

Problem Set 5 - Symmetry ¹

Tip: Don't use brute force on these problems.

1. A tosses n fair coins and B tosses $(n - 1)$ fair coins. Find the probability that A gets strictly more heads than B .
2. A, B, C, D are different points on a plane. $AB = AD, CB = CD$. Prove that $AC \perp BD$.
3. The Dice Master rolls a $D20$ (a fair 20-side die) and one Player rolls three $D6$'s (a fair 6-side die) at the same time. The one with the highest total takes 1 dollar from the others (do nothing when tied). Is this a fair game (meaning: is it true that both players have an equal chance to win.) (Optional question: What if there are more Players in the same game (they all roll three $D6$'s)?)
4. A and B take turns putting quarters on a rectangle table. Each coin is fully laid on the table, not being stacked on other coins. A goes first and the player not able to place a coin loses. Who has the winning strategy?
5. Five integers are chosen uniformly at random in the range $[0, 99]$. Let $X_1 \leq X_2 \leq X_3 \leq X_4 \leq X_5 \leq X_6$ be the selected numbers in order. Determine the expected value of $X_4 - X_2$.
6. $X_1, X_2, \dots$ are independent and X_n is a uniformly chosen integer within 1 to n . Find the probability that $\sum_{n=1}^{640} X_n$ is divisible by 491.
7. For positive integers m, n , consider the 2-player game $G(m, n)$ where players take turns selecting integer points in $[1, m] \times [1, n]$ subject to the following rules: The point (x, y) selected at each step is not allowed to be $\leq$ any previously selected point. (Here we say $(x, y) \leq (x', y')$ if $x \leq x'$ and $y \leq y'$.) The game ends when (m, n) is selected, and the player who selects (m, n) is the loser. Determine the set of all pairs (m, n) of positive integers for which the first player has a winning strategy in the game $G(m, n)$.
8. Prove that

$$\int_0^{\pi/2} \ln(\sin x) dx = -\frac{\pi \ln 2}{2}$$
9. (a) A, B, C, D are independently and uniformly chosen on a circle. Find the probability that AB intersects CD . (b) A, B, C, D, E, F are independently and uniformly chosen on a circle. Find the probability that the triangles ABC and DEF are disjoint.
10. (*Putnam) Four points are independently and uniformly chosen on a unit sphere. Find the probability that the center of the sphere is inside the tetrahedron formed by the four chosen points. You may think about the two dimensional version first.
11. (*Putnam) Prove that there is no way to partition the unit disk into two congruent sets.

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