Recap and Homework 2/20/2020
We covered associated primes, but didn't get very far.

Theorem. Every Noetherian module M over a Noetherian ring R has a finite filtration with successive quotients isomorphic to R/p_i for prime ideals p_i.

Proof. In class.

Definition. In the above setting, a prime p in R is called an associated prime of M iff there exists a submodule of M isomorphic to R/p (equivalently, there exists $x \in M$ with $\mathrm{Ann}(x) = p$.) We denote by $\mathrm{Ass}(M)$ the set of associated primes of M.

Lemma. For a short exact sequence of R-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

we have

$$\mathrm{Ass}(M_1) \subseteq \mathrm{Ass}(M_2) \subseteq \Big(\mathrm{Ass}(M_1) \cup \mathrm{Ass}(M_3)\Big).$$

Proof. In class.

Corollary. For a Noetherian module M over a Noetherian ring R the set $\mathrm{Ass}(M)$ is finite.

Proof. In class.

Definition. In the above setup, a zero divisor of R in M is an element $r \in R$ such that there exists a nonzero $x \in M$ with $rx = 0$.

Theorem. The set of zero divisors of M is

$$\bigcup_{p \in \mathrm{Ass}(M)} p$$

in the above setup.

Proof. In class.

Definition. In the above setup, an element r of R is nilpotent on M iff $r^n M = 0$ for some positive integer n.

Theorem. The set of nilpotent elements on M (which is by definition the radical $\sqrt{\mathrm{Ann}(M)}$ of the annihilator of M) is

$$\bigcap_{p \in \mathrm{Ass}(M)} p$$

in the above setup.

Proof. I proved

$$\bigcap_{p \in \mathrm{Ass}(M)} p \supseteq \sqrt{\mathrm{Ann}(M)}$$

in class (easy). I made a bit of an associated prime of myself trying to prove the other inclusion. We will return to it next time.

Homework.

1. (a) Find two different filtrations with prime quotients of the module $M = \mathbb{C}[x, y]/\langle x^2 y\rangle$ over $R = \mathbb{C}[x, y]$.

(b) Find $\mathrm{Ass}(M)$.

2. Find $\mathrm{Ass}(M)$ for $M = \mathbb{C}[x, y]/\langle x^2, xy\rangle$ over $R = \mathbb{C}[x, y]$.

3. For an ideal I of R and an element $a \in R$ we define $I : a$ to be the set of $x \in R$ such that $ax \in I$.

(a) Prove that $I : a$ is an ideal.

(b) Prove that a prime ideal p of R can be written in the form $I : a$ if and only if it is an associated prime of R/I.

4. Let M be an R-module, and let S be a multiplicatively closed subset of R. Prove that if $p \in \mathrm{Ass}(M)$ and $p \cap S = \emptyset$, then $S^{-1}p$ is an associated prime of $S^{-1}M$.