

Problem Set 6- Inequalities ¹

1. Prove that $n! \geq (n/e)^n$ and that $n! \leq (n+1) \left(\frac{n+1}{e}\right)^n$.
2. Suppose that $a_1, a_2, \dots$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} a_n$ converges. Prove that for any $p > 1/2$, $\sum_{n \geq \infty} \sqrt{a_n}/n^p$ also converges.
3. Let $x_1, \dots, x_n$ be positive real numbers and k a positive integer. Prove $\frac{1}{n} \sum_i x_i^k \leq \frac{\sum_i x_i^{k+1}}{\sum_i x_i}$
4. Let $a_1, a_2, \dots, a_n$ be positive real numbers and let s denote their sum. Show that $(1+a_1)(1+a_2)\dots(1+a_n) \leq \sum_{i=0}^n s^i/i!$.
5. Let $a_1, \dots, a_n$ be real numbers that are not all 0. Determine the minimum possible value of $\sum x_i^2$ over all choices of real numbers $x_1, \dots, x_n$ such that $\sum_i a_i x_i = 1$.
6. For n positive real numbers with minimum m and maximum M , let A and G denote their arithmetic and geometric means. Prove that $A - G \leq (\sqrt{M} - \sqrt{m})^2/n$.
7. Let $x_1, \dots, x_n$ be real numbers in $[0, \pi]$ Let x be their average. Prove that:

$$\prod_{i=1}^n \sin(x_i)/x_i \leq (\sin(x)/x)^n.$$
8. If a, b, c are positive reals with $abc = 1$. Show that:

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

9. If a, b, c are positive reals, show that:

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{1+abc}.$$

10. (*Putnam) Let $a_1, \dots, a_n, b_1, \dots, b_n$ be nonnegative real numbers. Show $(a_1 \cdots a_n)^{1/n} + (b_1 \cdots b_n)^{1/n} \leq [(a_1 + b_1) \cdots (a_n + b_n)]^{1/n}$.
11. (*Putnam) Prove or disprove: If x and y are real numbers with $y \geq 0$ and $y(y+1)^2 \leq (x+1)^2$ then $y(y-1) \leq x^2$.

¹Version 6-30-2025