];
EqsFPPC3Zdeg6 = ns.AllMons6;
```

Out[∘]= 1927.76

Out[∘]= 1925.1

```
EqsFPPC3Z = Join[EqsFPPC3Zdeg5, EqsFPPC3Zdeg6];
Length[EqsFPPC3Z]
EqsFPPC3Z = Table[Numerator[Factor[EqsFPPC3Z[[k]]]], {k, 1, 38}];
(* The name of the file needs to be
 adjusted according to your situation. *)
fname = "Desktop/FPP2020/EqsFPPC3Z";
OpenWrite[fname];
Write[fname, EqsFPPC3Z];
Close[fname];
```

Out[∘]= 38

```
In[•]:= sls = NSolve[Join[Table[0 == EqsFPPC3Z[[k]], {k, 1, 38}],
          {Z2 == 1, Z3 == 0, Z4 == 0, Z5 == 1, Z6 == 0, Z7 == 0, Z8 == 1, Z9 == 0,
           Z10 == 0, Z11 == 1, Z12 == 0}], 600];
        Length[sls]
        SingPtZ = RootApproximant[VarsZ /. sls[[1]]]

Out[•]= 1

Out[•]= {0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1}
```

We will try to make the variables nicer. Our first goal is to have good first four variables.

```
In[•]:= CyclicZ = {Z2 → Z3, Z3 → Z4, Z4 → Z2, Z5 → Z6, Z6 → Z7,
           Z7 → Z5, Z8 → Z9, Z9 → Z10, Z10 → Z8, Z11 → Z12, Z12 → Z13, Z13 → Z12};
        fixedpts = NSolve[Join[Join[Table[0 == EqsFPPC3Z[[k]], {k, 1, 38}],
             Table[VarsZ[[k]] == (VarsZ[[k]] /. CyclicZ), {k, 1, 14}]], {Z1 == 1}], 600];
        Length[fixedpts]
        FirstFourFixed =
         Table[RootApproximant[{Z1, Z2, Z3, Z4} /. fixedpts[[k + 1]], 12], {k, 1, 3}]
        w = Exp[2 Pi I / 3]
        RootReduce[(0.346... - 0.313... i) w + (0.173... + 1.22... i) w^2]
        RootReduce[(0.346... - 0.313... i) w^2 + (0.173... + 1.22... i) w]
        FullSimplify[
         Solve[{a1 FirstFourFixed[[1]][[1]] + a2 FirstFourFixed[[1]][[2]] == 1}][[1]]]

Out[•]= 6
```

Out[•]= $\left\{\left\{1, \frac{-7232-6409\,i\,\sqrt{2}}{22815}, \frac{-7232-6409\,i\,\sqrt{2}}{22815}, \frac{-7232-6409\,i\,\sqrt{2}}{22815}\right\},\right.$
$\{1, 0.346\ldots-0.313\ldots i, 0.346\ldots-0.313\ldots i, 0.346\ldots-0.313\ldots i\},$
$\left.\{1, 0.173\ldots+1.22\ldots i, 0.173\ldots+1.22\ldots i, 0.173\ldots+1.22\ldots i\}\right\}$

Out[•]= $e^{\frac{2 i \pi}{3}}$

Out[•]= $\frac{478046747-96008522\,i\,\sqrt{2}}{448337565}$

Out[•]= $\frac{-710395495-190928201\,i\,\sqrt{2}}{448337565}$

Out[•]= $\left\{a2 \to \frac{22815\,(-1+a1)}{7232+6409\,i\,\sqrt{2}}\right\}$

In[•]:=
```
(* We can not make these value too nice,
but we can try to make FirstFourFixed[[1]][[2]] and the w-
 traces of the others as small as possible.
   Specifically, we want to use a1,
a2 in Q(i √2) to get the following three values smaller. *)
```

$$\text{wantsmall} = \text{FullSimplify}\left[\left\{a1 + \frac{-7232 - 6409\, i\, \sqrt{2}}{22\,815}\, a2,\ -a1 + \frac{478\,046\,747 - 96\,008\,522\, i\, \sqrt{2}}{448\,337\,565}\, a2,\ -a1 + \frac{-710\,395\,495 - 190\,928\,201\, i\, \sqrt{2}}{448\,337\,565}\, a2\right\}\right]$$

$$\text{AllOptions} = \text{FullSimplify}\left[\left\{\text{wantsmall} /. \{a1 \to 1, a2 \to 0\},\ \text{wantsmall} /. \left\{a1 \to i\sqrt{2}, a2 \to 0\right\},\ \text{wantsmall} /. \{a2 \to 1, a1 \to 0\},\ \text{wantsmall} /. \left\{a2 \to i\sqrt{2}, a1 \to 0\right\}\right\}\right]$$

Out[•]=
$$\left\{a1 + \frac{\left(-7232 - 6409\, i\, \sqrt{2}\right) a2}{22\,815},\ -a1 + \frac{\left(478\,046\,747 - 96\,008\,522\, i\, \sqrt{2}\right) a2}{448\,337\,565},\ -a1 + \frac{\left(-710\,395\,495 - 190\,928\,201\, i\, \sqrt{2}\right) a2}{448\,337\,565}\right\}$$

Out[•]=
$$\left\{\{1, -1, -1\},\ \left\{i\sqrt{2}, -i\sqrt{2}, -i\sqrt{2}\right\},\ \left\{\frac{-7232 - 6409\, i\, \sqrt{2}}{22\,815},\ \frac{478\,046\,747 - 96\,008\,522\, i\, \sqrt{2}}{448\,337\,565},\ \frac{-710\,395\,495 - 190\,928\,201\, i\, \sqrt{2}}{448\,337\,565}\right\},\ \left\{\frac{2\left(6409 - 3616\, i\, \sqrt{2}\right)}{22\,815},\ \frac{192\,017\,044 + 478\,046\,747\, i\, \sqrt{2}}{448\,337\,565},\ \frac{381\,856\,402 - 710\,395\,495\, i\, \sqrt{2}}{448\,337\,565}\right\}\right\}$$

