Recap and Homework 2/24/2020

We covered associated primes and primary decomposition. All the rings and modules are assumed to be Noetherian.

Theorem. The set of nilpotent elements on M (which is by definition the radical $\sqrt{\mathrm{Ann}(M)}$ of the annihilator of M) is

$$\bigcap_{p\in \mathrm{Ass}(M)} p.$$

Proof. Finished in class.

Definition. A module M over R is called co-primary if $\mathrm{Ass}(M)$ consists of a single element. It is equivalent to the statement that every zero divisor on M is nilpotent on M.

Definition. An ideal I of R is called primary iff R/I is co-primary. This is equivalent to the following statement.

$$\text{If } ab \in I \text{ and } b \notin I, \text{ then } a^n \in I \text{ for some } n.$$

More generally, we call a submodule N of M primary if M/N is co-primary.

The main result of the class was the following.

Theorem. For every Noetherian module M there exist primary submodules $M_1, \ldots, M_n$ such that

$$0 = \bigcap_{i=1}^{n} M_i.$$

Proof. In class. The main idea was to consider submodules of M that can not be written as intersections of two strictly larger submodules. Noetherian property of M shows that 0 can be written as an intersection of such ideals. Then we can show that each of these ideals is primary.

Theorem. Let p be an associated prime of M. If $0 = \bigcap_{i=1}^{n} M_i$ with primary M_i, then one of the M/M_i has p as its (unique) associated prime.

Proof. In class. Main idea: the natural map $M \to \bigoplus_i M/M_i$ is injective.

Future plans. Next time we will argue that one can find a primary decomposition that involves exactly one M_i for each associated prime of M. We will also discuss uniqueness of lack thereof of such M_i.

Homework. All modules and rings are assumed Noetherian.

1. Let M_1 and M_2 be two primary submodules of a module M, with

$$\mathrm{Ass}(M/M_1) = \mathrm{Ass}(M/M_2) = \{p\}.$$

Prove that $M_1 \cap M_2$ is primary with $\mathrm{Ass}(M/M_1 \cap M_2) = \{p\}$.

2. Let I be a primary ideal of R and p the corresponding associated prime. Prove that $p \supseteq I \supseteq p^n$ for some n.

3. Prove that if m is a maximum ideal of R and an ideal I satisfies $m \supseteq I \supseteq m^n$ for some n, then I is primary. **The statement fails if one only assumes that m is prime. In fact, powers of prime ideals are not always primary!**

4. Let $R = \mathbb{Z}$ and let $M = \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}$.

(a) Prove that $\mathrm{Ass}(M) = \{2\mathbb{Z}, 3\mathbb{Z}, \{0\}\}$ (easy).

(b) Find a primary decomposition of M in the form $0 = M_1 \cap M_2 \cap M_3$ with M/M_i having the above three associated primes, in this order.

(c) Give another example with different (as submodules) M_1 and M_2.