Problem Set 10-Geometry, with hints

1. A and B are two points on the same side of a line L in the plane. What is the shortest way to go from A to B touching L along the way?

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function with the property that $f(A) + f(B) + f(C) + f(D) = 0$ for any four points A, B, C and D that form the vertices of a square. What is f?

3. Given any five points on the surface of a sphere in $\mathbb{R}^3$, show that there is a closed hemisphere containing at least four of them.

4. Let $p_1, \ldots, p_n$ be distinct points in the closed unit disc in the plane. Let d_k be the distance from p_k to the nearest other point. Show that $\sum_{k=1}^{n} (d_k)^2 \leq 16$.

5. Given is a finite set of spherical planets in $\mathbb{R}^3$, all of the same radius and no two intersecting. On the surface of each planet consider the set of points not visible from any other planet. Prove that the total area of these sets is equal to the surface area of one planet.

6. For $i = 1, 2$ let T_i be a triangle with side lengths a_i, b_i, c_i, and area A_i. Suppose that $a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2$, and that T_2 is an acute triangle. Does it follow that $A_1 \leq A_2$?

7. Suppose A, B and C are three distinct non-collinear points with integer coordinates such that the lengths $|AB|$, $|BC|$ and $|CA|$ are all integers. What's the smallest possible value of $|AB|$?

8. Let S be a finite set of points in the plane with the property that the area of any triangle comprised of three points from S is at most 1. Show that S can be covered by a single triangle of area at most 4.

9. Suppose that a set $S \subset \mathbb{R}^2$ is covered by a finite collection of discs. Show that one can choose a subset $D_1, \ldots, D_n$ of these discs which are pairwise disjoint, such that S is still covered by $\cup_{i=1}^{n} 3D_i$, where $3D_i$ is the disc with same centre as D_i but of thrice the radius.

10. Suppose S is a collection of n points in the plane such that the line through any two points of S passes through a third point of S. Show that all the points of S are collinear.

11. Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A windmill is a process that starts with a line ℓ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the pivot P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of $\mathcal{S}$. This process continues indefinitely. Show that we can choose a point P in $\mathcal{S}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

12. Prove that any $a \times b$ rectangle can be cut with finitely many straight lines in such a way that the resulting pieces can be rearranged to form a $1 \times ab$ rectangle.