```
In[•]:= PossibleLinComb =
   {{a1 → 1, a2 → 0}, {a1 → ⅈ √2, a2 → 0}, {a2 → 1, a1 → 0}, {a2 → ⅈ √2, a1 → 0}};
ToBeSmall = {};
For[i = 1, i ≤ 4, i++,
   row = wantsmall /. PossibleLinComb[[i]];
   newrow = {};
   For[j = 1, j ≤ 3, j++,
    AppendTo[newrow,
      (-1 / 2) SeriesCoefficient[MinimalPolynomial[row[[j]] + 100 + 100 ⅈ √2, t],
         {t, 0, 1}] / SeriesCoefficient[
         MinimalPolynomial[row[[j]] + 100 + 100 ⅈ √2, t], {t, 0, 2}] - 100];
    AppendTo[newrow, (-1 / 2) SeriesCoefficient[MinimalPolynomial[
          row[[j]] / ⅈ √2 + 100 + 100 ⅈ √2, t], {t, 0, 1}] / SeriesCoefficient[
         MinimalPolynomial[row[[j]] + 100 + 100 ⅈ √2, t], {t, 0, 2}] - 100];
   ];
   AppendTo[ToBeSmall, newrow];
  ];
ToBeSmall
```

Out[•]=

$$\left\{\{1, 0, -1, 0, -1, 0\}, \{0, 2, 0, -2, 0, -2\},\right.$$
$$\left\{-\frac{7232}{22\,815}, -\frac{986}{1755}, \frac{478\,046\,747}{448\,337\,565}, -\frac{192\,017\,044}{448\,337\,565}, -\frac{142\,079\,099}{89\,667\,513}, -\frac{381\,856\,402}{448\,337\,565}\right\},$$
$$\left.\left\{\frac{986}{1755}, -\frac{14\,464}{22\,815}, \frac{192\,017\,044}{448\,337\,565}, \frac{956\,093\,494}{448\,337\,565}, \frac{381\,856\,402}{448\,337\,565}, -\frac{284\,158\,198}{89\,667\,513}\right\}\right\}$$

In[*]:=
```
(* we want to find a linear combination of these four vectors
 in dimension 6 space that would have small integer values *)
MatrixRank[ToBeSmall]
{u, r, v} = SmithDecomposition[ToBeSmall];
vt = Transpose[v];
SmallBestRels = {};
For[i = 1, i ≤ Length[ToBeSmall], i++,
  If[Mod[i, 10] == 0, Print[i];];
  basisvec = Table[0, {k, 1, Length[ToBeSmall[[1]]]}];
  basisvec[[i]] = 1;
  AppendTo[SmallBestRels, LinearSolve[vt, basisvec]];
 ];
ByteCount[SmallBestRels]
SmallerBestRels = LatticeReduce[SmallBestRels];
ByteCount[SmallerBestRels]
```

Out[*]= 4

Out[*]= 928

Out[*]= 304

In[∘]:=
```
SmallerBestRels
({b1, b2, b3, b4} /.
  Solve[{b1, b2, b3, b4}.ToBeSmall == {13, -35, 27, 54, -14, -90}][[1]])
{a1, a2} /.
  {{a1 → 1, a2 → 0}, {a1 → ⅈ √2, a2 → 0}, {a2 → 1, a1 → 0}, {a2 → ⅈ √2, a1 → 0}};
FullSimplify[wantsmall /. {a1 → 114916/23859 - 185699/47718 ⅈ √2,
    a2 → 243097205/12430539 + 636441325/24861078 ⅈ √2 }];

