Problem Set 8. Invariants and Monovariants

1. We start with an $m \times n$ rectangular chocolate bar internally subdivided into an array of mn squares. We start by picking up the chocolate bar and breaking it along one of its internal lines. We then pick up a piece and repeat this process until there's nothing left to break apart. Show that regardless of how we go about this process, it always takes the same number of steps.

2. A bug starts at the number 1. Each turn, if she is at x, she can either jump to x^2, to $10 + x$, or to $3x - 4$. Is it possible, after some series of hops, for her to ever land on 2020?

3. Given n red points and n blue points in the plane, show that we can draw n non-intersecting line segments, each having one red endpoint and one blue endpoint.

4. Some people are in a building with several rooms. Each minute, one person leaves a room and moves to another that has at least as many people. Show that eventually all of the people are in a single room.

5. A real number is written in each square of an $n \times n$ array. We can pick any row we like and multiply all the entries in that row by -1. We can do the same with any column. Prove that after performing these operations a finite number of times, we can make it so that all the row sums and all the column sums are positive.

6. Several positive integers are written on a blackboard (possibly with repetition). One can erase any two distinct integers and write their greatest common divisor and least common multiple instead. Prove that eventually you will end up with a set of numbers such that every number is a divisor of every number larger than it.

7. A deck contains cards numbered $1, 2, \ldots, n$ in some order. We repeat the following: when the card on the top of the deck is numbered k, reverse the order of the first k cards. Show that 1 is eventually on the top of the deck.

8. Start with three stones at coordinates $(0, 0)$, $(1, 0)$, and $(0, 1)$. At each step, you may remove one of the stones with coordinates (x, y) and instead put down two stones, one at $(x + 1, y)$ and another at $(x, y + 1)$, assuming there are no stones at these coordinates. Can you devise a finite sequence of steps after which all three of the initially occupied coordinates will be empty?

9. Four congruent right triangles are given. One can cut one of them along the altitude and repeat the operation several times with the newly obtained triangles. Prove that no matter how we perform a finite number of cuts, we can always find among the triangles two that are congruent.

10. An infection spreads among the squares of an $n \times n$ checkerboard in the following manner: if a square has two or more infected neighbours, then it becomes infected itself. Neighbours are orthogonal only, so each square has at most four neighbours. Prove that you cannot infect the whole board if you begin with $n - 1$ infected squares.