MATH 642:491

Problem Set 7-Polynomials [1]

1. (a) Let $P(X)$ be a quadratic polynomial with real coefficients. If $P(0)$, $P(1)$ and $P(2)$ are all integers, show that P maps every integer to an integer.
   (b) Show that there are quadratic polynomials P such that $P(0)$, $P(1)$ and $P(3)$ are integers, but P does not map every integer to an integer.
2. $P(X)$ is a polynomial such that $P(1) = 1$ and $P(-1) = 2$. What is the remainder when you divide $P(X)$ by $X^2 - 1$?
3. Let $P^{[n]}(x)$ denote the polynomial defined by the recurrence $P^{[n]}(x) = P(x)$ and for $n > 1$, $P^{[n]}(x) = P(P^{(n-1)}(x)$. Let S_n be the set of all polynomials P with complex coefficients such that $P^{[n]}(x) = P(x)$. Determine S_n.
4. Suppose $P(X) \in \mathbb{C}[X]$ is such that for every $x \in \mathbb{R}$, $P(x) \in \mathbb{R}$. Show that all the coefficients of $P(X)$ are real numbers.
5. Find all complex numbers a, b such that the roots of $x^2 + ax + b$ are $\{a, b\}$.
6. Let $P(X)$ be a real polynomial that is non-negative for all $X \in \mathbb{R}$. Prove that, for some k, there are real polynomials $f_1(X), \ldots, f_k(X)$ such that $P(X) = \sum_{i=1}^{k} (f_i(X))^2$.
7. Show that the polynomial
$$P(X) = X^n + X^{n-1} + X^{n-2} + a_3X^{n-3} + a_4X^{n-4} + \ldots + a_n$$
cannot have all its roots real.
8. Prove that if $f(X)$ is a polynomial with integer coefficients, and there exists an integer k such that none of the integers $f(1), f(2), \ldots, f(k)$ are divisible by k, then f has no integral roots.
9. Let $f(X)$ be a polynomial, and suppose that $f(X) + f'(X) > 0$ for all $X \in \mathbb{R}$. Prove that $f(X) > 0$ for all $X \in \mathbb{R}$.
10. Let $f(X)$ be a non-zero polynomial with integer coefficients. Can it be the case that $f(n)$ is prime for all sufficiently large $n \in \mathbb{N}$?
11. Let $P(X)$ be a polynomial of degree n such that $P(X) = Q(X)P''(X)$, where $Q(X)$ is a quadratic polynomial. Show that if $P(X)$ has at least two distinct roots, then it must have n distinct roots.
12. Let $P(X) = c_nX^n + c_{n-1}X^{n-1} + \cdots + c_0$ be a polynomial with integer coefficients. Suppose that r is a rational root of P. For $j \in \{1, \ldots, n\}$ let $a_j = \sum_{i=0}^{j-1} c_{n-i}r^{j-i}$. Prove that a_i is an integer for all i.
$$c_nr, c_nr^2 + c_{n-1}r, c_nr^3 + c_{n-1}r^2 + c_{n-2}r, \ldots, c_nr^n + c_{n-1}r^{n-1} + \cdots + c_1r$$
are all integers.

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Hint. Work backwards from a_n. $a_n = P(r) - c_0 = -c_0$. For $j < n$, $a_j = \frac{1}{r}a_{j+1} - c_{n-j}$. Write $r = a/b$ where a and b are relatively prime and show that a_{j+1} is divisible by p.

13. Suppose that real numbers $a_0, a_1, \ldots, a_n$, and some $x \in (0,1)$ satisfy

$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \cdots + \frac{a_n}{1-x^{n+1}} = 0.$$

Then show that there is some $y \in (0,1)$ such that

$$a_0 + a_1 y + \cdots + a_n y^n = 0.$$

14. Let $P(X) = X^r + a_1 X^{r-1} + \ldots + a_{r-1}X + a_r$ be a polynomial with complex coefficients such that $P(X)$ divides $X^n - 1$. Show that $|a_i| \leq \binom{r}{i}$.

15. Show that $X^{100} - X - 5$ cannot be written as the product of two polynomials of degree at least one with integer coefficients.