Table[RootApproximant[
  {Z1, a1 Z1 + a2 Z2, a1 Z1 + a2 Z3, a1 Z1 + a2 Z4} /. {a1 -> 114916/23859 - 185699/47718 ⅈ √2,
     a2 → 243097205/12430539 + 636441325/24861078 ⅈ √2 } /. fixedpts[[k + 1]], 12], {k, 1, 3}]
(* So this looks pretty good. How do we make a change of variables so that new Z-
 s are related to old Z-s this way *)
ZtoNewZ = FullSimplify[Solve[{Z1n, Z2n, Z3n, Z4n} ==
       ({Z1, a1 Z1 + a2 Z2, a1 Z1 + a2 Z3, a1 Z1 + a2 Z4} /. {a1 -> 114916/23859 - 185699/47718 ⅈ √2,
          a2 → 243097205/12430539 + 636441325/24861078 ⅈ √2 }), {Z1, Z2, Z3, Z4}]][[1]];
NewZtoZ = {Z1n → Z1, Z2n → a1 Z1 + a2 Z2, Z3n → a1 Z1 + a2 Z3, Z4n → a1 Z1 + a2 Z4} /.
   {a1 -> 114916/23859 - 185699/47718 ⅈ √2, a2 → 243097205/12430539 + 636441325/24861078 ⅈ √2 };
```

Out[∘]=
```
{{-1, 0, 1, 0, 1, 0}, {0, -1, 0, 1, 0, 1},
 {13, -35, 27, 54, -14, -90}, {-35, -26, 54, -54, -90, 28}}
```

Out[∘]=
$$\left\{\frac{114916}{23859}, -\frac{185699}{47718}, \frac{243097205}{12430539}, \frac{636441325}{24861078}\right\}$$

Out[∘]=
$$\left\{\left\{1, \frac{1}{2}\left(26 - 35\, i\sqrt{2}\right), \frac{1}{2}\left(26 - 35\, i\sqrt{2}\right), \frac{1}{2}\left(26 - 35\, i\sqrt{2}\right)\right\},\right.$$
$$\{1, 22.9\ldots + 0.892\ldots\, i, 22.9\ldots + 0.892\ldots\, i, 22.9\ldots + 0.892\ldots\, i\},$$
$$\left.\{1, -35.9\ldots + 24.6\ldots\, i, -35.9\ldots + 24.6\ldots\, i, -35.9\ldots + 24.6\ldots\, i\}\right\}$$

```
(* Let us see if the equation of degree 12 looks good in these Z-s *)
(* The name of the file needs to be
 adjusted according to your situation. *)
fname = "Desktop/FPP2020/Eq12";
OpenRead[fname];
Eq12S = Read[fname];
Close[fname];
```

In[∘]:=
```
ptsZnew = {};
For[numpt = 1, numpt ≤ 500, numpt++,
  If[Mod[numpt, 100] == 20, Print[numpt]];
  pt = {S1, S2, S3, S4} /. NSolve[{0 == Eq12S, S1 == 1,
        S2 == RandomInteger[1000] / 1000 + I RandomInteger[1000] / 1000,
        S3 == RandomInteger[1000] / 1000 + I RandomInteger[1000] / 1000}, 600][[1]];
  AppendTo[ptsZnew, NewZtoZ /. NSolve[({S1, S2, S3, S4} /. StoZ) == pt][[1]]];
 ];
```

```
20
120
220
320
420
```

In[∘]:=
```
MonsZ12n = {};
For[i1 = 0, i1 ≤ 12, i1++,
  For[i2 = 0, i2 ≤ 12 - i1, i2++,
   For[i3 = 0, i3 ≤ 12 - i1 - i2, i3++,
    AppendTo[MonsZ12n, Z1n^i1 Z2n^i2 Z3n^i3 Z4n^(12 - i1 - i2 - i3)];
   ]]];
BigMatrix = MonsZ12n /. ptsZnew;

ns = NullSpace[BigMatrix];
Length[ns]
Accuracy[ns]
nsr = Block[{$MaxExtraPrecision = 1000},
   RootApproximant[RowReduce[Chop[ns, 10^(-500)]], 2]];
Eq12Znew = Numerator[Factor[nsr[[1]].MonsZ12n]];
ByteCount[Eq12S]
ByteCount[Eq12Znew]
```

Out[∘]= 1

Out[∘]= 591.149

Out[∘]= 380 000

Out[∘]= 340 768

Let us just write equations in new variables W and see if these are at all shorter.

In[∘]:=
```
VarsW = {W1, W2, W3, W4, W5, W6, W7, W8, W9, W10, W11, W12, W13, W14};
WtoZ = Join[{W1 → Z1n, W2 → Z2n, W3 → Z3n, W4 → Z4n} /. NewZtoZ,
   Table[VarsW[[k]] → VarsZ[[k]] / (1.61… × 10^-4 + 3.79… × 10^-3 ⅈ), {k, 5, 14}]];
PtsOnXinW = {};

