MATH 640:491

Problem Set 2 (More introductory Problems.)[1]

1. Find the smallest positive integer having exactly 100 positive divisors.

2. Let $a_1, a_2, \ldots$ be an arbitrary sequence of distinct real numbers lying in the interval $(0,1)$. Starting from the interval $(0,1)$ we successively remove each of the numbers from the interval. After removing $a_1, \ldots, a_{i-1}$ the set of points remaining consists of i disjoint intervals. Exactly one of those intervals contains a_i and removing a_i splits that interval into two intervals, for a total of $i+1$ intervals. Define the ith score s_i to be the length of the interval that contained a_i multiplied by the product of the lengths of the two smaller intervals that were created. Define the total score to be the (infinite) sun of all of the scores.

   How should you choose the infinite sequence $a_1, a_2, \ldots$ so as to maximize the score. What is the maximum score possible?

3. Suppose we define the *linear size* of a box B, denote $LS(B)$ to be the sum of its length, width and height. Is it possible to construct two boxes B, C where $LS(B) < LS(C)$ and C fits inside of B (where we are allowed to rotate C to fit inside of B).

4. Given 6 points in the plane, show that the ratio of the maximum distance between two of them to the minimum distance between two of them is at least $\sqrt{3}$.

5. Evaluate $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}}$. (More formally, let $x_1 = \sqrt{2}$, and let $x_{n+1} = \sqrt{2}^{x_n}$ for each $n \geq 1$. Determine whether $\lim_{n\to\infty} x_n$ exists, and if it does find it.)

6. Standard U.S. coins are pennies (1 cent), nickels (5 cents), dimes (10 cents), quarters (25 cents), half dollars (50 cents) and dollars (100 cents). Say that an integer k is *convertible* provided that it is possible to find a collection of k coins that adds up to a dollar. What is the smallest positive integer that is not convertible?

   So there is no cricial configuration with fewer than 76 coins. It can be checked that 77 is not convertible and so 77 is the least integer that is not convertible.

7. Evaluate the determinant of the $n \times n$ matrix whose (i,j)th entry is $a^{|i-j|}$.

8. For a nonnegative integer n, let $f(n)$ be the number of ways to express n as a sum of powers of 2, where each power of 2 is used at most twice. For example, $f(6) = 3$ since $6 = 4+2 = 4+1+1 = 2+2+1+1$. (We define $f(0) = 1$). For a positive integer n. let $r(n) = f(n)/f(n-1)$. The function r is a rather amazing function: it is a one-to-one correspondence (bijection) between the set of positive integers and the set of positive rational numbers! Prove this.

[1] Version 6-30-2025