For[j = 1, j ≤ Length[PtsOnXinZ], j++,
  AppendTo[PtsOnXinW,
    N[Table[VarsW[[k]] → (VarsW[[k]] /. WtoZ) /. PtsOnXinZ[[j]], {k, 1, 14}], 2000]];
 ];

AllMons5 = {};
For[i = 1, i ≤ 4, i++,
 For[j = 5, j ≤ 14, j++, AppendTo[AllMons5, VarsW[[i]] × VarsW[[j]]];]];
AllMons6 = {};
For[i = 5, i ≤ 14, i++,
 For[j = 5, j ≤ i, j++, AppendTo[AllMons6, VarsW[[i]] × VarsW[[j]]];]];
For[i = 1, i ≤ 4, i++, For[j = 1, j ≤ i, j++,
   For[k = 1, k ≤ j, k++, AppendTo[AllMons6, VarsW[[i]] × VarsW[[j]] × VarsW[[k]]];]]];

ns = RowReduce[NullSpace[Table[AllMons5[[k]] /. PtsOnXinW[[l]],
      {l, 1, Length[PtsOnXinW]}, {k, 1, Length[AllMons5]}]]];
ns = Chop[ns, 10^(-Round[Accuracy[ns]] + 5)];
Accuracy[ns]
ns = Block[{$MaxExtraPrecision = 400}, RootApproximant[ns, 2]];
EqsFPPC3Wdeg5 = ns.AllMons5;
ns = RowReduce[NullSpace[Table[AllMons6[[k]] /. PtsOnXinW[[l]],
      {l, 1, Length[PtsOnXinW]}, {k, 1, Length[AllMons6]}]]];
ns = Chop[ns, 10^(-Round[Accuracy[ns]] + 5)];
Accuracy[ns]
ns = Block[{$MaxExtraPrecision = 400}, RootApproximant[ns, 2]];
EqsFPPC3Wdeg6 = ns.AllMons6;
EqsFPPC3W = Join[Table[Numerator[Factor[EqsFPPC3Wdeg5[[k]]]], {k, 1, 9}],
   Table[Numerator[Factor[EqsFPPC3Wdeg6[[k]]]], {k, 1, 29}]];
```

Out[∘]= 1928.39

Out[∘]= 1917.98

In[∘]:=
```
sls = NSolve[
   Join[Table[0 == EqsFPPC3W[[k]], {k, 1, 38}], {W1 == 0, W2 == 1, W3 == 0, W4 == 0}], 500];
RootApproximant[VarsW /. sls]
```

Out[∘]= {{0, 1, 0, 0, -1, 0, 0, -1, 0, 0, -1, 0, 0, -1}, {0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1}}

Let us not try to make this nice but just go with what we have. Let us find a bunch of points on this surface with high accuracy.

In[ ]:=
```
SetOfPointsW = {};
prec = 6000;
(* to be skipped the second time around *)
numpts = 200;
For[i = 1, i ≤ numpts, i++,
  If[Mod[i, numpts / 10] == 0, Print[i]];
  rpt1 = (RandomInteger[{1, 100 000}] + I RandomInteger[{1, 100 000}]) / 100 000;
  rpt2 = (RandomInteger[{1, 100 000}] + I RandomInteger[{1, 100 000}]) / 100 000;
  Eqs = Join[{W1 == 1, W2 + W3 + W4 == rpt1, W5 + W6 + W7 == rpt2},
    Table[0 == EqsFPPC3W[[j]], {j, 1, 38}]];
  sols = NSolve[Eqs, 200];
  fr = FindRoot[Table[Eqs[[k]], {k, 1, 14}],
    Table[{VarsW[[j]], VarsW[[j]] /. sols[[1]]}, {j, 1, 14}],
    MaxIterations → 100, WorkingPrecision → prec, AccuracyGoal → prec];
  AppendTo[SetOfPointsW, fr];
 ];
(* quality control *)
For[i = 1, i ≤ numpts, i++,
  If[Chop[Sum[Abs[EqsFPPC3W[[j]]] /. SetOfPointsW[[i]], {j, 1, 38}]] ≠ 0,
    Print["warning, point ", i, " looks fishy"];];];
```

```
20
40
60
80
100
120
140
160
180
200
```

```
(* The name of the file needs to be
 adjusted according to your situation. *)
fname = "Desktop/FPP2020/SetOfPointsW";
OpenWrite[fname];
Write[fname, SetOfPointsW];
Close[fname];

fname = "Desktop/FPP2020/EqsFPPC3W";
OpenWrite[fname];
Write[fname, EqsFPPC3W];
Close[fname];
```